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Some Game-Theoretic Characterizations for Rapid Filters on ω *

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Abstract Infinite game is a power tool in studying various objects and finding descriptions of some properties of filters in mathematics. Game-theoretic characterizations for meager filters, Q-filters and Ramsey filters were obtained by Tomek Bartoszynski, Claude Laflamme and Marion Scheepers. In this paper, the authors obtain two game-theoretic characterizations for rapid filters on ω.

Keywords Rapid filter, Infinite game, Game-theoretic characterization
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1 Introduction

The natural numbers will be denoted by ω. X is a nonempty subset of ω. $\mathcal{P}(X)$ denotes the collection of all subsets of X. $[X]^\omega$ denotes the collection of all infinite subsets of X and $[X]^{<\omega}$ denotes the collection of all finite subsets of X. $[X]^{\leq k} = \{s \in [X]^{<\omega} : |s| \leq k\}$ for every $k < \omega$. ${}^{<\omega}X$ denotes the collection of all finite sequences of X.

A filter is a collection of subsets of ω closed under finite intersections, supersets and containing all co-finite sets. For a filter $\mathcal{F}$, $\mathcal{F}^* = \{\omega \setminus X : X \in \mathcal{F}\}$ is called an ideal and $\mathcal{F}^+ = \mathcal{P}(\omega) \setminus \mathcal{F}^*$ is called a co-ideal. The Fréchet filter is the collection of co-finite sets, denoted by $\mathfrak{Fr}$. In particular, $\mathfrak{Fr}^* = [\omega]^{<\omega}$ and $\mathfrak{Fr}^+ = [\omega]^\omega$. $\mathcal{F}$ is a Q-filter if for every partition $\{s_k : k \in \omega\}$ of ω into finite sets, there is $X \in \mathcal{F}$ such that $|X \cap s_k| \leq 1$ for all $k \in \omega$ (X is called a selector). $\mathcal{F}$ is meager means $\mathcal{F}$ is a meager set in topology space $\mathcal{P}(\omega)$.

Let $\mathcal{F}$ be a filter on ω. We say that $\mathcal{F}$ is rapid if for every partition $\{s_k : k \in \omega\}$ of ω into finite sets, there is $X \in \mathcal{F}$ such that $|X \cap s_k| \leq k$ for all $k \in \omega$. Apparently, every Q-filter is a rapid filter.

Mokobodzki in [2, Theorem 4.6.4] shows that every rapid filter is nonmeasurable and does not have the Baire property. Judah and Shelah in [2, Theorem 4.6.7] show that there exists a model for ZFC in which there is no rapid filter with arbitrary lager continuum. So to obtain a rapid filter, one needs some extra assumptions beyond ZFC, and the games $G_{1.5}(\mathcal{F})$ and $G(\mathfrak{Fr}, [\omega]^{\leq f_0}, \mathcal{F})$ (which are defined in Section 3) are determined in above mentioned Judah and Shelah's model.

Infinite game is a power tool in studying some filters, Tomek Bartoszynski, Claude Laflamme and Marion Scheepers obtain some game-theoretic characterizations for meager filters, Q-filters

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and Ramsey filters.

The main result in the paper is that games $G_{1.5}(\mathcal{F})$ and $G(\mathfrak{Fr}, [\omega]^{\leq I}, \mathcal{F})$ are game-theoretic characterizations for rapid filters.

2 Infinite Games and Filter Games

Infinite game is a power tool in studying various objects in mathematics. Please refer to [1, 4] for topological games and descriptive set theory games, respectively.

A and B are nonempty set, φ is a formula. A infinite game $G(A, B, \varphi)$ is played by two players ONE and TWO as follows: At stage $k < \omega$, ONE chooses $a_k \in A$ and TWO responds with a $b_k \in B$. At the end of the game, TWO wins the game if the formula $\varphi(a_0, b_0, a_1, b_1, \cdots)$ is true. Otherwise, ONE wins.

