Chin. Ann. Math. Ser. B
44(2), 2023, 289–298
DOI: 10.1007/s11401-023-0015-z

Chinese Annals of Mathematics, Series B

Weighted Composition Operators from the Bloch Spaces to Weighted Hardy Spaces on Bounded Symmetric Domains*

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Abstract Let $\mathbb{B}_E$ be a bounded symmetric domain realized as the unit open ball of JB*-triples. The authors will characterize the bounded weighted composition operator from the Bloch space $\mathcal{B}(\mathbb{B}_E)$ to weighted Hardy space $H_v^\infty(\mathbb{B}_E)$ in terms of Kobayashi distance. The authors also give a sufficient condition for the compactness, and also give the upper bound of its essential norm. As a corollary, they show that the boundedness and compactness are equivalent for composition operator from $\mathcal{B}(\mathbb{B}_E)$ to $H^\infty(\mathbb{B}_E)$, when E is a finite dimension JB*-triple. Finally, they show the boundedness and compactness of weighted composition operators from $\mathcal{B}(\mathbb{B}_E)$ to $H_{v,0}^\infty(\mathbb{B}_E)$ are equivalent when E is a finite dimension JB*-triple.

Keywords Weighted composition operators, Bloch functions, Holomorphic functions, Bounded symmetric domains, Kobayashi distance
2000 MR Subject Classification 47B38, 32A18, 32M15

1 Introduction

Weighted composition operators on the Bloch spaces of the unit disk $\mathbb{D} \subset \mathbb{C}$ were studied by Ohno-Zhao [17], where the boundedness and compactness were characterized. Later, Chen-Stević-Zhou [4] and Allen-Colonna [2] studied such operators on Bloch spaces defined on unit polydisk and bounded homogeneous domains, respectively. Ohno [16] and Hosokawa-Izuchi-Ohno [13] studied the weighted composition operator from the Bloch space to H^∞ in the one-dimensional case, and by Li-Stević [15] in the case of unit ball of $\mathbb{C}^n$. The weighted composition operators on weighted Hardy space $H_v^\infty(\mathbb{D})$ were studied by Contreras-Hernández-Díaz [9] and by Galindo-Lindström [11].

Stević [18] gave the norm of weighted composition operators from the Bloch space to the weighted Hardy space H_v^∞ of the unit ball. Allen [1] studied such operators from the Bloch space to the weighted Hardy space H_v^∞ on bounded homogeneous domains, and he also characterized the compact weighted composition operators in case of unit polydisk.

It is natural to study such operators on domains in infinite variables, which is very different from finite-variables case. For a Hilbert ball, Blasco-Galindo-Miralles [3] defined the Bloch functions on it. Chu-Hamada-Honda-Kohr [6] defined the Bloch functions on any bounded

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*This work was supported by the National Natural Science Foundation of China (No. 12171251).

symmetric domain in terms of the Kobayashi metric instead of the Bergman metric for the finite dimensional case. They also gave criterion for the boundedness and compactness of the composition operators between the Bloch spaces, which extends many known results for the finite dimensional domains.

In this paper, we will use the Bloch norm defined in [6], which is defined on arbitrary bounded symmetric domain realized as an open unit ball of a JB*-triple. We will give a brief introduction of Bloch spaces $\mathcal{B}(\mathbb{B}_E)$ and weighted Hardy spaces $H_v^\infty(\mathbb{B}_E)$ in bounded symmetric domains in Section 2. In Section 3, we will characterize the boundedness of weighted composition operators $C_{\psi,\varphi}$ from $\mathcal{B}(\mathbb{B}_E)$ to $H_v^\infty(\mathbb{B}_E)$. We also study the compactness of $C_{\psi,\varphi}$, and give the upper bounded of its essential norm. Then we show the boundedness and compactness of composition operators from $\mathcal{B}(\mathbb{B}_E)$ to Hardy space $H^\infty(\mathbb{B}_E)$ are equivalent. In Section 4, we study the weighted composition operators from $\mathcal{B}(\mathbb{B}_E)$ to $H_{v,0}^\infty(\mathbb{B}_E)$, and show that the boundedness and compactness are equivalent.

