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Extension Operators Preserving Biholomorphic Mappings on Hartogs Domains*

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Abstract In this paper, the authors extend the Roper-Suffridge operator on the generalized Hartogs domains. They mainly research the properties of the extended operator. By the characteristics of Hartogs domains and the geometric properties of subclasses of spirallike mappings, they obtain the extended Roper-Suffridge operator preserving almost starlikeness of complex order λ, almost spirallikeness of type β and order α, parabolic spirallikeness of type β and order ρ on the Hartogs domains in different conditions. They conclude that the corresponding extension operator preserves the same geometric invariance on the unit ball B^n in $\mathbb{C}^n$. The conclusions provide a new approach to study these geometric mappings in $\mathbb{C}^n$.

Keywords Roper-Suffridge operator, Spirallike mappings, Starlike mappings
2000 MR Subject Classification 32A30, 30C45

1 Introduction

There are great differences between the theory of several complex variables and the theory of one complex variable. To achieve the generation of the theory of one complex variable in higher dimensional complex spaces, scholars began to study some biholomorphic mappings with particular geometric characters, such as starlike mappings, convex mappings and spirallike mappings.

Biholomorphic mappings with particular geometric characters are important research objects. It is an important and difficult problem in the geometric theory of several complex variables to construct particular geometric mappings. From the different geometric features of mappings, some scholars defined different subclasses of biholomorphic mappings and studied their properties. Feng, Liu and Ren [1] introduced three kinds of subclasses of spirallike mappings-almost spirallike mappings of type β and order α, spirallike mappings of type β and order α, strongly spirallike mappings of type β and order α on the unit ball in complex Banach

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spaces. Cai and Liu [2] introduced strong and almost spirallike mappings of type β and order α on bounded starlike and circular domains. Feng, Zhang and Chen [3] introduced parabolic and spirallike mappings of type β and order ρ on bounded starlike and circular domains. In addition, lots of subclasses of biholomorphic mappings with particular geometric characters were introduced in spaces of several complex variables (see [4–6]).

It is difficult to find the concrete examples of these geometric mappings in higher dimensional complex spaces, while easier in the complex plane. Let $H(D)$ denote the set of holomorphic functions on D. The introduction of Roper-Suffridge operator (see [7])

$$\phi_n(f)(z) = \Big(f(z_1), \sqrt{f'(z_1)}z_0\Big)',$$

(where $z = (z_1, z_0) \in B^n, z_1 \in D, z_0 = (z_2, \cdots, z_n) \in \mathbb{C}^{n-1}, f(z_1) \in H(D), \sqrt{f'(0)} = 1$) sets up a bridge between biholomorphic mappings and biholomorphic functions, and offers a powerful way to construct and research these geometric mappings. Many scholars extended the Roper-Suffridge operator on different domains in different spaces. Applying the extended operators, we can construct these geometric mappings through biholomorphic functions with the same geometric properties. By far, there have been many graceful results about the Roper-Suffridge operator and its extensions (see [8–14]).

The generalized Roper-Suffridge operators have different properties on different domains. This makes us to research them on more general domains. Moreover, a variety of geometric mappings are constantly emerging, especially several subclasses of spirallike mappings which have attracted the attention of the majority of scholars. Therefore, it is essential to study the properties of the Roper-Suffridge extension operators preserving these subclasses of spirallike mappings on more general domains.

In 2004, Zhao, Zhang and Yin [15] studied the Einstein-K(a)hler metric on Cartan-Hartogs domain of the second type. In 2009, Wang, Ahn and Park [16] introduced a kind of Hartogs domains, constructed their circumscribed Hermite ellipsoids with minimum volume, and discussed their applications on the extreme value problems. In 2011, Wang and Liu [17] introduced new Hartogs domains and discussed Lu Qi-Keng's problem on the Hartogs domain

$$\Omega = \{(\xi, z, w) \in \mathbb{C}^{N_0+N_1+N_2}:\ \|\xi\|^{2K} < (1-\|z\|^2)^L(1-\|w\|^2),\ K, L > 0\}.$$

In 2015, Pan and Wang [18] discussed Bergman-Hartogs domains based on bounded symmetric domains

$$\Omega = \Big\{(w_{(1)}, \cdots, w_{(r)}, z) \in \mathbb{C}^{m_1} \times \cdots \mathbb{C}^{m_r} \times E : \sum_{j=1}^{r} \|w_{(j)}\|^{2p_j} < K_E(z,z)^{-q}\Big\},$$

where $K_E(z,z)$ is the Bergman kernel function on E, $r \in \mathbb{Z}^+$ and $p_1, \cdots, p_r > 1$, $q > 0$. They discussed the holomorphic automorphism groups of this kind of domains and pointed out that the domains aren't holomorphic equivalent to the unit ball. In 2016, Tang [19] studied the properties of Roper-Suffridge extension operators on the Bergman-Hartogs domain

$$\Omega^{B_n}_{p,q} = \{(w,z) \in \mathbb{C}^m \times B^n : \|w\|^{2p} < K_{B^n}(z,z)^{-q}\}, \quad w = (w_1, \cdots, w_m),\ z = (z_1, \cdots, z_n).$$

Now we introduce a kind of generalized Hartogs domain

$$\Omega_N = \{(\xi_{(1)}, \cdots, \xi_{(r)}, z, w) \in \mathbb{C}^{m_1} \times \cdots \times \mathbb{C}^{m_r} \times B^{N_1}(0,1) \times B^{N_2}(0,1) :$$
$$\|\xi_{(1)}\|^{2s_1} + \cdots + \|\xi_{(r)}\|^{2s_r} < (1-\|z\|^2)^l (1-\|w\|^2)^t,\ s_i > 0\ (i=1,\cdots,r), l \geq 0, t \geq 0\}$$