In the infinite game $G(A, B, \varphi)$, a map $\sigma : {}^{<\omega}B \to A$ is called a strategy for ONE. Similarly, a map $\tau : {}^{<\omega}A \to B$ is called a strategy for TWO.

We say ONE has a winning strategy if there is a strategy σ for ONE such that for every $x_1, x_2, \cdots, x_n, \cdots \in B$, $\neg\varphi(\sigma(\emptyset), x_1, \sigma(\langle x_1\rangle), x_2, \sigma(\langle x_1, x_2\rangle), x_3, \cdots)$ is true, σ is called a winning strategy for ONE. Similarly, TWO has a winning strategy if there is a strategy τ for TWO such that for every $y_1, y_2, \cdots, y_n, \cdots \in A$, $\varphi(y_1, \tau(\langle y_1\rangle), y_2, \tau(\langle y_1, y_2\rangle), \cdots)$ is true, τ is called a winning strategy for TWO. A game $G(A, B, \varphi)$ is determined if either ONE or TWO has a winning strategy, and $G(A, B, \varphi)$ is undetermined if both ONE and TWO have no winning strategy. Two games are equivalent if a player has a winning strategy in one game if and only if the same player has a winning strategy in the other game.

We will be interested in some filter games which are some specific infinite games, and the formula φ is related to filters.

$\mathcal{F}$ is a filter. The game $G(\mathfrak{Fr}, \omega, \mathcal{F})$ (see [5]) is played by two players ONE and TWO as follows: At stage $k < \omega$, ONE chooses $X_k \in \mathfrak{Fr}$ and TWO responds with $x_k \in X_k$. At the end of the game, TWO wins the game if $\{x_k : k \in \omega\} \in \mathcal{F}$. Otherwise, ONE wins.

$\mathcal{F}$ is a filter. The game $G(\mathfrak{Fr}, [\omega]^{<\omega}, \mathcal{F})$ (see [5]) is played by two players ONE and TWO as follows: At stage $k < \omega$, ONE chooses $X_k \in \mathfrak{Fr}$ and TWO responds with $s_k \in [X_k]^{<\omega}$. At the end of the game, TWO wins the game if $\bigcup_{k\in\omega} s_k \in \mathcal{F}$. Otherwise, ONE wins.

$\mathcal{F}$ is a filter. The game $G_1(\mathcal{F})$ (see [3]) is played by two players ONE and TWO as follows: At stage $k < \omega$, ONE chooses $m_k \in \omega$ and TWO responds with $n_k \in \omega$. At the end of the game, TWO wins the game if:

(1) $n_0 < n_1 < \cdots < n_k < \cdots$ and

(2) there are infinitely many $k < \omega$ such that $m_k < n_k$ and

(3) $\{n_0, n_1, \cdots, n_k, \cdots\} \in \mathcal{F}$.

Otherwise, ONE wins.

$\mathcal{F}$ is a filter. The game $G_2(\mathcal{F})$ (see [3]) is played by two players ONE and TWO as follows: At stage $k < \omega$, ONE chooses $m_k \in \omega$ and TWO responds with $n_k \in \omega$. At the end of the game, TWO wins the game if:

(1) $n_0 < n_1 < \cdots < n_k < \cdots$ and

(2) for all but finitely many $k < \omega$ such that $m_k < n_k$ and

(3) $\{n_0, n_1, \cdots, n_k, \cdots\} \in \mathcal{F}$.

Otherwise, ONE wins.

3 Game-Theoretic Characterizations for Filters

Theorem 3.1 (see [3]) *$\mathcal{F}$ is a filter, the games $G(\mathfrak{Fr},\omega,\mathcal{F})$ and $G_2(\mathcal{F})$ are equivalent, and*

(1) ONE *has a winning strategy if and only if $\mathcal{F}$ is not a Q-filter.*

(2) TWO *has no winning strategy.*

Theorem 3.2 (see [5]) *$\mathcal{F}$ is a filter, the games $G(\mathfrak{Fr},[\omega]^{<\omega},\mathcal{F})$ and $G_1(\mathcal{F})$ are equivalent, and*

(1) ONE *has a winning strategy if and only if $\mathcal{F}$ is meager.*

(2) TWO *has no winning strategy.*

Next, we will give some game-theoretic characterizations of rapid filter.