2 Preliminaries

A bounded symmetric domain is a bounded open connected subset D in a complex Banach space such that each point $a \in D$ is an isolated fixed point of an involutive holomorphic bijection $s_a : D \to D$ with a holomorphic inverse s_a^{-1}. Finite dimensional bounded symmetric domains have been classfied by É. Cartan. For the infinite dimensional cases, Kaup's Riemann mapping theorem asserts that a bounded symmetric domain is biholomorphic to the open unit ball of a JB*-triples (see [14]). Recall that a JB*-triples is a complex Banach space E together with a Jordan triple product $(x, y, z) \in E \times E \times E \mapsto \{x, y, z\} \in E$ which is symmetric and linear in the outer variables, but conjugate linear in the middle variable, and satisfies

(i) $\{x, y, \{a, b, c\}\} = \{\{x, y, a\}, b, c\} - \{a, \{y, x, b\}, c\} + \{a, b, \{x, y, c\}\}$;

(ii) the box operator $a\Box a$ is a hermitian operator on E and has non-negative spectrum;

(iii) $\|a\Box a\| = \|a\|^2$

for any $a, b, c, x, y \in E$, where $a\Box b$ is defined by

$$a\Box b(\cdot) = \{a, b, \cdot\}$$

and satisfies $\|a\Box b\| \leq \|a\|\|b\|$. The open ball of a JB*-triples is a bounded symmetric domain. We refer to [5] for further relevant details and references.

A C*-algebra $\mathcal{A}$ is a JB*-triple with the Jordan triple product

$$\{a, b, c\} = \frac{1}{2}(ab^*c + cb^*a), \quad \forall\, a, b, c \in \mathcal{A}.$$

In particular, the von Neumann algebra $L(H)$ of bounded linear operators on a Hilbert space H is a JB*-triple.

From now on, let $\mathbb{B}_E$ be the unit open ball of a JB*-triple E. If Ω is a domain of a complex Banach space, let $H(\mathbb{B}_E, \Omega)$ denote the set of all holomorphic mappings from $\mathbb{B}_E$ to Ω. For every $x, y \in E$, the Bergman operator $B(x, y) \in L(E)$ is defined by

$$B(x, y)z = z - 2(x\Box y)(z) + \{x, \{y, z, y\}, x\}, \quad z \in E$$

Let $\mathbb{B}_E$ be the unit open ball of E. For each $a \in \mathbb{B}_E$, the Möbius transformation g_a defined by

$$g_a(z) = a + B(a,a)^{\frac{1}{2}}(I_E + z\Box a)^{-1}(z), \quad \forall\, z \in \mathbb{B}_E$$

is a biholomorphic mapping of $\mathbb{B}_E$ onto itself with $g_a(0) = a, g_a(-a) = 0$ and $g_{-a} = g_a^{-1}$. Here, $L(E)$ denotes the set of continuous linear operators on E, and I_E is the identity. Let

$$\kappa(z,x) = \|Dg_{-z}(z)(x)\|, \quad \forall\, z \in \mathbb{B}_E,\ x \in E$$

be the infinitesimal Kobayashi metric for $\mathbb{B}_E$. Let $\mathrm{Aut}(\mathbb{B}_E)$ denote the family of all holomorphic antomorphisms of $\mathbb{B}_E$.

Bloch functions on bounded homogeneous domain in $\mathbb{C}^n$ were first defined by Hahn (see [12, 19]). Chu-Hamada-Honda-Kohr [6] defined the Bloch functions on a bounded symmetric domain of any dimension.

Definition 2.1 (see [6, Section 3]) *$f \in H(\mathbb{B}_E, \mathbb{C})$ is called a Bloch function if*

$$\sup\{Q_f(z) : z \in \mathbb{B}_E\} < \infty,$$

where

$$Q_f(z) = \sup_{x \in X \setminus \{0\}} \frac{|Df(z)x|}{\kappa(z,x)}.$$

The space $\mathcal{B}(\mathbb{B}_E)$ of all Bloch function on $\mathbb{B}_E$ is a complex Banach space with the Bloch norm

$$\|f\|_{\mathcal{B}(\mathbb{B}_E)} = |f(0)| + \|f\|_{\mathcal{B}(\mathbb{B}_E),s},$$

where the Bloch semi-norm of f is defined by

$$\|f\|_{\mathcal{B}(\mathbb{B}_E),s} = \sup_{g \in \mathrm{Aut}(\mathbb{B}_E)} \|D(f \circ g)(0)\|,$$

which is equal to $\sup\{Q_f(z) : z \in \mathbb{B}_E\}$.

