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On the Range of Certain ASH Algebras of Real Rank Zero*

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(*Dedicated to Prof. Guihua Gong on the occasion of his 60th birthday*)

Abstract In this paper, the authors consider the range of a certain class of ASH algebras in [An, Q., Elliott, G. A., Li, Z. and Liu, Z., The classification of certain ASH C*-algebras of real rank zero, *J. Topol. Anal.*, **14**(1), 2022, 183–202], which is under the scheme of the Elliott program in the setting of real rank zero C*-algebras. As a reduction theorem, they prove that all these ASH algebras are still the AD algebras studied in [Dadarlat, M. and Loring, T. A., Classifying C^*-algebras via ordered, mod-p K-theory, *Math. Ann.,* **305**, 1996, 601–616].

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1 Introduction

The history of the classification of amenable C*-algebras, begins with the UHF algebras of Glimm (see [7, 15]), and the AF algebras of Bratteli ([4] by diagrams, and also Elliott [11] by dimension group). And later, a very tidy result was given as Effros-Handelman-Shen Theorem (see [8]), which showed that all the unperforated Riesz groups just coincide with all dimension groups of AF algebras. Such studies can be the consideration of the question in the case of real rank zero. A C*-algebra is said to have real rank zero, if the set of invertible self-adjoint elements is dense in the set of self-adjoint elements. This was also the setting for the classification of AT algebras considered in [12]. Particularly, we wish to mention the exciting breakthrough that all simple separable unital $\mathcal{Z}$-stable C*-algebras can be classified by the Elliott invariant provided the universal coefficient theorem (UCT for short) holds (see [13, 17–18, 25]). So the direction to consider the real rank zero setting, which are not necessarily simple is the next natural restriction in the classification theory.

In a number of articles [5–6, 9, 12, 14], particular attention has been given to the case of inductive limits of dimension drop interval algebras (they called them AD algebras). The case of ordinary interval "no dimension drops" is easy, since a real rank zero inductive limit of such

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algebras must be an AF algebra. These papers consider the case of what might be known as classical dimension drop interval algebras with the same dimension drop at both ends of the interval. More recent papers (see [19, 20, 23–24]) go further, establish classification in the simple case for inductive limits of (finite direct sums of) generalized dimension drop interval algebras (possibly different dimension drops at the two endpoints). And recently, the authors, Elliott and Li classified such inductive limits (we call them $A\mathcal{JS}$ algebras) in the sense of real rank zero (see [2]), which seems larger than the class considered in [6] (the AD algebras above). But there is no evidence that the new class is really larger than the classical one.

It was shown in [10] (see also [26]) that an $A\mathcal{JS}$ algebra of real rank zero with trivial K_1-group is an AF algebra and its invariant stays in the range of Effros-Handelman-Shen Theorem (see [8]). Then how about those $A\mathcal{JS}$ algebras with non-trivial torsion K_1-groups, which are of course not AF or AT algebras?

In this paper, we point out that a generalized dimension drop algebra is always KK-equivalent to a classical dimension drop algebra, and hence, they have same K-theoretical invariant as group; however, they may have different Dadalat-Loring orders (see [6]). And we develop a series of reduction tricks, which shows that after composing a "large" map, the differences of the orders disappear. That is to say, we prove that an $A\mathcal{JS}$ algebra of real rank zero is always an AD algebra and such two classes of algebras have the same range of invariants.

2 Notations and Preliminaries

In this section, we collect some necessary definitions and set up notations for the convenience of readers.

Definition 2.1 (see [12]) *The classical dimension drop interval algebra refers to the* C^*-*algebra*

$$I_p = \{f \in \mathrm{M}_p(C_0(0,1]) :\ f(1) = \lambda \cdot 1_p,\ 1_p \textit{ is the identity of } \mathrm{M}_p\},$$

and the C^*-*algebra* $\widetilde{I}_p$ *obtained by adjoining a unit to* I_p.

We use $\gcd(m_0, m_1)$ to denote the greatest common divisor of m_0 and m_1.