and the following extended operator

$$\begin{aligned} F(\xi, z, w) = \Big(&\Big(\frac{f(z_1)}{z_1}\Big)^{\delta_1}\Big(\frac{f(w_1)}{w_1}\Big)^{\gamma_1}\xi_{(1)}, \cdots, \Big(\frac{f(z_1)}{z_1}\Big)^{\delta_r}\Big(\frac{f(w_1)}{w_1}\Big)^{\gamma_r}\xi_{(r)}, \\ &f(z_1) + \frac{f(z_1)}{z_1}\sum_{j=2}^{N_1} P_j(z_j), \Big(\frac{f(z_1)}{z_1}\Big)^{\frac{1}{\varepsilon_2}} z_2, \cdots, \Big(\frac{f(z_1)}{z_1}\Big)^{\frac{1}{\varepsilon_{N_1}}} z_{N_1}, \\ &f(w_1) + \frac{f(w_1)}{w_1}\sum_{j=2}^{N_2} Q_j(w_j), \Big(\frac{f(w_1)}{w_1}\Big)^{\frac{1}{\eta_2}} w_2, \cdots, \Big(\frac{f(w_1)}{w_1}\Big)^{\frac{1}{\eta_{N_2}}} w_{N_2}\Big)', \end{aligned} \tag{1.1}$$

where $f \in H(D)$, the principal value branches of the power functions are chosen, P_j is the homogeneous polynomial of z_j with degree ε_j and Q_j is the homogeneous polynomial of w_j with degree η_j ($\varepsilon_j, \eta_j \geq 1$). The geometric properties preserved by (1.1) for $\xi_{(i)} = 0$ and $w_j = 0$ are the same as that preserved by the following operator

$$F(z) = \Big(f(z_1) + \frac{f(z_1)}{z_1}\sum_{j=2}^{N_1} P_j(z_j), \Big(\frac{f(z_1)}{z_1}\Big)^{\frac{1}{\varepsilon_2}} z_2, \cdots, \Big(\frac{f(z_1)}{z_1}\Big)^{\frac{1}{\varepsilon_{N_1}}} z_{N_1}\Big)',$$

where $\varepsilon_j \geq 2$ ($j = 2, \cdots, N_1$).

In this article, we mainly research the properties of the operator (1.1) on Hartogs domain Ω_N. We mainly discuss the geometric invariance of (1.1) preserving almost starlikeness of complex order λ, almost spirallikeness of type β and order α, parabolic spirallikeness of type β and order ρ on Ω_N in different conditions. We draw a conclusion that the corresponding extension operator preserves the same geometric invariance on B^n in $\mathbb{C}^n$.

2 Definitions and Lemmas

In the following, let D denote the unit disk in $\mathbb{C}$, B^n denote the unit ball in $\mathbb{C}^n$ and $DF(z)$ denote the Fréchet derivative of F at z.

To get the main results, we need the following definitions and lemmas.

Definition 2.1 (see [20]) *Suppose that Ω is a bounded starlike and circular domain in $\mathbb{C}^n$, and the Minkowski functional $\rho(z)$ is C^1 except for a lower-dimensional manifold. Let $F(z)$ be a normalized locally biholomorphic mapping on Ω, and let*

$$\Re\Big[(1-\lambda)\frac{2}{\rho(z)}\frac{\partial \rho}{\partial z}(z)(DF(z))^{-1}F(z)\Big] \geq -\Re\lambda, \quad z \in \Omega \setminus \{0\},$$

where $\lambda \in \mathbb{C}, \Re\lambda \leq 0$. Then $F(z)$ is called an almost starlike mapping of complex order λ on Ω.

Setting $\lambda = \frac{\alpha}{\alpha - 1}, \alpha \in [0,1)$ in Definition 2.1, we obtain the definition of almost starlike mappings of order α on Ω.

Definition 2.2 (see [6]) *Suppose that Ω is a bounded starlike and circular domain in $\mathbb{C}^n$, and the Minkowski functional $\rho(z)$ is C^1 except for a lower-dimensional manifold. Let $F(z)$ be a normalized locally biholomorphic mapping on Ω and $\alpha \in [0,1)$, $\beta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Then $F(z)$ is said to be an almost spirallike mapping of type β and order α on Ω if*

$$\Re\Big[\mathrm{e}^{-\mathrm{i}\beta}\frac{2}{\rho(z)}\frac{\partial\rho}{\partial z}(z)[DF(z)]^{-1}F(z)\Big] \geq \alpha\cos\beta.$$

Setting $\alpha = 0$ and $\beta = 0$ in Definition 2.2, respectively, we get the corresponding definitions of spirallike mappings of type β, almost starlike mappings of order α.

Definition 2.3 (see [3]) *Suppose that Ω is a bounded starlike and circular domain in $\mathbb{C}^n$, and the Minkowski functional $\rho(z)$ is C^1 except for a lower-dimensional manifold. Let $F(z)$ be a normalized locally biholomorphic mapping on Ω and $\rho \in [0,1)$, $\beta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Then $F(z)$ is said to be a parabolic and spirallike mapping of type β and order ρ on Ω if*

$$\Big|\mathrm{e}^{-\mathrm{i}\beta}\frac{2}{\rho(z)}\frac{\partial\rho}{\partial z}(z)(DF(z))^{-1}F(z) - (1-\mathrm{i}\sin\beta)\Big| < (1-2\rho) + \Re\Big\{\mathrm{e}^{-\mathrm{i}\beta}\frac{2}{\rho(z)}\frac{\partial\rho}{\partial z}(z)(DF(z))^{-1}F(z)\Big\}.$$

Setting $n = 1$ in Definitions 2.1–2.3, we get the corresponding almost starlike functions of complex order λ, almost spirallike functions of type β and order α, parabolic and spirallike functions of type β and order ρ in the complex plane.