We will denote by ρ the Kobayashi distance on $\mathbb{B}_E$, which is the integral form of the Kobayashi metric κ. For any $a, b \in \mathbb{B}_E$, we have that

$$\rho(a,b) = \tanh^{-1} \|g_{-a}(b)\|,$$

where g_{-a} is the Möbius transformation induced by $-a$. Chu-Hamada-Honda-Kohr [7, Lemma 2.2] showed that a holomorphic function f on $\mathbb{B}_E$ is a Bloch function if and only if it is a Lipschits mapping as a function from $\mathbb{B}_E$, equipped with the Kobayashi distance, to $\mathbb{C}$ in the Euclidean distance.

A weight v on $\mathbb{B}_E$ is a continuous and strictly positive function satisfying

$$0 < \inf_{z \in \mathbb{B}_E} v(z) \le \sup_{z \in \mathbb{B}_E} v(z) < \infty.$$

The weighted Hardy space of holomorphic functions is defined as

$$H_v^\infty(\mathbb{B}_E) = \Big\{ f \in H(\mathbb{B}_E, \mathbb{C}) : \|f\|_v := \sup_{z \in \mathbb{B}_E} v(z)|f(z)| < \infty \Big\}.$$

The space $H_v^\infty(\mathbb{B}_E)$ is a Banach space. If $v \equiv 1$ on $\mathbb{B}_E$, then it is just the space $H^\infty(\mathbb{B}_E)$ of bounded holomorphic functions on $\mathbb{B}_E$ with the sup norm $\|\cdot\|_\infty$.

3 Main Results

For $\psi \in H(\mathbb{B}_E)$ and $\varphi \in H(\mathbb{B}_E, \mathbb{B}_E)$, we can define the weighted composition operator $C_{\psi,\varphi}$ from the Bloch space $\mathcal{B}(\mathbb{B}_E)$ to the weighted Hardy space $H_v^\infty(\mathbb{B}_E)$ by

$$C_{\psi,\varphi}(f)(z) = \psi(z) f(\varphi(z)), \quad \forall\, f \in \mathcal{B}(\mathbb{B}_E),\ z \in \mathbb{B}_E.$$

At first, we will give characterization of boundedness of weighted composition operators. For any $z \in \mathbb{B}_E$, it follows from [7, Lemma 2.2] that

$$\begin{aligned}|f(z)| &\le |f(0)| + |f(z) - f(0)| \\ &\le |f(0)| + \|f\|_{\mathcal{B}(\mathbb{B}_E),s}\,\rho(z,0) \\ &\le \max\{1, \rho(z,0)\}\|f\|_{\mathcal{B}}.\end{aligned}$$

This implies that the evaluation functional $\delta_z \in \mathcal{B}(\mathbb{B}_E)^*$ at z is bounded and $\|\delta_z\|_{\mathcal{B}(\mathbb{B}_E)^*} \le \max\{1, \rho(z,0)\}$.

Theorem 3.1 *The weighted composition operator $C_{\psi,\varphi} : \mathcal{B}(\mathbb{B}_E) \to H_v^\infty(\mathbb{B}_E)$ is bounded if and only if*

$$\omega_{\psi,\varphi,v} := \sup_{z \in \mathbb{B}_E} v(z)|\psi(z)|\|\delta_{\varphi(z)}\|_{\mathcal{B}(\mathbb{B}_E)^*} < \infty. \tag{3.1}$$

In this case, $\|C_{\psi,\varphi}\| = \omega_{\psi,\varphi,v}$.

Proof Suppose that $C_{\psi,\varphi}$ is bounded. For any $z \in \mathbb{B}_E$ and $f \in \mathcal{B}(\mathbb{B}_E)$ with $\|f\|_{\mathcal{B}(\mathbb{B}_E)} \le 1$, we have that

$$v(z)|\psi(z)||\delta_{\varphi(z)}(f)| = v(z)|C_{\psi,\varphi}(f)(z)| \le \|C_{\psi,\varphi}(f)\|_v \le \|C_{\psi,\varphi}\|.$$

By taking the sup for all f with $\|f\|_{\mathcal{B}(\mathbb{B}_E)} \le 1$, one can derive that

$$v(z)|\psi(z)|\|\delta_{\varphi(z)}\|_{\mathcal{B}(\mathbb{B}_E)^*} \le \|C_{\psi,\varphi}\|.$$

This shows that $\omega_{\psi,\varphi,v} \le \|C_{\psi,\varphi}\|$.