Definition 2.2 (see [19]) *A Jiang-Su block, denoted by* $I[m_0, m, m_1]$, *is the unital* C^*-*algebra*

$$I[m_0, m, m_1] = \{f \in \mathrm{M}_m(\mathrm{C}([0,1])) : f(0) = a_0 \otimes 1_{m/m_0}, f(1) = a_1 \otimes 1_{m/m_1}\},$$

where $\gcd(m_0, m_1) = 1$, m_0, m_1 *divide* m, a_0 *and* a_1 *(for a given* f*) belong to* M_{m_0} *and* M_{m_1}, *respectively, and* $1_{m/m_0}$ *and* $1_{m/m_1}$ *are the identity elements of* M_{m/m_0} *and* M_{m/m_1}, *respectively.*

For convenience, we denote by D the class of all matrix algebras and all $M_r(\widetilde{I}_p)$ and at the same time, denote by $\mathcal{JS}$ the class of all matrix algebras and all $M_r(I[m_0, m, m_1])$ (generalized dimension drop algebra). Denote the inductive limit algebras of direct-sums in D and $\mathcal{JS}$ by AD algebras and $A\mathcal{JS}$ algebras, respectively. Note that $\widetilde{I}_p = I[1, p, 1]$, which means AD algebras are always $A\mathcal{JS}$ algebras. Note that every algebra in the classes AD and A$\mathcal{JS}$ is not necessarily simple.

Remark 2.1 The K-theoretic information in a generalized dimension drop interval algebra is summarized as follows. For $I[m_0, m, m_1]$, one has the short exact sequence

$$0 \to \mathrm{M}_m(\mathrm{C}_0(0,1)) \xrightarrow{\iota} I[m_0, m, m_1] \xrightarrow{\pi_0 \oplus \pi_1} \mathrm{M}_{m/m_0} \oplus \mathrm{M}_{m/m_1} \to 0,$$

where ι is the embedding map.

Then we have the six-term exact sequence

$$0 \to \mathrm{K}_0(I[m_0, m, m_1]) \xrightarrow{(\pi_0 \oplus \pi_1)_*} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{(\frac{m}{m_0}, -\frac{m}{m_1})} \mathbb{Z} \xrightarrow{\iota_*} \mathrm{K}_1(I[m_0, m, m_1]) \to 0.$$

Hence

$$\mathrm{K}_0(I[m_0, m, m_1]) = \mathbb{Z}, \quad \mathrm{K}_1(I[m_0, m, m_1]) = \mathbb{Z}_p,$$

where $p = \gcd\left(\frac{m}{m_0}, \frac{m}{m_1}\right)$.

Definition 2.3 *Let $A = I[m_0, m, m_1]$, $B = I[n_0, n, n_1]$. Denote by $C(A, B)$ the set of all the commutative diagrams*:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{K}_0(A) \arrow[r, "(\pi_0 \oplus \pi_1)_*"] \arrow[d, "\lambda^{0*}"'] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, "(\frac{m}{m_0}, -\frac{m}{m_1})"] \arrow[d, "\lambda^0"'] & \mathbb{Z} \arrow[r, "\iota_*"] \arrow[d, "\lambda^1"'] & \mathrm{K}_1(A) \arrow[r] \arrow[d, "\lambda^{1*}"'] & 0 \\
0 \arrow[r] & \mathrm{K}_0(B) \arrow[r, "(\pi'_0 \oplus \pi'_1)_*"'] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, "(\frac{n}{n_0}, -\frac{n}{n_1})"'] & \mathbb{Z} \arrow[r, "\iota'_*"'] & \mathrm{K}_1(B) \arrow[r] & 0
\end{tikzcd}$$