Lemma 2.1 (see [21]) *Let $\rho(\xi, z, w)$ be the Minkowski functional of Ω_N. Denote $\rho(\xi, z, w) = \rho$. Then*

$$\begin{aligned}
\frac{\partial\rho}{\partial\xi_{ij}} &= \frac{s_i}{2}\|\xi_{(i)}\|^{2(s_i-1)}\overline{\xi_{ij}}\rho^{3-2s_i}\Big\{\sum_{k=1}^{r}\|\xi_{(k)}\|^{2s_k}s_k\rho^{2-2s_k}\\
&\quad+\Big(1-\Big\|\frac{z}{\rho}\Big\|^2\Big)^{l-1}\Big(1-\Big\|\frac{w}{\rho}\Big\|^2\Big)^{t-1}\Big[l\|z\|^2\Big(1-\Big\|\frac{w}{\rho}\Big\|^2\Big)+t\|w\|^2\Big(1-\Big\|\frac{z}{\rho}\Big\|^2\Big)\Big]\Big\}^{-1},\\
\frac{\partial\rho}{\partial z_{j_1}} &= \frac{\rho l}{2}\Big(1-\Big\|\frac{z}{\rho}\Big\|^2\Big)^{l-1}\Big(1-\Big\|\frac{w}{\rho}\Big\|^2\Big)^{t}\overline{z_{j_1}}\Big\{\sum_{k=1}^{r}\|\xi_{(k)}\|^{2s_k}s_k\rho^{2-2s_k}\\
&\quad+\Big(1-\Big\|\frac{z}{\rho}\Big\|^2\Big)^{l-1}\Big(1-\Big\|\frac{w}{\rho}\Big\|^2\Big)^{t-1}\Big[l\|z\|^2\Big(1-\Big\|\frac{w}{\rho}\Big\|^2\Big)+t\|w\|^2\Big(1-\Big\|\frac{z}{\rho}\Big\|^2\Big)\Big]\Big\}^{-1},\\
\frac{\partial\rho}{\partial w_{j_2}} &= \frac{\rho t}{2}\Big(1-\Big\|\frac{w}{\rho}\Big\|^2\Big)^{t-1}\Big(1-\Big\|\frac{z}{\rho}\Big\|^2\Big)^{l}\overline{w_{j_2}}\Big\{\sum_{k=1}^{r}\|\xi_{(k)}\|^{2s_k}s_k\rho^{2-2s_k}\\
&\quad+\Big(1-\Big\|\frac{z}{\rho}\Big\|^2\Big)^{l-1}\Big(1-\Big\|\frac{w}{\rho}\Big\|^2\Big)^{t-1}\Big[l\|z\|^2\Big(1-\Big\|\frac{w}{\rho}\Big\|^2\Big)+t\|w\|^2\Big(1-\Big\|\frac{z}{\rho}\Big\|^2\Big)\Big]\Big\}^{-1}.
\end{aligned}$$

When $(\xi, z, w) \in \partial\Omega_N$, we have $\rho = 1$ and

$$\begin{aligned}
\frac{\partial\rho}{\partial\xi_{ij}} &= \frac{s_i\|\xi_{(i)}\|^{2(s_i-1)}\overline{\xi_{ij}}}{2(\triangle_1+\|z\|^2\triangle_2+\|w\|^2\triangle_3)},\\
\frac{\partial\rho}{\partial z_{j_1}} &= \frac{\triangle_2\overline{z_{j_1}}}{2(\triangle_1+\|z\|^2\triangle_2+\|w\|^2\triangle_3)}, \quad \frac{\partial\rho}{\partial w_{j_2}} = \frac{\triangle_3\overline{w_{j_2}}}{2(\triangle_1+\|z\|^2\triangle_2+\|w\|^2\triangle_3)},
\end{aligned}$$

where $i = 1, \cdots, r$, $j = 1, \cdots, m_i$, $j_1 = 1, \cdots, N_1$, $j_2 = 1, \cdots, N_2$,

$$\triangle_1 = \sum_{i=1}^{r}\|\xi_{(i)}\|^{2s_i}s_i, \quad \triangle_2 = l(1-\|z\|^2)^{l-1}(1-\|w\|^2)^t, \quad \triangle_3 = t(1-\|w\|^2)^{t-1}(1-\|z\|^2)^l.$$

Lemma 2.2 *Let* $(\xi,z,w)\in\partial\Omega_N$, $F(\xi,z,w)$ *be the function defined by* (1.1) *with* $f\in H(D)$. *Then*

$$\frac{2\partial\rho}{\partial(\xi,z,w)}(\xi,z,w)(DF(\xi,z,w))^{-1}F(\xi,z,w)=\frac{H}{\triangle_1+\|z\|^2\triangle_2+\|w\|^2\triangle_3},$$

where

$$\begin{aligned}H=&\sum_{i=1}^{r}s_i\|\xi_{(i)}\|^{2s_i}\Big\{1-\delta_i\frac{z_1f'-f}{z_1f'}\Big[1+\frac{1}{z_1}\sum_{j=2}^{N_1}(1-\varepsilon_j)P_j(z_j)\Big]\\&-\gamma_i\frac{w_1f'-f}{w_1f'}\Big[1+\frac{1}{w_1}\sum_{j=2}^{N_2}(1-\eta_j)Q_j(w_j)\Big]\Big\}\\&+\triangle_2\Big\{|z_1|^2\frac{f}{z_1f'}\Big[1+\frac{1}{z_1}\sum_{j=2}^{N_1}(1-\varepsilon_j)P_j(z_j)\Big]\\&+\sum_{j=2}^{N_1}|z_j|^2\Big[1-\frac{1}{\varepsilon_j}\frac{z_1f'-f}{z_1f'}\Big(1+\frac{1}{z_1}\sum_{k=2}^{N_1}(1-\varepsilon_k)P_k(z_k)\Big)\Big]\Big\}\\&+\triangle_3\Big\{|w_1|^2\frac{f}{w_1f'}\Big[1+\frac{1}{w_1}\sum_{j=2}^{N_2}(1-\eta_j)Q_j(z_j)\Big]\\&+\sum_{j=2}^{N_2}|w_j|^2\Big[1-\frac{w_1f'-f}{\eta_jw_1f'}\Big(1+\frac{1}{w_1}\sum_{k=2}^{N_2}(1-\eta_k)Q_k(w_k)\Big)\Big]\Big\}.\end{aligned}$$