$\mathcal{F}$ is a filter. The game $G_{1.5}(\mathcal{F})$ is played by two players ONE and TWO as follows: At stage $k<\omega$, ONE chooses $m_k\in\omega$ and TWO responds with $n_k\in\omega$. At the end of the game, TWO wins the game if:

(Condition 1) $n_0<n_1<\cdots<n_k<\cdots$ and

(Condition 2) for each $k<\omega$, there is $i\in\left[\frac{k(k-1)}{2}+1,\frac{k(k+1)}{2}+1\right]$, such that $m_i<n_i$ and

(Condition 3) $\{n_0,n_1,\cdots,n_k,\cdots\}\in\mathcal{F}$.

Otherwise, ONE wins.

Theorem 3.3 *Fix a filter $\mathcal{F}$ and consider the game $G_{1.5}(\mathcal{F})$. Then*

(1) ONE *has a winning strategy if and only if $\mathcal{F}$ is not a rapid filter.*

(2) TWO *has no winning strategy.*

Proof We first deal with player ONE. Suppose that $\mathcal{F}$ is a rapid filter and to show that ONE does not have a winning strategy.

Consider a strategy σ for ONE. We may assume that for all $x_1<x_2<\cdots<x_n$, the strategy σ satisfies

$$\max\{x_1,x_2,\cdots,x_n\}+1<\sigma(\langle x_1,x_2,\cdots,x_n\rangle)$$

and

$$\sigma(\langle x_1\rangle)<\sigma(\langle x_1,x_2\rangle)<\cdots<\sigma(\langle x_1,x_2,\cdots,x_n\rangle).$$

Define g by $g(1)=\sigma(\emptyset)$, and

$$g(n+1)=\max\{\sigma(\langle j_1,\cdots,j_i\rangle):i\le n+1,j_1<\cdots<j_i\le n+1\}+g(n),\quad n\in\omega.$$

Let $h(1)=\sigma(\emptyset)+1$, and $h(n+1)=g(h(n))$ for each n. Consider the partition $\{I_n:n\in\omega\}$, where

$$I_n=\Big[h\Big(\frac{n(n+1)}{2}\Big),h\Big(\frac{n(n+1)}{2}+n+1\Big)\Big).$$

Since $\mathcal{F}$ is rapid filter, there is an $X\in\mathcal{F}$ such that $|X\cap I_n|\le n$ for each n. Assume that $|X\cap I_n|=n$. Then fix $N\in\omega$, $|X\cap I_N|=N$ and

$$I_N=\bigcup_{i=p}^{q}[h(i),h(i+1)),\quad p=\frac{N(N+1)}{2},\quad q=\frac{N(N+1)}{2}+N.$$

Since there are $N+1$ intervals in I_N. So there is $j\in[p,q]$ such that

$$X\cap[h(j),h(j+1))=\emptyset.$$

Write $X = \{x_1, x_2, \cdots, x_n, \cdots\}$ in increasing order. Then

Case 1 If $h(j+1) \le \min(X \cap I_N)$, then $x_{\frac{N(N-1)}{2}} \le h(j)$, and

$$\sigma(\langle x_1, x_2, \cdots, x_{\frac{N(N-1)}{2}}\rangle) < g(x_{\frac{N(N-1)}{2}}) \le g(h(j)) = h(j+1) \le x_{\frac{N(N-1)}{2}+1}.$$

Case 2 $\exists\, l \in [\frac{N(N-1)}{2}+1, \frac{N(N+1)}{2}]$, such that $x_l < h(j) < h(j+1) \le x_{l+1}$, and

$$\sigma(\langle x_1, x_2, \cdots, x_l\rangle) < g(x_l) \le g(h(j)) = h(j+1) \le x_{l+1}.$$

So $(\sigma(\emptyset), x_1, \sigma(\langle x_1\rangle), x_2, \sigma(\langle x_1, x_2\rangle), x_3, \cdots)$ means that σ is not a winning strategy of ONE.