(λ^0, λ^1 *are* 2×2 *matrix and integer, respectively*) *and by $M(A, B)$ the subset of $C(A, B)$ of all the commutative diagrams*:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{K}_0(A) \arrow[r, "(\pi_0 \oplus \pi_1)_*"] \arrow[d, "0"'] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, "(\frac{m}{m_0}, -\frac{m}{m_1})"] \arrow[d, "\mu^0"'] & \mathbb{Z} \arrow[r, "\iota_*"] \arrow[d, "\mu^1"'] & \mathrm{K}_1(A) \arrow[r] \arrow[d, "0"'] & 0 \\
0 \arrow[r] & \mathrm{K}_0(B) \arrow[r, "(\pi'_0 \oplus \pi'_1)_*"'] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, "(\frac{n}{n_0}, -\frac{n}{n_1})"'] & \mathbb{Z} \arrow[r, "\iota'_*"'] & \mathrm{K}_1(B) \arrow[r] & 0,
\end{tikzcd}$$

such that there exists $\mu \in \mathrm{Hom}(\mathrm{K}_1(SM_m), \mathrm{K}_0(M_{n_0} \oplus M_{n_1}))$ satisfying $\mu^0 = \mu \circ \left(\frac{m}{m_0}, -\frac{m}{m_1}\right)$, $\mu^1 = \left(\frac{n}{n_0}, -\frac{n}{n_1}\right) \circ \mu$.

In particular, we point out that $C(A, B)$ is an Abelian group and $M(A, B)$ is its subgroup.

We quote the following result from [1] (see the notations and more details there). This is our technical tool for calculating.

Theorem 2.1 (see [1, Theorem 2.9]) *Given two Elliott-Thomsen algebras A and B. Then we have a natural isomorphism of groups*:

$$\mathrm{KK}(A, B) \cong C(A, B)/M(A, B).$$

In fact, for elements $\lambda \in C(A, B)$ and $\eta \in C(B, E)$, there is also an natural product $\lambda \times \eta \in C(A, E)$, which induces the Karsparov-product on the quotient groups.

For two matrices S and T, we say $S \geq T$, if $S - T$ has no negative entry.

Definition 2.4 *Given two Elliott-Thomsen algebras A and B. Let $\lambda \in C(A,B)$ be the following diagram*:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{K}_0(A) \arrow[r, "\pi_*"] \arrow[d, "\lambda^{0*}"'] & \mathrm{K}_0(F_1) \arrow[r, "\alpha-\beta"] \arrow[d, "\lambda^0"'] & \mathrm{K}_1(SF_2) \arrow[r, "\iota_*"] \arrow[d, "\lambda^1"'] & \mathrm{K}_1(A) \arrow[r] \arrow[d, "\lambda^{1*}"'] & 0 \\
0 \arrow[r] & \mathrm{K}_0(B) \arrow[r, "\pi'_*"'] & \mathrm{K}_0(F'_1) \arrow[r, "\alpha'-\beta'"'] & \mathrm{K}_1(SF'_2) \arrow[r, "\iota'_*"'] & \mathrm{K}_1(B) \arrow[r] & 0\,.
\end{tikzcd}$$

Let us say that λ is positive *or $\lambda \geq 0$ if $\lambda^0 \geq 0$, i.e., λ^0 has no negative entry.*

By Theorem 2.1, we denote $\mathrm{KK}(\lambda)$ by the KK-class corresponding to the diagram λ.

Theorem 2.2 (see [1, Theorem 3.8, 2, Proposition 4.3]) *Let $A, B \in \mathcal{JS}$. A diagram $\lambda \in C(A,B)$ can be lifted to a homomorphism if and only if λ is positive.*

Furthermore, if λ is postive and

$$\mathrm{KK}(\lambda)([1_A]) \leq [1_B],$$

then λ can be lifted to a homomorphism from A to B; if

$$\mathrm{KK}(\lambda)([1_A]) = [1_B],$$

then we can lift λ to a unital homomorphism.

3 Main Result

In this section, we will prove that an $A\mathcal{JS}$ algebra of real rank zero is an AD algebra, the main result is Theorem 3.1, a reduction theorem (In fact, even in the simple case, the reduction theorem also plays an important role (see [16])).