Proof From (1.1) we have $DF(\xi,z,w)=$

$$\begin{pmatrix}(\frac{f}{z_1})^{\delta_1}(\frac{f}{w_1})^{\gamma_1}I_{m_1}&\cdots&0_{m_1\times m_r}&v_1&0&\cdots&0&\sigma_1\\\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots\\0_{m_r\times m_1}&\cdots&(\frac{f}{z_1})^{\delta_r}(\frac{f}{w_1})^{\gamma_r}I_{m_r}&v_r&0&\cdots&0&\sigma_r\\0_{1\times m_1}&\cdots&0_{1\times m_r}&u_1&\frac{f}{z_1}P_2'(z_2)&\cdots&\frac{f}{z_1}P_{N_1}'(z_{N_1})&0_{1\times N_2}\\0_{1\times m_1}&\cdots&0_{1\times m_r}&u_2&(\frac{f}{z_1})^{\frac{1}{\varepsilon_2}}&\cdots&0&0_{1\times N_2}\\\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots\\0_{1\times m_1}&\cdots&0_{1\times m_r}&u_{N_1}&0&\cdots&(\frac{f}{z_1})^{\frac{1}{\varepsilon_{N_1}}}&0_{1\times N_2}\\0_{N_2\times m_1}&\cdots&0_{N_2\times m_r}&0_{N_2\times 1}&0_{N_2\times 1}&\cdots&0_{N_2\times 1}&J\end{pmatrix},$$

where $I_{m_i}(i=1,\cdots,r)$ denotes the identity matrix of m_i-row and m_i-column, $0_{p\times q}$ denotes the null matrix of p-row and q-column ($p,q\in\mathbb{Z}^+$), and

$$v_i=\begin{pmatrix}\delta_i\big(\frac{f(z_1)}{z_1}\big)^{\delta_i}\frac{z_1f'(z_1)-f(z_1)}{z_1f(z_1)}\big(\frac{f(w_1)}{w_1}\big)^{\gamma_i}\xi_{i1}\\\cdots\cdots\\\delta_i\big(\frac{f(z_1)}{z_1}\big)^{\delta_i}\frac{z_1f'(z_1)-f(z_1)}{z_1f(z_1)}\big(\frac{f(w_1)}{w_1}\big)^{\gamma_i}\xi_{im_i}\end{pmatrix},\quad i=1,\cdots,r,$$

$$\sigma_i=\begin{pmatrix}\gamma_i\big(\frac{f(z_1)}{z_1}\big)^{\delta_i}\frac{w_1f'(w_1)-f(w_1)}{w_1f(w_1)}\big(\frac{f(w_1)}{w_1}\big)^{\gamma_i}\xi_{i1},0_{1\times(N_2-1)}\\\cdots\cdots\\\gamma_i\big(\frac{f(z_1)}{z_1}\big)^{\delta_i}\frac{w_1f'(w_1)-f(w_1)}{w_1f(w_1)}\big(\frac{f(w_1)}{w_1}\big)^{\gamma_i}\xi_{im_i},0_{1\times(N_2-1)}\end{pmatrix},\quad i=1,\cdots,r,$$

$$u_1=f'(z_1)+\frac{z_1f'(z_1)-f(z_1)}{z_1^2}\sum_{j=2}^{N_1}P_j(z_j),$$

$$u_j=\frac{1}{\varepsilon_j}\Big(\frac{f(z_1)}{z_1}\Big)^{\frac{1}{\varepsilon_j}}\frac{z_1f'(z_1)-f(z_1)}{z_1f(z_1)}z_j,\quad j=2,\cdots,N_1,$$

$$J=\begin{pmatrix} f'(w_1)+\frac{w_1f'(w_1)-f(w_1)}{w_1^2}\sum\limits_{j=2}^{N_2}Q_j(w_j) & \frac{f(w_1)}{w_1}Q_2'(w_2) & \cdots & \frac{f(w_1)}{w_1}Q_{N_2}'(w_{N_2}) \\ \frac{1}{\eta_2}\big(\frac{f(w_1)}{w_1}\big)^{\frac{1}{\eta_2}}\frac{w_1f'(w_1)-f(w_1)}{w_1f(w_1)}w_2 & \big(\frac{f(w_1)}{w_1}\big)^{\frac{1}{\eta_2}} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{\eta_{N_2}}\big(\frac{f(w_1)}{w_1}\big)^{\frac{1}{\eta_{N_2}}}\frac{w_1f'(w_1)-f(w_1)}{w_1f(w_1)}w_{N_2} & 0 & \cdots & \big(\frac{f(w_1)}{w_1}\big)^{\frac{1}{\eta_{N_2}}} \end{pmatrix}.$$

Let

$$(DF(\xi,z,w))^{-1}F(\xi,z,w)=H_0=(h_1,\cdots,h_r,h_{r+1},\cdots,h_{r+N_1},h_{r+N_1+1},\cdots,h_{r+N_1+N_2})'.$$

Then $DF(\xi,z,w)H_0=F(\xi,z,w)$. By direct calculation we get

$$\begin{cases} h_i=\xi_{(i)}\Big\{1-\delta_i\dfrac{z_1f'-f}{z_1f'}\Big[1+\dfrac{1}{z_1}\sum\limits_{j=2}^{N_1}(1-\varepsilon_j)P_j(z_j)\Big]-\gamma_i\dfrac{w_1f'-f}{w_1f'}\Big[1+\dfrac{1}{w_1}\sum\limits_{j=2}^{N_2}(1-\eta_j)Q_j(w_j)\Big]\Big\}, \\ h_{r+1}=\dfrac{f(z_1)}{f'(z_1)}\Big[1+\dfrac{1}{z_1}\sum\limits_{j=2}^{N_1}(1-\varepsilon_j)P_j(z_j)\Big], \\ h_{r+N_1+1}=\dfrac{f(w_1)}{f'(w_1)}\Big[1+\dfrac{1}{w_1}\sum\limits_{j=2}^{N_2}(1-\eta_j)Q_j(w_j)\Big], \\ h_{r+j}=z_j\Big\{1-\dfrac{1}{\varepsilon_j}\dfrac{z_1f'-f}{z_1f'}\Big[1+\dfrac{1}{z_1}\sum\limits_{k=2}^{N_1}(1-\varepsilon_k)P_k(z_k)\Big]\Big\},\quad j=2,\cdots,N_1, \\ h_{r+N_1+k}=w_j\Big\{1-\dfrac{1}{\eta_j}\dfrac{w_1f'-f}{w_1f'}\Big[1+\dfrac{1}{w_1}\sum\limits_{k=2}^{N_2}(1-\eta_k)Q_k(w_k)\Big]\Big\},\quad j=2,\cdots,N_2, \end{cases}$$

where $\xi_{(i)}=(\xi_{i1},\cdots,\xi_{im_i})$ $(i=1,\cdots,r)$. Thus, from Lemma 2.1 we obtain the desired conclusion.