Next assume that $\mathcal{F}$ is not a rapid filter, and there is a partition $\{s_n : n \in \omega\}$ of ω such that

$$\forall X \in \mathcal{P}(\omega), \quad \forall n \in \omega, \quad |X \cap s_n| \le n \to X \in \mathcal{F}^c.$$

To construct a strategy σ for ONE. $\sigma(\emptyset) = 0$, and in the k-th inning, TWO chooses n_k, suppose $k \in \big[\frac{N(N-1)}{2}, \frac{N(N+1)}{2}\big)$, let

$$\sigma(\langle n_1, n_2, \cdots, n_k\rangle) = m_{k+1} > \max\,\{n_k\} \cup \{m_k\} \cup (s_1 \cup \cdots \cup s_{k+2N+1}). \qquad (*)$$

Then, let $X = \{n_1, n_2, \cdots, n_k, \cdots\}$, if X satisfies (Condition 1)–(Condition 2) , then $k \in \omega$, $|X \cap s_k| \le k$.

(Since for any $k \in \omega$, find the largest $l \in \omega$ ($l \in \big[\frac{N(N-1)}{2}, \frac{N(N+1)}{2}\big)$) such that $l + 2N + 1 \le k$, i.e., if $l_0 > l$ and $l_0 \in \big[\frac{M(M-1)}{2}, \frac{M(M+1)}{2}\big)$, then $l_0 + 2M + 1 > k$. So either $l + 2N + 1 = k$ or $l + 2N + 1 < k$ and $l = \frac{N(N+1)}{2} - 1$, i.e., $l + 1 = \frac{N(N+1)}{2} \in \big[\frac{N(N+1)}{2}, \frac{(N+1)(N+2)}{2}\big)$, while $(l+1) + 2(N+1) + 1 = l + 2N + 4 > k$.

If $l + 2N + 1 = k$, then by $(*)$, $m_{l+1} >$max s_k. As $l + 1 \in \big[\frac{N(N-1)}{2} + 1, \frac{N(N+1)}{2} + 1\big)$ and X satisfies (Condition 2), $\exists\ l^* \in \big[\frac{N(N+1)}{2} + 1, \frac{(N+1)(N+2)}{2} + 1\big]$ s.t. $m_{l^*} < n_{l^*}$, i.e., $n_{l^*} > m_{l^*} > m_{l+1} >$max s_k. So $X \cap s_k \subset \{n_1, \cdots, n_{l^*-1}\}$, i.e.,

$$|X \cap s_k| \le l^* - 1 = l + l^* - 1 - l \le l + \frac{(N+1)(N+2)}{2} + 1 - 1 - \frac{N(N-1)}{2} = l + 2N + 1 \le k.$$

If $l + 2N + 1 < k$, then $l + 1 = \frac{N(N+1)}{2}$ and $l + 2 = \frac{N(N+1)}{2} + 1$, so $\exists\ l^* \in \big[\frac{N(N+1)}{2} + 1, \frac{(N+1)(N+2)}{2} + 1\big]$ s.t. $m_{l^*} < n_{l^*}$, i.e., $n_{l^*} > m_{l^*} \ge m_{l+2} >$max s_k. So $X \cap s_k \subset \{n_1, \cdots, n_{l^*-1}\}$, i.e.,

$$\begin{aligned} |X \cap s_k| &\le l^* - 1 = l + l^* - 1 - l \le l + \frac{(N+1)(N+2)}{2} + 1 - 1 - \Big(\frac{N(N+1)}{2} - 1\Big) \\ &= l + N + 2 \le l + 2N + 1 \le k. \end{aligned}$$

Let $Y = X \setminus (s_1 \cup s_2 \cup s_3 \cup s_4 \cup s_5)$, so $k \in \omega$, $|Y \cap s_k| \le k$ and $Y \in \mathcal{F}^c$, and then $X \in \mathcal{F}^c$.)