Lemma 3.1 *Let $A = M_r(I[m_0, m, m_1])$ and let $\lambda \in C(A, \widetilde{I}_q)$ be the following diagram*:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{K}_0(A) \arrow[r, "\pi_*"] \arrow[d, "\lambda^{0*}"'] & \mathbb{Z}\oplus\mathbb{Z} \arrow[r, "(\frac{m}{m_0}, -\frac{m}{m_1})"] \arrow[d, "\lambda^0"'] & \mathbb{Z} \arrow[r, "\iota_*"] \arrow[d, "\lambda^1"'] & \mathrm{K}_1(A) \arrow[r] \arrow[d, "\lambda^{1*}"'] & 0 \\
0 \arrow[r] & \mathrm{K}_0(B) \arrow[r, "\pi'_*"'] & \mathbb{Z}\oplus\mathbb{Z} \arrow[r, "(q,-q)"'] & \mathbb{Z} \arrow[r, "\iota'_*"'] & \mathrm{K}_1(B) \arrow[r] & 0.
\end{tikzcd}$$

Suppose that

$$\lambda^0 \begin{pmatrix} m_0 \\ m_1 \end{pmatrix} \geq \begin{pmatrix} 2m \\ 2m \end{pmatrix},$$

then there exists a homomorphism $\gamma : A \to M_s(\widetilde{I}_q)$ for some integer s such that $\mathrm{KK}(\gamma) = \mathrm{KK}(\lambda)$.

Proof Write $\lambda^0 = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, then by assumption, we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} m_0 \\ m_1 \end{pmatrix} \geq \begin{pmatrix} 2m \\ 2m \end{pmatrix}.$$

Set

$$k_0 = \left[\frac{bm_1 - am_0}{2m}\right] \quad \text{and} \quad k_1 = \left[\frac{dm_1 - cm_0}{2m}\right],$$

where $[x]$ is the largest integer smaller than x, then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} k_0 \\ k_1 \end{pmatrix} \left(\frac{m}{m_0}, -\frac{m}{m_1}\right) \geq 0.$$

Let ζ be the following diagram in $C(A, \widetilde{I}_q)$,

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{K}_0(A) \arrow[r, "\pi_*"] \arrow[d, "\zeta^{0*}"'] & \mathbb{Z}\oplus\mathbb{Z} \arrow[r, "(\frac{m}{m_0}, -\frac{m}{m_1})"] \arrow[d, "\zeta^0"'] & \mathbb{Z} \arrow[r, "\iota_*"] \arrow[d, "\zeta^1"'] & \mathrm{K}_1(A) \arrow[r] \arrow[d, "\zeta^{1*}"'] & 0 \\
0 \arrow[r] & \mathrm{K}_0(B) \arrow[r, "\pi'_*"'] & \mathbb{Z}\oplus\mathbb{Z} \arrow[r, "(q,-q)"'] & \mathbb{Z} \arrow[r, "\iota'_*"'] & \mathrm{K}_1(B) \arrow[r] & 0,
\end{tikzcd}$$

where

$$\zeta^0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} k_0 \\ k_1 \end{pmatrix} \left(\frac{m}{m_0}, -\frac{m}{m_1}\right)$$

and

$$\zeta^1 = \lambda^1 + k_0 q - k_1 q.$$

It is easy to check that $\zeta - \lambda \in M(A, \widetilde{I}_q)$. By Theorems 2.1–2.2, we can lift ζ to a homomorphism $\gamma : A \to M_s(\widetilde{I}_q)$ for some integer s. Then we have $\mathrm{KK}(\gamma) = \mathrm{KK}(\zeta) = \mathrm{KK}(\lambda)$.

Lemma 3.2 *Let $A = M_r(I[m_0, m, m_1])$, $B = M_s(I[n_0, n, n_1])$. Suppose that $\psi : A \to B$ is a homomorphism satisfying that*

$$[\psi(1_A)] \geq 2mr \begin{pmatrix} n_0 \\ n_1 \end{pmatrix} \quad \text{in } K_0(B).$$