Lemma 2.3 (see [22]) *Suppose that $\Omega\in\mathbb{C}^n$ is a bounded starlike and circular domain, and the Minkowski functional $\rho(z)$ is C^1 except for a lower-dimensional manifold. Then, for any $z=(z_1,\cdots,z_n)\in\Omega\setminus\Omega_0$, we have*

$$2\Re\frac{\partial\rho(z)}{\partial z}z=\rho(z),\quad \frac{\partial\rho}{\partial z}(\lambda z)=\frac{\partial\rho(z)}{\partial z}(\lambda\geq 0),\quad \frac{\partial\rho}{\partial z}(\mathrm{e}^{\mathrm{i}\theta}z)=\mathrm{e}^{-\mathrm{i}\theta}\frac{\partial\rho(z)}{\partial z}(\theta\in R).$$

3 The Invariance of Almost Starlike Mappings of Complex Order λ

Theorem 3.1 *Let $f(z_1)$ be an almost starlike function of complex order λ on D with $\lambda\in\mathbb{C}$ and $\Re\lambda\leq 0$. Let $F(\xi,z,w)$ be the function denoted by (1.1) with $\delta_i,\gamma_i,\delta_i+\gamma_i\in[0,1]$ $(i=1,\cdots,r)$, $\varepsilon_j\geq 4$ $(j=2,\cdots,N_1)$ and $\eta_j\geq 4$ $(j=2,\cdots,N_2)$. Let the power functions take the principal value branches. Then $F(\xi,z,w)$ is an almost starlike mapping of complex order λ on Ω_N provided that*

$$\|P_j\|\leq\frac{2}{(8a+|1-\lambda|)\varepsilon_j},\quad \|Q_j\|\leq\frac{2}{(8b+|1-\lambda|)\eta_j},$$

where

$$a=\max_{1\le i\le r}\Big\{1,\frac{s_i\delta_i}{l}\Big\},\quad b=\max_{1\le i\le r}\Big\{1,\frac{s_i\gamma_i}{t}\Big\}.$$

Proof By Definition 2.1, we need only to prove

$$\Re\Big[(1-\lambda)\frac{2}{\rho(\xi,z,w)}\frac{\partial\rho(\xi,z,w)}{\partial(\xi,z,w)}(DF(\xi,z,w))^{-1}F(\xi,z,w)+\lambda\Big]\ge 0. \tag{3.1}$$

Let $(\xi,z,w)=\zeta(\eta,\lambda,\mu)=|\zeta|\mathrm{e}^{\mathrm{i}\theta}(\eta,\lambda,\mu)$, where $(\eta,\lambda,\mu)\in\partial\Omega_N$, $\zeta\in\overline{D}\setminus\{0\}$. By Lemma 2.3 we have

$$\begin{aligned}&\frac{2}{\rho(\xi,z,w)}\frac{\partial\rho(\xi,z,w)}{\partial(\xi,z,w)}(DF(\xi,z,w))^{-1}F(\xi,z,w)\\&=\frac{2}{\rho(|\zeta|\mathrm{e}^{\mathrm{i}\theta}(\eta,\lambda,\mu))}\frac{\partial\rho}{\partial(\xi,z,w)}(|\zeta|\mathrm{e}^{\mathrm{i}\theta}(\eta,\lambda,\mu))(DF(\zeta\eta,\zeta\lambda,\zeta\mu))^{-1}F(\zeta\eta,\zeta\lambda,\zeta\mu)\\&=\frac{2}{|\zeta|}\frac{\mathrm{e}^{-\mathrm{i}\theta}\partial\rho}{\partial(\xi,z,w)}(\eta,\lambda,\mu)(DF(\zeta\eta,\zeta\lambda,\zeta\mu))^{-1}F(\zeta\eta,\zeta\lambda,\zeta\mu)\\&=\frac{2\partial\rho}{\partial(\xi,z,w)}(\eta,\lambda,\mu)\frac{(DF(\zeta\eta,\zeta\lambda,\zeta\mu))^{-1}F(\zeta\eta,\zeta\lambda,\zeta\mu)}{\zeta}.\end{aligned}$$

Obviously $\frac{2\partial\rho}{\partial(\xi,z,w)}(\eta,\lambda,\mu)\frac{(DF(\zeta\eta,\zeta\lambda,\zeta\mu))^{-1}F(\zeta\eta,\zeta\lambda,\zeta\mu)}{\zeta}$ is holomorphic with respect to ζ, and thus the left-hand side of (3.1) is harmonic. From the minimum principle of harmonic functions, we need only to prove that (3.1) holds for $(\xi,z,w)\in\partial\Omega_N$, thus $\rho(\xi,z,w)=1$.