So $X \in \mathcal{F}^c$ and σ is a winning strategy for ONE.

TWO has no winning strategy in $G_{1.5}(\mathcal{F})$ because TWO has no winning strategy in $G_1(\mathcal{F})$.

$\mathcal{F}$ is a filter and $f \in \omega^\omega$. The game $G(\mathfrak{Fr}, [\omega]^{\le f}, \mathcal{F})$ is played by two players ONE and TWO as follows: At stage $k < \omega$, ONE chooses $X_k \in \mathfrak{Fr}$ and TWO responds with $s_k \in [X_k]^{\le f(k)}$. At the end of the game, TWO wins the game if $\bigcup_{k \in \omega} s_k \in \mathcal{F}$. Otherwise, ONE wins.

Theorem 3.4 *Fix a filter $\mathcal{F}$, $f_0 \in \omega^\omega$, $f_0(0) = 2$, $f_0(n) = 4n + 1$, $n \in \omega \setminus \{0\}$. Consider the game $G(\mathfrak{Fr}, [\omega]^{\leq f_0}, \mathcal{F})$. Then*

(1) ONE *has a winning strategy if and only if $\mathcal{F}$ is not a rapid filter.*

(2) TWO *has no winning strategy.*

Proof We first deal with player ONE. Suppose $\mathcal{F}$ is not a rapid filter. Then there is a partition of ω into finite sets $\{I_n : n \in \omega\}$, such that

$$\forall X \in \mathcal{P}(\omega), \quad \forall n \in \omega, \quad |X \cap I_n| \leq n \to X \in \mathcal{F}^c.$$

To construct a strategy σ for ONE. Let

$$\sigma(\emptyset) = \omega \setminus (I_0 \cup \cdots \cup I_{f_0(0)}).$$

TWO chooses $s_0 \in [\sigma(\emptyset)]^{\leq f_0(0)}$, $|s_0| \leq f_0(0)$ and let

$$\sigma(\langle s_0 \rangle) = \sigma(\emptyset) \setminus (I_0 \cup \cdots \cup I_{f_0(0)+f_0(1)}).$$

TWO chooses $s_1 \in [\sigma(\langle s_0 \rangle)]^{\leq f_0(1)}$, $|s_1| \leq f_0(1)$ and let

$$\sigma(\langle s_0, s_1 \rangle) = \sigma(\langle s_0 \rangle) \setminus (I_0 \cup \cdots \cup I_{f_0(0)+f_0(1)+f_0(2)}).$$

$$\cdots$$

Finally, let $X = \bigcup_{k \in \omega} s_k$. Then $|X \cap I_n| \leq n$. So $X \in \mathcal{F}^c$ and σ is a winning strategy of ONE.

Suppose ONE has a winning strategy σ. Without loss of generality. ONE plays co-finite sets of the form (n, ∞). Then construct a winning strategy $\widetilde{\sigma}$ for ONE in the game $G_{1.5}(\mathcal{F})$. Without loss of generality, assume in the game $G_{1.5}(\mathcal{F})$ TWO picks $n_{k+1} > n_k$, $k \in \omega$.

In the game $G_{1.5}(\mathcal{F})$, TWO may choose $n_i \leq m_i$, but TWO must choose many $n_k > m_k$ as the winning (condition 2) in $G_{1.5}(\mathcal{F})$. So we can recursively construct $\widetilde{\sigma}$. Suppose TWO chooses $n_k > m_k$ in $G_{1.5}(\mathcal{F})$ where $\sigma(\langle s_0, s_1, \cdots, s_l \rangle) = (m_k, \infty)$, then by σ ONE can get some new $M \in \omega$ in $G_{1.5}(\mathcal{F})$ as ONE will give (M, ∞) by σ. Until TWO chooses the next $n_{K+1} > m_{K+1}$, $K \geq k$. Let $s_{l+1} = \{n_k, \cdots, n_K\}$ and $\sigma(\langle s_0, s_1, \cdots, s_l, s_{l+1} \rangle) = (m_{k+1}, \infty)$, and so on.