Then there exists an algebra E in class D *such that ψ can be factored through E in the sense of* KK, *i.e., there exist homomorphisms $\gamma : A \to E$ and $\iota : E \to B$ such that*

$$\mathrm{KK}(\gamma) \times \mathrm{KK}(\iota) = \mathrm{KK}(\psi).$$

Proof Let $q = \gcd(\frac{n}{n_0}, \frac{n}{n_1})$ and $\rho \in C(\widetilde{I}_q, B)$ be the positive diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{K}_0(\widetilde{I}_q) \arrow[r, "\pi_*"] \arrow[d] & \mathbb{Z}\oplus\mathbb{Z} \arrow[r, "(q,-q)"] \arrow[d, "\begin{pmatrix} x & y \\ z & t \end{pmatrix}"'] & \mathbb{Z} \arrow[r, "\iota_*"] \arrow[d, "1"'] & \mathrm{K}_1(\widetilde{I}_q) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{K}_0(B) \arrow[r, "\pi'_*"'] & \mathbb{Z}\oplus\mathbb{Z} \arrow[r, "(\frac{n}{n_0}, -\frac{n}{n_1})"'] & \mathbb{Z} \arrow[r, "\iota'_*"'] & \mathrm{K}_1(B) \arrow[r] & 0,
\end{tikzcd}$$

where $x, y, z, t \geq 0$ satisfy that

$$x \cdot \frac{n}{n_0} - z \cdot \frac{n}{n_1} = q, \quad y \cdot \frac{n}{n_0} - t \cdot \frac{n}{n_1} = -q$$

and

$$\begin{pmatrix} x \\ z \end{pmatrix} + \begin{pmatrix} y \\ t \end{pmatrix} = \begin{pmatrix} n_0 \\ n_1 \end{pmatrix}.$$

Then we have

$$xt - zy = x(z+t) - z(x+y) = xn_1 - zn_0 = 1,$$

hence

$$\det\begin{pmatrix} x & y \\ z & t \end{pmatrix} = 1 \quad \text{and} \quad \begin{pmatrix} x & y \\ z & t \end{pmatrix}^{-1}\begin{pmatrix} n_0 \\ n_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

By Theorem 2.2, ρ can be lifted to a unital homomorphism embedding map ι from $M_s(\widetilde{I}_q)$ to B such that

$$K_0(\iota)\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} n_0 \\ n_1 \end{pmatrix} \quad \text{in } K_0(B).$$

In particular, the embedding map ι is of the following form

$$M_s(\widetilde{I}_q) \ni f(t) \overset{\iota}{\longmapsto} u \cdot \mathrm{diag}\{f(t), \underbrace{f(0), \cdots, f(0)}_{z\cdot\frac{n}{n_1}}, \underbrace{f(1), \cdots, f(1)}_{y\cdot\frac{n}{n_0}}\} \cdot u^* \in B,$$

where u is a certain unitary in $M_{sn}(C[0,1])$ (one can see more details in [1, Lemma 3.6]).

Denote $\rho^{-1} \in C(B, \widetilde{I}_q)$ to be the following diagram:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathrm{K}_0(B) & \xrightarrow{\pi'_*} & \mathbb{Z}\oplus\mathbb{Z} & \xrightarrow{(\frac{n}{n_0}, -\frac{n}{n_1})} & \mathbb{Z} & \xrightarrow{\iota'_*} & \mathrm{K}_1(B) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \begin{pmatrix} x & y \\ z & t \end{pmatrix}^{-1}} & & \downarrow{\scriptstyle 1} & & \downarrow & & \\
0 & \longrightarrow & \mathrm{K}_0(\widetilde{I}_q) & \longrightarrow & \mathbb{Z}\oplus\mathbb{Z} & \xrightarrow[(q,-q)]{} & \mathbb{Z} & \longrightarrow & \mathrm{K}_1(\widetilde{I}_q) & \longrightarrow & 0,
\end{array}$$

which may not be a positive diagram, while $\rho \times \rho^{-1}$ is

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathrm{K}_0(\widetilde{I}_q) & \longrightarrow & \mathbb{Z}\oplus\mathbb{Z} & \xrightarrow{(q,-q)} & \mathbb{Z} & \longrightarrow & \mathrm{K}_1(\widetilde{I}_q) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}} & & \downarrow{\scriptstyle 1} & & \downarrow & & \\
0 & \longrightarrow & \mathrm{K}_0(\widetilde{I}_q) & \longrightarrow & \mathbb{Z}\oplus\mathbb{Z} & \xrightarrow[(q,-q)]{} & \mathbb{Z} & \longrightarrow & \mathrm{K}_1(\widetilde{I}_q) & \longrightarrow & 0.
\end{array}$$