Let

$$q(z_1)=(1-\lambda)\frac{f(z_1)}{z_1f'(z_1)}+\lambda. \tag{3.2}$$

Since $f(z_1)$ is an almost starlike function of complex order λ on D, we have $\Re q(z_1)\ge 0$. Let $g(z_1)=\frac{q(z_1)-1}{q(z_1)+1}$. Then $g(z_1)\in H(D),|g(z_1)|<1,g(0)=0$. By Schwarz lemma we get $|g(z_1)|\le|z_1|$, therefore

$$|q(z_1)-1|\le\frac{2|z_1|}{1-|z_1|}. \tag{3.3}$$

By Lemma 2.2 and (3.2), we get

$$\begin{aligned}&(\triangle_1+\|z\|^2\triangle_2+\|w\|^2\triangle_3)\Big[(1-\lambda)\frac{2}{\rho(\xi,z,w)}\frac{\partial\rho(\xi,z,w)}{\partial(\xi,z,w)}(DF(\xi,z,w))^{-1}F(\xi,z,w)+\lambda\Big]\\&=(1-\lambda)H+\lambda(\triangle_1+\|z\|^2\triangle_2+\|w\|^2\triangle_3)\\&=\triangle_1\Big[1-\delta_i-\gamma_i+\delta_iq(z_1)+\gamma_iq(w_1)+\delta_i(q(z_1)-1)\frac{1}{z_1}\sum_{j=2}^{N_1}(1-\varepsilon_j)P_j(z_j)+\gamma_i(q(w_1)-1)\frac{1}{w_1}\\&\cdot\sum_{j=2}^{N_2}(1-\eta_j)Q_j(w_j)\Big]+\triangle_2\Big\{|z_1|^2q(z_1)+|z_1|^2[q(z_1)-\lambda]\frac{1}{z_1}\sum_{j=2}^{N_1}(1-\varepsilon_j)P_j(z_j)+\sum_{j=2}^{N_1}|z_j|^2\Big[1-\frac{1}{\varepsilon_j}\\&+\frac{1}{\varepsilon_j}q(z_1)+\frac{1}{\varepsilon_j}(q(z_1)-1)\frac{1}{z_1}\sum_{k=2}^{N_1}(1-\varepsilon_k)P_k(z_k)\Big]\Big\}+\triangle_3\Big\{|w_1|^2q(w_1)+|w_1|^2[q(w_1)-\lambda]\frac{1}{w_1}\\&\cdot\sum_{j=2}^{N_2}(1-\eta_j)Q_j(w_j)+\sum_{j=2}^{N_2}|w_j|^2\Big[1-\frac{1}{\eta_j}+\frac{1}{\eta_j}q(w_1)+\frac{1}{\eta_j}(q(w_1)-1)\frac{1}{w_1}\sum_{k=2}^{N_2}(1-\eta_k)Q_k(w_k)\Big]\Big\}\end{aligned}$$

$$= \triangle_1\Big[1-\delta_i-\gamma_i+\delta_i q(z_1)+\gamma_i q(w_1)+\delta_i(q(z_1)-1)\frac{1}{z_1}\sum_{j=2}^{N_1}(1-\varepsilon_j)P_j(z_j)+\gamma_i(q(w_1)-1)\frac{1}{w_1}$$

$$\cdot\sum_{j=2}^{N_2}(1-\eta_j)Q_j(w_j)\Big]+\triangle_2\Big\{|z_1|^2q(z_1)+|z_1|^2[(q(z_1)-1)+(1-\lambda)]\frac{1}{z_1}\sum_{j=2}^{N_1}(1-\varepsilon_j)P_j(z_j)$$

$$+\sum_{j=2}^{N_1}|z_j|^2\Big[1-\frac{1}{\varepsilon_j}+\frac{1}{\varepsilon_j}q(z_1)+\frac{1}{\varepsilon_j}(q(z_1)-1)\frac{1}{z_1}\sum_{k=2}^{N_1}(1-\varepsilon_k)P_k(z_k)\Big]\Big\}$$

$$+\triangle_3\Big\{|w_1|^2q(w_1)+|w_1|^2[(q(w_1)-1)+(1-\lambda)]\frac{1}{w_1}\sum_{j=2}^{N_2}(1-\eta_j)Q_j(w_j)$$

$$+\sum_{j=2}^{N_2}|w_j|^2\Big[1-\frac{1}{\eta_j}+\frac{1}{\eta_j}q(w_1)+\frac{1}{\eta_j}(q(w_1)-1)\frac{1}{w_1}\sum_{k=2}^{N_2}(1-\eta_k)Q_k(w_k)\Big]\Big\}.$$

For $\varepsilon_j,\eta_j\ge 4$, by direct calculation we have

$$\begin{cases}\dfrac{s_i\delta_i}{l}(1-\|z\|^2)+|z_1|^2+\sum\limits_{j=2}^{N_1}\frac{|z_j|^2}{\varepsilon_j}\le\dfrac{s_i\delta_i}{l}+\Big(1-\dfrac{s_i\delta_i}{l}\Big)\|z\|^2\le a,\\ \dfrac{s_i\gamma_i}{t}(1-\|w\|^2)+|w_1|^2+\sum\limits_{j=2}^{N_2}\frac{|w_j|^2}{\eta_j}\le\dfrac{s_i\gamma_i}{t}+\Big(1-\dfrac{s_i\gamma_i}{t}\Big)\|w\|^2\le b,\end{cases}\tag{3.4}$$

where

$$a=\max_{1\le i\le r}\Big\{1,\frac{s_i\delta_i}{l}\Big\},\quad b=\max_{1\le i\le r}\Big\{1,\frac{s_i\gamma_i}{t}\Big\}.$$

In addition, $(\xi,z,w)\in\partial\Omega_N$ follows $\sum\limits_{i=1}^{r}\|\xi_{(i)}\|^{2s_i}=(1-\|z\|^2)^l(1-\|w\|^2)^t$, then we have

$$\triangle_2=\frac{l}{1-\|z\|^2}\sum_{i=1}^{r}\|\xi_{(i)}\|^{2s_i},\quad \triangle_3=\frac{t}{1-\|w\|^2}\sum_{i=1}^{r}\|\xi_{(i)}\|^{2s_i}.\tag{3.5}$$