On the other hand, assume that $\omega_{\psi,\varphi,v} < \infty$. For any $f \in \mathcal{B}(\mathbb{B}_E)$ with $\|f\|_{\mathcal{B}(\mathbb{B}_E)} \le 1$ and any $z \in \mathbb{B}_E$, one can derive that

$$v(z)|C_{\psi,\varphi} f(z)| \le v(z)|\psi(z)|\|\delta_{\varphi(z)}\|_{\mathcal{B}(\mathbb{B}_E)^*} \le \omega_{\psi,\varphi,v}.$$

This implies that

$$\|C_{\psi,\varphi}(f)\|_v = \sup_{z \in \mathbb{B}_E} v(z)|C_{\psi,\varphi}(f)(z)| \le \omega_{\psi,\varphi,v}.$$

Therefore, $C_{\psi,\varphi}$ is bounded and $\|C_{\psi,\varphi}\| = \omega_{\psi,\varphi,v}$. This completes the proof.

For $a \in \mathbb{B}_E \setminus \{0\}$, the set

$$T(a) = \{\ell_a \in E^* : \ell_a(a) = \|a\|, \|\ell_a\| = 1\}$$

of support functionals of a is nonempty by the Hahan-Banach Theorem. Let $f_a(z) = \iota(\ell_a(z))$ for all $z \in \mathbb{B}_E$, where

$$\iota(\zeta) := \tanh^{-1}(\zeta) = \frac{1}{2}\log\frac{1+\zeta}{1-\zeta}, \quad \forall\, \zeta \in \mathbb{D}.$$

Then $f_a \in \mathcal{B}(\mathbb{B}_E)$ and $\|f_a\|_{\mathcal{B}(\mathbb{B}_E)} = \|\iota\|_{\mathcal{B}(\mathbb{D}),s} = 1$ (see Chu et al. [7, Example 2.5]). Note that $f_a(0) = 0$, then it follows that

$$\begin{aligned}\|\delta_a\|_{\mathcal{B}(\mathbb{B}_E)^*} &\geq |\delta_a(f_a)| \\ &= \iota(\ell_a(a)) = \frac{1}{2}\log\frac{1+\|a\|}{1-\|a\|} \\ &= f_a(a) - f_a(0) = \rho(a,0).\end{aligned} \tag{3.2}$$

Therefore we have that $\rho(a,0) \leq \|\delta_a\|_{\mathcal{B}(\mathbb{B}_E)^*} \leq \max\{1, \rho(a,0)\}$ when $a \in \mathbb{B}_E \setminus \{0\}$.

So we can generalize [1, Theorem 3.1] and [8, Theorem 4.3] to JB*-triples.

Corollary 3.1 *Let $\psi \in H(\mathbb{B}_E)$ and $\varphi \in H(\mathbb{B}_E, \mathbb{B}_E)$. Then the weighted composition operator $C_{\psi,\varphi} : \mathcal{B}(\mathbb{B}_E) \to H_v^\infty(\mathbb{B}_E)$ is bounded if and only if*

$$\varpi_{\psi,\varphi,,v} := \max\Big\{\|\psi\|_v, \sup_{z\in\mathbb{B}_E} v(z)|\psi(z)|\rho(\varphi(z),0)\Big\} < \infty. \tag{3.3}$$

In this case, $\|C_{\psi,\varphi}\| = \varpi_{\psi,\varphi,v}$.

Proof Note that when $C_{\psi,\varphi}$ is bounded, since the constant function 1 belongs to $\mathcal{B}(\mathbb{B}_E)$, then $\|C_{\psi,\varphi}(1)\|_v = \|\psi\|_v$. Then it follows from Theorem 3.1, one can conclude this corollary. This completes the proof.

Recall that a sequence $(f_n)_{n\in\mathbb{N}}$ of functions on a domain $D \subset E$ converges locally uniformly to a function f if and only if it converges uniformly on every closed ball strictly contained in D (see [10]).