By Theorem 2.1, there is a diagram $\lambda \in C(A, B)/M(A, B)$ inducing the KK class of ψ:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathrm{K}_0(A) & \xrightarrow{\pi_*} & \mathbb{Z}\oplus\mathbb{Z} & \xrightarrow{(\frac{m}{m_0}, -\frac{m}{m_1})} & \mathbb{Z} & \xrightarrow{\iota_*} & \mathrm{K}_1(A) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \lambda^{0*}} & & \downarrow{\scriptstyle \lambda^0} & & \downarrow{\scriptstyle \lambda^1} & & \downarrow{\scriptstyle \lambda^{1*}} & & \\
0 & \longrightarrow & \mathrm{K}_0(B) & \xrightarrow[\pi'_*]{} & \mathbb{Z}\oplus\mathbb{Z} & \xrightarrow[(\frac{n}{n_0}, -\frac{n}{n_1})]{} & \mathbb{Z} & \xrightarrow[\iota'_*]{} & \mathrm{K}_1(B) & \longrightarrow & 0.
\end{array}$$

By assumption,

$$r \cdot \lambda^0 \begin{pmatrix} m_0 \\ m_1 \end{pmatrix} = kr\begin{pmatrix} n_0 \\ n_1 \end{pmatrix} \geq 2mr\begin{pmatrix} n_0 \\ n_1 \end{pmatrix}.$$

Then we have $k \geq 2m$ and

$$\begin{pmatrix} x & y \\ z & t \end{pmatrix}^{-1} \lambda^0 \begin{pmatrix} m_0 \\ m_1 \end{pmatrix} = k\begin{pmatrix} x & y \\ z & t \end{pmatrix}^{-1}\begin{pmatrix} n_0 \\ n_1 \end{pmatrix} = \begin{pmatrix} k \\ k \end{pmatrix} \geq \begin{pmatrix} 2m \\ 2m \end{pmatrix}.$$

By Lemma 3.1, the KK class induced by $\lambda \times \rho^{-1} \in C(A, \widetilde{I}_q)$ can be lifted as a homomorphism $\gamma : A \to M_s(\widetilde{I}_q)$, and now we have

$$\mathrm{KK}(\gamma) \times \mathrm{KK}(\iota) = \mathrm{KK}(\psi).$$

Let $E = M_s(\widetilde{I}_q)$, we achieve the proof.

The following lemma is [3, Corollary 4.3].

Lemma 3.3 *Let A be a semiprojective* C**-algebra generated by a finite or countable set $\mathcal{G} = \{x_1, x_2, \cdots\}$ with $\lim_{j\to\infty} \|x_j\| = 0$ if $\mathcal{G}$ is infinite. Then there is a $\delta > 0$ such that, whenever B is a* C**-algebra, and ϕ_0 and ϕ_1 are homomorphisms from A to B with $\|\phi_0(x_j) - \phi_1(x_j)\| < \delta$ for all j, then ϕ_0 and ϕ_1 are homotopic.*

We will need the decomposition theorem (see [2, Theorem 3.8, 21]).

Lemma 3.4 *Let $A, B \in \mathcal{JS}$, $F \subset A$ be a finite set, $\varepsilon > 0$, L be a positive integer, there exists a finite set $G \subset A_{sa}$ such that if a homomorphism $\psi : A \to B$ satisfies that*

$$\psi(G) \subset_{1/6} \{f \in B \mid f\, has\ finite\ spectrum\},$$

then there exists a projection $p \in B$ and a homomorphism $\nu : A \to (1-p)B(1-p)$ with finite dimensional image such that

(1) $[\nu(1_A)] \geq L \cdot [p]$ in $K_0(B)$,

(2) $\|\psi(f) - p\psi(f)p - \nu(f)\| < \varepsilon$, $\forall\, f \in F$.