Since $\delta_i+\gamma_i\in[0,1]$ and $\varepsilon_j,\eta_j\ge 4$, from (3.3)–(3.5) we obtain

$$(\triangle_1+\|z\|^2\triangle_2+\|w\|^2\triangle_3)\Re\Big[(1-\lambda)\frac{2}{\rho(\xi,z,w)}\frac{\partial\rho(\xi,z,w)}{\partial(\xi,z,w)}(DF(\xi,z,w))^{-1}F(\xi,z,w)+\lambda\Big]$$

$$\ge\triangle_1\Big[1-\delta_i-\gamma_i-\delta_i\frac{2}{1-|z_1|}\sum_{j=2}^{N_1}(\varepsilon_j-1)|P_j(z_j)|-\gamma_i\frac{2}{1-|w_1|}\sum_{j=2}^{N_2}(\eta_j-1)|Q_j(w_j)|\Big]$$

$$+\triangle_2\Big\{-|z_1|\Big(\frac{2|z_1|}{1-|z_1|}+|1-\lambda|\Big)\sum_{j=2}^{N_1}(\varepsilon_j-1)|P_j(z_j)|+\sum_{j=2}^{N_1}|z_j|^2\Big[1-\frac{1}{\varepsilon_j}-\frac{1}{\varepsilon_j}\frac{2}{1-|z_1|}$$

$$\cdot\sum_{k=2}^{N_1}(\varepsilon_k-1)|P_k(z_k)|\Big]\Big\}+\triangle_3\Big\{-|w_1|\Big(\frac{2|w_1|}{1-|w_1|}+|1-\lambda|\Big)\sum_{j=2}^{N_1}(\eta_j-1)|Q_j(w_j)|$$

$$+\sum_{j=2}^{N_2}|w_j|^2\Big[1-\frac{1}{\eta_j}-\frac{1}{\eta_j}\frac{2}{1-|w_1|}\sum_{k=2}^{N_2}(\eta_k-1)|Q_k(w_k)|\Big]\Big\}$$

$$=\sum_{i=1}^{r}\|\xi_{(i)}\|^{2s_i}\Big\{s_i(1-\delta_i-\gamma_i)+\frac{l}{1-\|z\|^2}\Big\{\sum_{j=2}^{N_1}(\varepsilon_j-1)\frac{|z_j|^2}{\varepsilon_j}-\frac{2}{1-|z_1|}\sum_{j=2}^{N_1}(\varepsilon_j-1)|P_j(z_j)|$$

$$\cdot\Big[\frac{s_i\delta_i}{l}(1-\|z\|^2)+|z_1|^2+\frac{|1-\lambda|}{2}|z_1|(1-|z_1|)+\sum_{j=2}^{N_1}\frac{|z_j|^2}{\varepsilon_j}\Big]\Big\}+\frac{t}{1-\|w\|^2}\Big\{\sum_{j=2}^{N_2}(\eta_j-1)\frac{|w_j|^2}{\eta_j}$$

$$-\frac{2}{1-|w_1|}\sum_{j=2}^{N_2}(\eta_j-1)|Q_j(w_j)|\Big[\frac{s_i\gamma_i}{t}(1-\|w\|^2)+|w_1|^2+\frac{|1-\lambda|}{2}|w_1|(1-|w_1|)+\sum_{j=2}^{N_2}\frac{|w_j|^2}{\eta_j}\Big]\Big\}\Big\}$$

$$\geq\sum_{i=1}^{r}\|\xi_{(i)}\|^{2s_i}\Big\{s_i(1-\delta_i-\gamma_i)+\frac{l}{1-\|z\|^2}\sum_{j=2}^{N_1}(\varepsilon_j-1)|z_j|^2\Big(\frac{1}{\varepsilon_j}-\frac{2|z_j|^2}{1-|z_1|}\|P_j\|\frac{8a+|1-\lambda|}{8}\Big)$$

$$+\frac{t}{1-\|w\|^2}\sum_{j=2}^{N_2}(\eta_j-1)|w_j|^2\Big(\frac{1}{\eta_j}-\frac{2|w_j|^2}{1-|w_1|}\|Q_j\|\frac{8b+|1-\lambda|}{8}\Big)\Big\}$$

$$\geq\sum_{i=1}^{r}\|\xi_{(i)}\|^{2s_i}\Big\{s_i(1-\delta_i-\gamma_i)+\frac{l}{1-\|z\|^2}\sum_{j=2}^{N_1}(\varepsilon_j-1)|z_j|^2\Big(\frac{1}{\varepsilon_j}-\frac{8a+|1-\lambda|}{2}\|P_j\|\Big)$$

$$+\frac{t}{1-\|w\|^2}\sum_{j=2}^{N_2}(\eta_j-1)|w_j|^2\Big(\frac{1}{\eta_j}-\frac{8b+|1-\lambda|}{2}\|Q_j\|\Big)\Big\}$$

$$\geq 0,$$

and the last inequality is due to

$$\|P_j\|\leq\frac{2}{(8a+|1-\lambda|)\varepsilon_j},\quad \|Q_j\|\leq\frac{2}{(8b+|1-\lambda|)\eta_j}.$$

Hence we get the desired conclusion.

From Theorem 3.1, we draw the following conclusion on the unit ball B^n in $\mathbb{C}^n$.

Corollary 3.1 *Let $f(z_1)$ be an almost starlike function of complex order λ on D with $\lambda\in\mathbb{C},\Re\lambda\leq 0$. Let*

$$F(z)=\Big(f(z_1)+\frac{f(z_1)}{z_1}\sum_{j=2}^{n}P_j(z_j),\Big(\frac{f(z_1)}{z_1}\Big)^{\frac{1}{\varepsilon_2}}z_2,\cdots,\Big(\frac{f(z_1)}{z_1}\Big)^{\frac{1}{\varepsilon_n}}z_n\Big)',$$

where $\varepsilon_j\geq 4$ $(j=2,\cdots,n)$ and the power functions take the principal value branches. Then $F(z)$ is an almost starlike mapping of complex order λ on B^n provided that $\|P_j\|\leq\frac{2}{(8+|1-\lambda|)\varepsilon_j}$.

Remark 3.1 Setting $\lambda=\frac{\alpha}{\alpha-1},\alpha\in[0,1)$ in Theorem 3.1 and Corollary 3.1, we get the corresponding results for almost starlike mappings of order α.