Lemma 3.1 *Assume that E is a finite dimensional JB*-triple and $C_{\psi,\varphi} : \mathcal{B}(\mathbb{B}_E) \to H_v^\infty(\mathbb{B}_E)$ is a weighted composition operator. Then the following statements are equivalent*:

(i) *$C_{\psi,\varphi}$ is compact.*

(ii) *For any bounded sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{B}(\mathbb{B}_E)$ converging to 0 locally uniformly in $\mathbb{B}_E$, there exists a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ such that $(\|C_{\psi,\varphi}(f_{n_k})\|_v)_{k\in\mathbb{N}}$ converges to 0.*

Proof For any bounded sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{B}(\mathbb{B}_E)$ converging to 0 locally uniformly in $\mathbb{B}_E$, since $C_{\psi,\varphi}$ is compact, there exists a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ such that $C_{\psi,\varphi}(f_{n_k}) \to f$ for some $f \in H_v^\infty(\mathbb{B}_E)$. This implies that, for any $z \in \mathbb{B}_E$, $\psi(z)f_{n_k}(\varphi(z)) \to f(z)$. Note that $(f_{n_k})_{k\in\mathbb{N}}$ in $\mathcal{B}(\mathbb{B}_E)$ converging to 0 locally uniformly in $\mathbb{B}_E$, one can derive that $f(z) = 0$. This shows that $\|C_{\psi,\varphi}(f_{n_k})\|_v \to 0$.

On the other hand, let $(g_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\mathcal{B}(\mathbb{B}_E)$ and without loss of generality, we can assume that $\|g_n\|_{\mathcal{B}(\mathbb{B}_E)} \leq 1$ for any $n \in \mathbb{N}$. Similar to the argument of [6, p.2432], there exists a subsequence $(g_{n_j})_{j\in\mathbb{N}}$ such that $(g_{n_j})_{j\in\mathbb{N}}$ locally uniformly converges to holomorphic function f_0 on $\mathbb{B}_E$ with $\|f\|_{\mathcal{B}(\mathbb{B}_E)} \leq 1$. For any $j \in \mathbb{N}$, put $h_j = g_{n_j} - f$. Then $(h_j)_{j\in\mathbb{N}}$ is a bounded sequence converging locally uniformly to 0. By the assumption, one can find a further subsequence $(h_{j_t})_{t\in\mathbb{N}}$ such that $\|C_{\psi,\varphi}(h_{j_t})\|_v \to 0$. This shows that the subsequence $(C_{\psi,\varphi}(g_{n_{j_t}}))_{t\in\mathbb{N}}$ converges to $C_{\psi,\varphi}(f)$ in $H_v^\infty(\mathbb{B}_E)$. This completes the proof.

Next, we will give a sufficient condition for the compactness of $C_{\psi,\varphi}$.

Theorem 3.2 *Suppose that E is a finite-dimensional JB*-triple. If*

$$\lim_{\|\varphi(z)\|\to 1} v(z)|\psi(z)|\|\delta_{\varphi(z)}\|_{\mathcal{B}(\mathbb{B}_E)^*} = 0, \tag{3.4}$$

then $C_{\psi,\varphi} : \mathcal{B}(\mathbb{B}_E) \to H_v^\infty(\mathbb{B}_E)$ is compact.

Proof By Lemma 3.1, it suffices to show that for any bounded sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{B}(\mathbb{B}_E)$ converging to 0 locally uniformly in $\mathbb{B}_E$, then there exists a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ such that $(\|\psi(f_{n_k}\circ\varphi)\|_v)_{k\in\mathbb{N}}$ converges to 0.