Lemma 3.5 *Let $A = M_r(I[m_0, m, m_1])$, $B = M_s(I[n_0, n, n_1])$. Then there exists a finite set $G \subset A_{sa}$ such that if a homomorphism $\psi : A \to B$ satisfies that*

$$\psi(G) \subset_{1/6} \{f \in B \mid f\, has\ finite\ spectrum\},$$

then there exists an algebra E in D *such that ψ can be factored through E in the sense of* KK.

Proof Let F be a finite set which generates A. Since A is semiprojective (see [22]), we choose δ as in Lemma 3.3 and a finite set G as in Lemma 3.4. Set $L = 2mr$, by Lemma 3.4, there exists a projection $p \in B$ and a homomorphism $\nu : A \to (1-p)B(1-p)$ with finite dimensional image such that

(1) $[\nu(1_A)] \geq 2mr \cdot [p]$ in $K_0(B)$,

(2) $\|\psi(f) - p\psi(f)p - \nu(f)\| < \delta$, $\forall\, f \in F$.

If $p = 0$, by Lemma 3.3, ψ and ν are homotopic, then ψ can be factored through a finite dimensional algebra in the sense of KK.

If $p \neq 0$, we have

$$[\psi(1_A)] \geq [\nu(1_A)] \geq 2mr \cdot [p] \geq 2mr \begin{pmatrix} n_0 \\ n_1 \end{pmatrix}.$$

By Lemma 3.2, there exists an algebra E in D such that ψ can be factored through E in the sense of KK.

The following result combines [2, Corollary 3.7, Theorem 5.8].

Lemma 3.6 *Let $A \in \mathcal{JS}$, $B = \varinjlim(B_n, \nu_{n,m})$ be an $A\mathcal{JS}$ algebra of real rank zero. Suppose that $\phi, \psi : A \to B_n$ are two homomorphisms with $[\phi] = [\psi]$ in $\mathrm{KK}(A, B_n)$, then for any finite set $F \subset A$ and $\varepsilon > 0$, there exists $r \geq n$ and a unitary $u \in B_r$ such that*

$$\|u \cdot \nu_{n,r} \circ \phi(f) \cdot u^* - \nu_{n,r} \circ \psi(f)\| < 30\varepsilon, \quad \forall f \in F.$$

Theorem 3.1 *Let $A = \lim(A_n, \psi_{n,m})$ be an $A\mathcal{JS}$ algebra of real rank zero, then A is an* AD *algebra of real rank zero.*

Proof We will prove the theorem by the intertwining argument.

Let $\varepsilon_n = \frac{1}{2^n}$. Let $B_1 = A_1$, $n_1 = 1$ and $\iota_1 : B_1 \to A_1$ be the identity map. Choose a finite set $\mathcal{G}_1 \subset (A_{n_1})_{sa}$ as in Lemma 3.5.

Since A has real rank zero, there exists $k_1 > n_1$ such that ψ_{n_1,k_1} satisfies that

$$\psi_{n_1,k_1}(\mathcal{G}_1) \subset_{1/6} \{f \in A_{k_1} \mid f \,\text{has finite spectrum}\}.$$

As both A_{n_1} and A_{k_1} are direct sums of blocks in $\mathcal{JS}$, the homomorphism ψ_{n_1,k_1} is also of the form as a direct sum of homomorphisms. We write $\psi^{i,j}_{n_1,k_1}$ as the map induced by ψ_{n_1,k_1} from the i^{th} copy $A^i_{n_1}$ of A_{n_1} to the j^{th} copy $A^j_{k_1}$ of A_{k_1}. Then for each i, j, we still have

$$\psi^{i,j}_{n_1,k_1}(\mathcal{G}_1) \subset_{1/6} \{f \in A^j_{k_1} \mid f \,\text{has finite spectrum}\}.$$

By Lemma 3.5, there exists a B_2 (a direct sum of blocks in D), a homomorphism $\gamma_{1,2} : A_1 \to B_2$ and a homomorphism $\iota : B_2 \to A_{k_1}$ such that $\mathrm{KK}(\psi_{n_1,k_1}) = \mathrm{KK}(\gamma_{1,2}) \times \mathrm{KK}(\iota)$.