For this, let $\widetilde{\sigma}(\emptyset)$ =min$(\sigma(\emptyset))$, and suppose

$$\sigma(\langle s_0, s_1, \cdots, s_l \rangle) = (M_1, \infty)$$

with that all these s_i's are disjoint, then let

$$\widetilde{\sigma}(\langle n_0, n_1, \cdots, n_k \rangle) = M_1,$$

where $(s_0 \cup s_1 \cup \cdots \cup s_l) = \{n_0, n_1, \cdots, n_k\}$. Suppose TWO chooses $n_{k+1} > M_1$, and

$$\sigma(\langle s_0, s_1, \cdots, s_l, \{n_{k+1}\} \rangle) = (M_2, \infty),$$

then let

$$\widetilde{\sigma}(\langle n_0, n_1, \cdots, n_k, n_{k+1} \rangle) = M_2.$$

Suppose TWO chooses $n_{k+2} \leq M_2$, and

$$\sigma(\langle s_0, s_1, \cdots, s_l, \{n_{k+1}, n_{k+2}\} \rangle) = (M_3, \infty),$$

then let

$$\widetilde{\sigma}(\langle n_0, n_1, \cdots, n_k, n_{k+1}, n_{k+2}\rangle) = M_3.$$

Suppose TWO chooses $n_{k+3} \le M_3$, and so on.

According to (Condition 2) of the definition of $G_{1.5}(\mathcal{F})$, there must be a $p \in \omega$ such that

$$\widetilde{\sigma}(\langle n_0, n_1, \cdots, n_k, n_{k+1}, n_{k+2}, \cdots, n_{k+p}\rangle) = M_{p+1},$$

and TWO has to choose $n_{k+p+1} > M_{p+1}$. For constructing s_{l+1}, we must prove $p \le f_0(l+1)$ and let $s_{l+1} = \langle n_{k+1}, \cdots, n_{k+p}\rangle$. Actually we only need to show it when $|s_i| = f_0(i)$, $i \le l$, because p will get maximum in this situation. So

$$k = \sum_{i \le l} f_0(i) = 2 + \frac{4l(l+1)}{2} + l = \frac{(2l+1)(2l+2)}{2} + 1,$$

then $k \in \left[\frac{2l(2l+1)}{2} + 1, \frac{(2l+1)(2l+2)}{2} + 1\right]$, and hence

$$\begin{aligned} p &\le \Big(\Big(\frac{(2l+2)(2l+3)}{2} + 1\Big) - \Big(\frac{(2l+1)(2l+2)}{2} + 1\Big)\Big) + 1 \\ &\quad + \Big(\Big(\frac{(2l+3)(2l+4)}{2} + 1\Big) - \Big(\frac{(2l+2)(2l+3)}{2} + 1\Big)\Big) - 1 \\ &= 2l + 2 + 1 + 2l + 3 - 1 = 4l + 5 = f_0(l+1) \end{aligned}$$

$\big($otherwise, $\neg\exists\, i \in \left[\frac{(2l+2)(2l+3)}{2} + 1, \frac{(2l+3)(2l+4)}{2} + 1\right]$, $m_i < n_i\big)$.

Let $s_{l+1} = \langle n_{k+1}, \cdots, n_{k+p}\rangle$, $|s_{l+1}| \le f_0(l+1)$, suppose

$$\sigma(\langle s_0, s_1, \cdots, s_l, s_{l+1}\rangle) = (M_{p+1}, \infty)$$

and

$$\widetilde{\sigma}(\langle n_0, n_1, \cdots, n_k, n_{k+1}, n_{k+2}, \cdots, n_{k+p}\rangle) = M_{p+1}.$$

Then TWO will choose $n_{k+p+1} > M_{p+1}$.

Since σ is a winning strategy, $\widetilde{\sigma}$ is a winning strategy for ONE in the game $G_{1.5}(\mathcal{F})$, and $\mathcal{F}$ is not a rapid filter.