Consider such $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{B}(\mathbb{B}_E)$ (we can assume that each norm is less than 1), for any $\varepsilon > 0$, there exists $r > 0$ such that

$$v(z)|\psi(z)|\|\delta_{\varphi(z)}\|_{\mathcal{B}(\mathbb{B}_E)^*} < \varepsilon$$

and hence

$$v(z)|\psi(z)||f_n(\varphi(z))| < \varepsilon, \quad \forall\, n \in \mathbb{N}$$

when $\|\varphi(z)\| > r$. Since $(f_n)_{n\in\mathbb{N}}$ converges to 0 locally uniformly on $\mathbb{B}_E$, there exists $K \in \mathbb{N}$ such that

$$|f_k(\varphi(z))| < \frac{\varepsilon}{\|\psi\|_v}$$

for all $k > K$ and $z \in \mathbb{B}_E$ with $\|\varphi(z)\| \le r$. Hence one can derive that

$$v(z)|\psi(z)||f_k(\varphi(z))| < \|\psi\|_v|f_k(\varphi(z))| < \varepsilon$$

for all $k > K$ and $z \in \mathbb{B}_E$ with $\|\varphi(z)\| \le r$. This shows that

$$v(z)|\psi(z)||f_k(\varphi(z))| < \varepsilon, \quad \forall\, k > K,\ z \in \mathbb{B}_E.$$

Therefore, we have that $\|C_{\psi,\varphi}(f_n)\| \to 0$. This completes the proof.

We can give a special sufficient condition for the compactness of $C_{\psi,\varphi}$.

Proposition 3.1 *Suppose that E is a finite-dimensional JB*-triple and $\|\varphi\|_\infty < 1$. If $C_{\psi,\varphi}$ is bounded, then it is compact.*

Proof Since $1 \in \mathcal{B}(\mathbb{B}_E)$, then $C_{\psi,\varphi}(1) = \psi$ belongs to $H_v^\infty(\mathbb{B}_E)$. For any bounded sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{B}(\mathbb{B}(E))$ converging to 0 locally uniformly, since $\|\varphi\|_\infty < 1$, we have that

$$\varphi(\mathbb{B}(E)) \subset \{x \in \mathbb{B}_E : \|x\| \le \|\varphi\|_\infty\},$$

which is strictly contained in $\mathbb{B}_E$. Then one can derive that

$$\|C_{\psi,\varphi}(f_n)\|_v \le \|\psi\|_v \sup_{z\in\varphi(\mathbb{B}_E)} |f_n(z)| \to 0.$$

It follows from Lemma 3.1 that $C_{\psi,\varphi}$ is compact. This completes the proof.

Remark 3.1 We do not know whether the converse of Theorem 3.2 is true. But use some ideas from [1], we can give an estimate of the essential norm of $C_{\psi,\varphi}$ in the following theorem.

Theorem 3.3 *Suppose that $C_{\psi,\varphi}:\mathcal{B}(\mathbb{B}_E)\to H_v^\infty(\mathbb{B}_E)$ is bounded, then the essential norm of $C_{\psi,\varphi}$ satisfies*

$$\|C_{\psi,\varphi}\|_e \le 2 \limsup_{\|\varphi(z)\|\to 1} v(z)|\psi(z)|\rho(\varphi(z),0).$$

Proof If $\|\varphi\|_\infty<1$, then it follows from Proposition 3.1 that the operator $C_{\psi,\varphi}$ is compact and hence $\|C_{\psi,\varphi}\|_e=0$. So we will assume $\|\varphi\|_\infty=1$ in the following argument.

Let $\delta\in(0,1)$ and $(\gamma_n)_{n\in\mathbb{N}}\subset(0,1)$ be a sequence of positive numbers which increasingly converges to 1. Then it follows from Proposition 3.1 that each operator $C_{\psi,\gamma_n\varphi}$ is compact. This implies that for any $m\in\mathbb{N}$, we have that

$$\begin{aligned}\|C_{\psi,\varphi}\|_e &\le \|C_{\psi,\varphi}-C_{\psi,\gamma_m\varphi}\|\\ &= \sup_{z\in\mathbb{B}_E}\sup_{\|f\|_{\mathcal{B}(\mathbb{B}_E)}\le 1} v(z)|\psi(z)||f(\varphi(z))-f(\gamma_m\varphi(z))|\\ &\le \sup_{\|\varphi(z)\|\le\delta}\sup_{\|f\|_{\mathcal{B}(\mathbb{B}_E)}\le 1} v(z)|\psi(z)||f(\varphi(z))-f(\gamma_m\varphi(z))|\\ &\quad+\sup_{\|\varphi(z)\|>\delta}\sup_{\|f\|_{\mathcal{B}(\mathbb{B}_E)}\le 1} v(z)|\psi(z)||f(\varphi(z))-f(\gamma_m\varphi(z))|. \end{aligned}\tag{3.5}$$