Set $\phi_{1,2} = \gamma_{1,2} \circ \iota_1$, then the following diagram commutes in the sense of KK

$$\begin{array}{ccc} A_{n_1} & \xrightarrow{\psi_{n_1,k_1}} & A_{k_1} \\ \uparrow \iota_1 & \searrow \gamma_{1,2} & \uparrow \iota \\ B_1 & \xrightarrow{\phi_{1,2}} & B_2. \end{array}$$

Choose finite sets $F_1 \subset A_1$ and $G_1 \subset B_1$ such that F_1 generates A_1, G_1 generates B_1 and $\iota_1(G_1) \subset F_1$, by Lemma 3.6, there exists $n_2 \geq k_1$ and $u \in A_{n_2}$ such that

$$\|\psi_{k_1,n_2} \circ \psi_{n_1,k_1}(f) - u(\psi_{k_1,n_2} \circ \iota \circ \gamma_{1,2}(f))u^*\| < 30\varepsilon_1$$

for any $f \in F_1$.

Set $\iota_2 = \psi_{k_1,n_2} \circ \iota$, then the following diagram:

$$\begin{array}{ccc} A_{n_1} & \xrightarrow{\psi_{n_1,n_2}} & A_{n_2} \\ & \searrow \gamma_{1,2} & \uparrow \iota_2 \\ & & B_2, \end{array}$$

almost commutes on F_1 to within $30\varepsilon_1$.

Now we choose a finite set $\mathcal{G}_2 \subset (A_{n_2})_{sa}$ as in Lemma 3.5, $F_2 \subset A_{n_2}$, $G_2 \subset B_2$ such that F_2 generates A_{n_2}, G_2 generates B_2 and

$$\gamma_{1,2}(F_1) \subset G_2, \quad \psi_{n_1,n_2}(F_1) \cup \iota(G_2) \subset F_2,$$

then we can find B_3 (a direct sum of blocks in D), $\gamma_{2,3} : A_{n_2} \to B_3$ and $\iota_3 : B_3 \to A_{n_3}$.

In general, we obtain the diagram:

$$\begin{array}{ccccccccc}
A_{n_1} & \xrightarrow{\psi_{n_1,n_2}} & A_{n_2} & \xrightarrow{\psi_{n_2,n_3}} & A_{n_3} & \to & \cdots & A \\
\uparrow \iota_1 & \searrow \gamma_{1,2} & \uparrow \iota_2 & \searrow \gamma_{2,3} & \uparrow \iota_3 & & & \uparrow \\
B_1 & \xrightarrow{\phi_{1,2}} & B_2 & \xrightarrow{\phi_{2,3}} & B_3 & \to & \cdots & B
\end{array}$$

with the following properties:

(i) $\gamma_{k,k+1}(F_k) \subset G_{k+1}$, and $\psi_{n_k,n_{k+1}}(F_k) \cup \iota_{k+1}(G_{k+1}) \subset F_{k+1}$;

(ii) F_k generates A_{n_k}, and G_k generates B_k;

(iii) $\phi_{k,k+1} = \gamma_{k,k+1} \circ \iota_k$ holds for every k;

(iv) For each k, there exists a unitary $u_{k+1} \in A_{n_{k+1}}$ such that

$$\|\psi_{k,k+1}(f) - u_{k+1}(\iota_k \circ \gamma_{k,k+1}(f))u_{k+1}^*\| < 30\varepsilon_k, \quad \forall f \in F_k.$$

Then by [12, 2.1–2.2], the above diagram defines an isomorphism from B to A. This ends the proof.

Combining the reduction theorem with the classification of AD algebra [6], we get a new proof of the following result.

Corollary 3.1 (see [2]) *Let A, B be two unital $\mathcal{AJS}$ algebras of real rank zero. Then A is isomorphic to B, if and only if*

$$(\underline{\mathrm{K}}(A), \underline{\mathrm{K}}^+(A), \Sigma A) \cong (\underline{\mathrm{K}}(B), \underline{\mathrm{K}}^+(B), \Sigma B).$$

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