TWO has no winning strategy in $G(\mathfrak{F}r, [\omega]^{\le f_0}, \mathcal{F})$ because TWO has no winning strategy in $G(\mathfrak{F}r, [\omega]^{<\omega}, \mathcal{F})$.

Corollary 3.1 *Fix a filter $\mathcal{F}$, $f_0 \in \omega^\omega$, $f_0(0) = 2$, $f_0(n) = 4n + 1$, $n \in \omega \setminus \{0\}$. The game $G(\mathfrak{F}r, [\omega]^{\le f_0}, \mathcal{F})$ and $G_{1.5}(\mathcal{F})$ are equivalent, and*

(1) ONE *has a winning strategy if and only if $\mathcal{F}$ is not a rapid filter.*

(2) TWO *has no winning strategy.*

Corollary 3.2 *Fix a filter $\mathcal{F}$, $f_0 \in \omega^\omega$, $f_0(0) = 2$, $f_0(n) = 4n + 1$, $n \in \omega \setminus \{0\}$. For every $f \in \omega^\omega$ such that f dominates f_0. Consider the game $G(\mathfrak{F}r, [\omega]^{\le f}, \mathcal{F})$. Then*

(1) ONE *has a winning strategy if and only if $\mathcal{F}$ is not a rapid filter.*

(2) TWO *has no winning strategy.*

Proof The proof is similar to that of Theorem 3.4, and we leave it to the reader.

For the rapid filter, we have the following conclusion.

Theorem 3.5 (see [2]) *$\mathcal{F}$ is a filter. The following conditions are equivalent:*

(1) *$\mathcal{F}$ is a rapid filter.*

(2) *There exists strictly increasing function $f \in \omega^\omega$ such that for every partition $\{s_k : k \in \omega\}$ of ω into finite sets, there is $X \in \mathcal{F}$ such that $|X \cap s_k| \leq f(k)$ for all $k \in \omega$.*

We will give a proof which likes game-theoretic characterizations for rapid filter. For convenience, we call $\mathcal{F}$ is f-rapid filter if $\mathcal{F}$ meets the condition (2) in Theorem 3.5. Then we have the following two claims.

Claim 3.1 $\mathcal{F}$ is a filter, for every strictly increasing function $f \in \omega^\omega$, the game $G_{1.5}(\mathcal{F}, f)$ is played by two players ONE and TWO as follows: At stage $k < \omega$, ONE chooses $m_k \in \omega$ and TWO responds with $n_k \in \omega$. At the end of the game, TWO wins the game if:

(1) $n_0 < n_1 < \cdots < n_k < \cdots$ and

(2) for each $k \in \omega$, there is $i \in \big[\sum\limits_{i \leq k-1} f(i) + 1, \sum\limits_{i \leq k} f(i) + 1\big]$, such that $m_i < n_i$, (let $f(-1) = 0$) and

(3) $\{n_0, n_1, \cdots, n_k, \cdots\} \in \mathcal{F}$.

Otherwise, ONE wins. Consider the game $G_{1.5}(\mathcal{F}, f)$, then

(1) ONE has a winning strategy if and only if $\mathcal{F}$ is not an f-rapid filter.

(2) TWO has no winning strategy.

Proof The proof is similar to that of Theorem 3.3. In the proof of supposing that $\mathcal{F}$ is a rapid filter and showing that ONE does not have a winning strategy, we let

$$I_n = \Big[h\Big(\sum_{i \leq n-1} f(i) + n\Big), h\Big(\sum_{i \leq n} f(i) + n + 1\Big)\Big).$$

In the proof of assuming that $\mathcal{F}$ is not a rapid filter and constructing a strategy σ for ONE, we let

$$\sigma(\langle n_1, n_2, \cdots, n_k\rangle) > \max\{n_k, (s_1 \cup \cdots \cup s_{f(N+1)+f(N+2)})\},$$

where TWO chooses n_k, and $k \in \big[\sum\limits_{i \leq N-1} f(i), \sum\limits_{i \leq N} f(i)\big)$.