On one hand, by [6, Corollary 3.5], one can derive that

$$\begin{aligned}&\sup_{\|z\|\le\delta}\sup_{\|f\|_{\mathcal{B}(\mathbb{B}_E)}\le 1} v(z)|\psi(z)||f(\varphi(z))-f(\gamma_m\varphi(z))|\\ &\le (1-\gamma_m)\sup_{\|\varphi(z)\|\le\delta}\sup_{\|f\|_{\mathcal{B}(\mathbb{B}_E)}\le 1} v(z)|\psi(z)|\sup_{\|z\|\le\delta}(\|\varphi(z)\|\|Df(z)\|)\\ &\le \delta(1-\gamma_m)\sup_{\|\varphi(z)\|\le\delta}\sup_{\|f\|_{\mathcal{B}(\mathbb{B}_E)}\le 1} v(z)|\psi(z)|\sup_{\|z\|\le\delta}\frac{\|f\|_{\mathcal{B}(\mathbb{B}_E),s}}{1-\|z\|^2}\\ &\le \frac{\delta}{1-\delta^2}(1-\gamma_m)\|\psi\|_v\sup_{\|f\|_{\mathcal{B}(\mathbb{B}_E)}\le 1}\|f\|_{\mathcal{B}(\mathbb{B}_E),s}\\ &\le \frac{\delta}{1-\delta^2}\|\psi\|_v\to 0 \quad \text{as } m\to\infty. \end{aligned}\tag{3.6}$$

On the other hand, it is easy to see that $f_\gamma\in\mathcal{B}(\mathbb{B}_E)$ and $\|f_\gamma\|_{\mathcal{B}(\mathbb{B}_E)}\le\|f\|_{\mathcal{B}(\mathbb{B}_E)}$, where $f_\gamma(z)=f(\gamma z)$ $(z\in\mathbb{B}_E)$ and $0<\gamma<1$. It follows from [7, Proposition 2.3] we have that

$$\begin{aligned}&v(z)|\psi(z)||f(\varphi(z))-f(\gamma_m\varphi(z))|\\ &\le v(z)|\psi(z)|\|f-f_{\gamma_m}\|_{\mathcal{B}(\mathbb{B}_E),s}\rho(\varphi(z),0).\end{aligned}\tag{3.7}$$

This implies that

$$\begin{aligned}&\sup_{\|\varphi(z)\|>\delta}\sup_{\|f\|_{\mathcal{B}(\mathbb{B}_E)}\le 1} v(z)|\psi(z)||f(\varphi(z))-f(\gamma_m\varphi(z))|\\ &\le 2\sup_{\|\varphi(z)\|>\delta} v(z)|\psi(z)|\rho(\varphi(z),0).\end{aligned}\tag{3.8}$$

By both (3.6) and (3.8), we have that

$$\|C_{\psi,\varphi}\|_e \le 2 \limsup_{\|\varphi(z)\|\to 1} v(z)|\psi(z)|\rho(\varphi(z),0).$$

This completes the proof.

When the weight v is the constant function 1, we can consider the composition operator C_φ from $\mathcal{B}(\mathbb{B}_E)$ to $H^\infty(\mathbb{B}_E)$, where $\varphi \in H(\mathbb{B}_E, \mathbb{B}_E)$. It follows from Corollary 3.1 that C_φ is bounded if and only if

$$\sup_{z\in\mathbb{B}_E} \rho(\varphi(z),0) < \infty.$$

Note that if $\varphi(0) = 0$, then $\|C_\varphi\| = 1$. It follows from the definition of Kobayashi distance that $\sup\limits_{z\in\mathbb{B}_E} \|\varphi(z)\| < 1$. So we can derive the following theorem.

Theorem 3.4 *Suppose that E is a finite dimensional JB*-triple. Then C_φ is bounded if and only if it is compact.*

Proof When C_φ is bounded, by the above argument and Proposition 3.1, we can derive that C_φ is compact. This completes the proof.