Claim 3.2 Fix a filter $\mathcal{F}$, for every strictly increasing function $f \in \omega^\omega$, let $F_f \in \omega^\omega$, $F_f(0) = f(0) + f(1) + 1$, $F_f(n) = f(2n) + f(2n+1)$, $n \in \omega \setminus \{0\}$. Consider the game $G(\mathfrak{Fr}, [\omega]^{\leq F_f}, \mathcal{F})$. Then

(1) ONE has a winning strategy if and only if $\mathcal{F}$ is not an f-rapid filter.

(2) TWO has no winning strategy.

Proof The proof is similar to that of Theorem 3.4.

Proof of Theorem 3.5 As F_f dominates f_0, and due to Corollary 3.2, $\mathcal{F}$ is a rapid filter if and only if $\mathcal{F}$ is an f-rapid filter.

Theorem 3.6 *Fix a filter $\mathcal{F}$, $I \in \omega^\omega$, $I(n) = n+1$, $n \in \omega$. Consider the game $G(\mathfrak{Fr}, [\omega]^{\leq I}, \mathcal{F})$. Then*

(1) ONE *has a winning strategy if and only if $\mathcal{F}$ is not a rapid filter.*

(2) TWO *has no winning strategy.*

Proof Let $f(0) = f(1) = 0$, $f(2) = f(3) = f(4) = 1$ and $f(4n+1) = f(4n+2) = f(4n+3) = f(4n+4) = n+1$, $n \in \omega \setminus \{0\}$ and the process is similar to Claims 3.1–3.2. To prove rapid and f-rapid are equivalent, if $\mathcal{F}$ is a rapid filter, for a partition $\{I_n : n \in \omega\}$, let $J_0 = I_0 \cup I_1$, $J_1 = I_2 \cup I_3 \cup I_4$ and $J_{n+1} = I_{4n+1} \cup I_{4n+2} \cup I_{4n+3} \cup I_{4n+4}$, $n \in \omega \setminus \{0\}$.

Then $F_f(0) = f(0) + f(1) + 1 = 1$, $F_f(1) = f(2) + f(3) = 2$, $F_f(2) = f(4) + f(5) = 3$, and for each $n \geq 3$, if $n = 2k+1$, $k \in \omega \setminus \{0\}$,

$$F_f(n) = f(2n) + f(2n+1) = f(4k+2) + f(4k+3) = k+1+k+1 = 2k+2 = n+1.$$

If $n = 2k+2$, $k \in \omega \setminus \{0\}$,

$$F_f(n) = f(2n) + f(2n+1) = f(4k+4) + f(4k+5) = k+1+k+2 = 2k+3 = n+1.$$

So $F_f(n) = n+1 = I(n)$, $n \in \omega$.

There are some games we will be interested in, $G(\mathfrak{Fr}, [\omega]^{\leq n}, \mathcal{F})$ for each $n \in \omega$. For this, we will give a definition that $\mathcal{F}$ is an n-Q-filter for each $n \in \omega$ if for every partition $\{s_k : k \in \omega\}$ of ω into finite sets, there is $X \in \mathcal{F}$ such that $|X \cap s_k| \leq n$ for all $k \in \omega$. Q-filters are n-Q-filters for each $n \in \omega$, and n-Q-filter are rapid filters for each $n \in \omega$. It is easy to see that if $\mathcal{F}$ is not an n-Q-filter, then ONE has a winning strategy in $G(\mathfrak{Fr}, [\omega]^{\leq n}, \mathcal{F})$ for each $n \in \omega$; and TWO has no winning strategy in $G(\mathfrak{Fr}, [\omega]^{\leq n}, \mathcal{F})$ for each $n \in \omega$. Concerning the game, we don't know the answers to the following question.

Question 3.1 Is $\mathcal{F}$ not an n-Q-filter when ONE has winning strategy in $G(\mathfrak{Fr}, [\omega]^{\leq n}, \mathcal{F})$ for each $n \in \omega$?

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