4 Weighted Composition Operators from $\mathcal{B}(\mathbb{B}_E)$ to $H^\infty_{v,0}(\mathbb{B}_E)$

$H^\infty_{v,0}(\mathbb{B}_E)$ is the closed subspace of $H^\infty_v(\mathbb{B}_E)$ consisting of the function f such that

$$\lim_{\|z\|\to 1} v(z)|f(z)| = 0.$$

We now discuss the boundedness and compactness of the weighted composition operator from the Bloch space to $H^\infty_{v,0}(\mathbb{B}_E)$.

Theorem 4.1 *Suppose that E is a finite dimensional JB*-triple, $\psi \in H(\mathbb{B}_E)$ and $\varphi \in H(\mathbb{B}_E, \mathbb{B}_E)$, then the following statements are equivalent:*

(a) $C_{\psi,\varphi} : \mathcal{B}(\mathbb{B}_E) \to H^\infty_{v,0}(\mathbb{B}_E)$ *is compact.*

(b) $C_{\psi,\varphi} : \mathcal{B}(\mathbb{B}_E) \to H^\infty_{v,0}(\mathbb{B}_E)$ *is bounded.*

(c) $\lim\limits_{\|z\|\to 1} v(z)|\psi(z)|\|\delta_{\varphi(z)}\|_{\mathcal{B}(\mathbb{B}_E)^*} = 0.$

Proof (a)$\Rightarrow$(b). Obvious.

(b)$\Rightarrow$(c). Suppose that there exists a positive number δ and a sequence $\{z_n\}$ in $\mathbb{B}_E$, such that $\|z_n\| \to 1$ and $v(z_n)|\psi(z_n)|\|\delta_{\varphi(z_n)}\|_{\mathcal{B}(\mathbb{B}_E)^*} > \delta$ for all $n \in \mathbb{N}$. Since

$$\|\delta_{\varphi(z_n)}\|_{\mathcal{B}(\mathbb{B}_E)} = \sup_{\|f_n\|_{\mathcal{B}(\mathbb{B}_E)}\le 1} |\delta_{\varphi(z_n)}(f_n)| = \sup_{\|f_n\|_{\mathcal{B}(\mathbb{B}_E)}\le 1} |f_n(\varphi(z_n))|.$$

By definition of the supremum, for each n, we may choose $f_n \in \mathcal{B}(\mathbb{B}_E)$, $\|f_n\|_{\mathcal{B}(\mathbb{B}_E)} \le 1$, such that

$$|f_n(\varphi(z_n))| > \|\delta_{\varphi(z_n)}\|_{\mathcal{B}(\mathbb{B}_E)^*} - \frac{\delta}{2}.$$

Then we have

$$v(z_n)|\psi(z_n)||f_n(\varphi(z_n))| > \delta - \frac{\delta}{2} v(z_n)|\psi(z_n)|.$$

If $f \in \mathcal{B}(\mathbb{B}_E)$ is a constant function c, and $C_{\psi,\varphi} f \in H^\infty_{v,0}(\mathbb{B}_E)$,

$$v(z_n)|\psi(z_n)||f(\varphi(z_n))| = cv(z_n)|\psi(z_n)|,$$

we can get $\psi \in H^\infty_{v,0}(\mathbb{B}_E)$. So $\lim\limits_{\|z_n\|\to 1} v(z_n)|\psi(z_n)| = 0$. Then for any $0 < \varepsilon < 1$, such that $\|z_n\| \to 1$ and $|v(z_n)||\psi(z_n)| < \varepsilon < 1$. Thus

$$v(z_n)|\psi(z_n)||f_n(\varphi(z_n))| > \delta - \frac{\delta}{2} v(z_n)|\psi(z_n)| > 0.$$

So we have

$$\lim_{\|z_n\|\to 1} v(z_n)|\psi(z_n)||f_n(\varphi(z_n))| \neq 0.$$

So $C_{\psi,\varphi} f_n \notin H^\infty_{v,0}(\mathbb{B}_E)$. But $C_{\psi,\varphi}$ is bounded, so $C_{\psi,\varphi} f_n \in H^\infty_{v,0}(\mathbb{B}_E)$, which contradicts the conclusion.

(c)⇒(a). By Theorem 3.2 and $H^\infty_{v,0}(\mathbb{B}_E)$ is the closed subspace of $H^\infty_v(\mathbb{B}_E)$, then we can have $C_{\psi,\varphi}$ is compact. This completes the proof.

Acknowledgement The authors would like to thank the referee for useful suggestions which improved the paper.

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