4 The Invariance of Almost Spirallike Mappings of Type β and Order α

Theorem 4.1 *Let $f(z_1)$ be an almost spirallike function of type β and order α with $\alpha\in[0,1)$ and $\beta\in\big(-\frac{\pi}{2},\frac{\pi}{2}\big)$. Let $F(\xi,z,w)$ be the function denoted by (1.1) with $\delta_i,\gamma_i,\delta_i+\gamma_i\in[0,1]$ $(i=1,\cdots,r)$, $\varepsilon_j\geq 4$ $(j=2,\cdots,N_1)$ and $\eta_j\geq 4$ $(j=2,\cdots,N_2)$, where the power functions take the principal value branches. Then $F(\xi,z,w)$ is an almost spirallike mapping of type β and* *order α on Ω_N provided that*

$$\|P_j\| \leq \frac{2(1-\alpha)\cos\beta}{[1+8a(1-\alpha)\cos\beta]\varepsilon_j}, \quad \|Q_j\| \leq \frac{2(1-\alpha)\cos\beta}{[1+8b(1-\alpha)\cos\beta]\eta_j},$$

where

$$a = \max_{1\leq i\leq r}\Big\{1, \frac{s_i\delta_i}{l}\Big\}, \quad b = \max_{1\leq i\leq r}\Big\{1, \frac{s_i\gamma_i}{t}\Big\}.$$

Proof By Definition 2.2, we need to prove

$$\Re\Big[e^{-i\beta}\frac{2}{\rho(\xi,z,w)}\frac{\partial\rho(\xi,z,w)}{\partial(\xi,z,w)}(DF(\xi,z,w))^{-1}F(\xi,z,w)-\alpha\cos\beta\Big] \geq 0. \tag{4.1}$$

Similar to Theorem 3.1, from the minimum principle of harmonic functions, we need only to prove that (4.1) holds for $(\xi,z,w) \in \partial\Omega_N$, thus $\rho(\xi,z,w)=1$.

Let

$$q(z_1) = e^{-i\beta}\frac{f}{z_1 f'} - \alpha\cos\beta. \tag{4.2}$$

Since $f(z_1)$ is an almost spirallike function of type β and order α on D. Then

$$\Re q(z_1) \geq 0, \quad q(0) = (1-\alpha)\cos\beta - i\sin\beta.$$

Therefore

$$\Big|\frac{q(z_1)-((1-\alpha)\cos\beta - i\sin\beta)}{q(z_1)+((1-\alpha)\cos\beta + i\sin\beta)}\Big| < 1.$$

By Schwarz lemma we obtain

$$\Big|\frac{q(z_1)-((1-\alpha)\cos\beta - i\sin\beta)}{q(z_1)+((1-\alpha)\cos\beta + i\sin\beta)}\Big| \leq |z_1|.$$

So we get

$$|q(z_1)-((1-\alpha)\cos\beta - i\ \sin\beta)| \leq \frac{2(1-\alpha)\cos\beta|z_1|}{1-|z_1|},$$

that is

$$|q(z_1)-(e^{-i\beta}-\alpha\cos\beta)| \leq \frac{2(1-\alpha)\cos\beta|z_1|}{1-|z_1|}. \tag{4.3}$$

By Lemma 2.2 and (4.2), we get

$$\begin{aligned}
&(\triangle_1+\|z\|^2\triangle_2+\|w\|^2\triangle_3)\Big[e^{-i\beta}\frac{2}{\rho(\xi,z,w)}\frac{\partial\rho(\xi,z,w)}{\partial(\xi,z,w)}(DF(\xi,z,w))^{-1}F(\xi,z,w)-\alpha\cos\beta\Big]\\
&= e^{-i\beta}H - \alpha\cos\beta(\triangle_1+\|z\|^2\triangle_2+\|w\|^2\triangle_3)\\
&= \triangle_1\Big\{e^{-i\beta}-\alpha\cos\beta+\delta_i[q(z_1)-(e^{-i\beta}-\alpha\cos\beta)]\Big[1+\frac{1}{z_1}\sum_{j=2}^{N_1}(1-\varepsilon_j)P_j(z_j)\Big]+\gamma_i[q(w_1)\\
&\quad -(e^{-i\beta}-\alpha\cos\beta)]\Big[1+\frac{1}{w_1}\sum_{j=2}^{N_2}(1-\eta_j)Q_j(w_j)\Big]\Big\}+\triangle_2\Big\{|z_1|^2q(z_1)+|z_1|^2[q(z_1)+\alpha\cos\beta]\frac{1}{z_1}\\
&\quad \cdot\sum_{j=2}^{N_1}(1-\varepsilon_j)P_j(z_j)+\sum_{j=2}^{N_1}|z_j|^2\Big[e^{-i\beta}-\alpha\cos\beta+\frac{1}{\varepsilon_j}(q(z_1)-(e^{-i\beta}-\alpha\cos\beta))\Big(1+\frac{1}{z_1}
\end{aligned}$$

$$\cdot\sum_{j=2}^{N_1}(1-\varepsilon_j)P_j(z_j)\Big)\Big]\Big\}+\triangle_3\Big\{|w_1|^2q(w_1)+|w_1|^2[q(w_1)+\alpha\cos\beta]\frac{1}{w_1}\sum_{j=2}^{N_2}(1-\eta_j)Q_j(w_j)$$

$$+\sum_{j=2}^{N_2}|w_j|^2\Big[\mathrm{e}^{-\mathrm{i}\beta}-\alpha\cos\beta+\frac{1}{\eta_j}(q(w_1)-(\mathrm{e}^{-\mathrm{i}\beta}-\alpha\cos\beta))\Big(1+\frac{1}{w_1}\sum_{j=2}^{N_2}(1-\eta_j)Q_j(w_j)\Big)\Big]\Big\}$$

$$=\triangle_1\Big\{(\mathrm{e}^{-\mathrm{i}\beta}-\alpha\cos\beta)(1-\delta_i-\gamma_i)+\delta_iq(z_1)+\gamma_iq(w_1)+\delta_i[q(z_1)-(\mathrm{e}^{-\mathrm{i}\beta}-\alpha\cos\beta)]\frac{1}{z_1}$$

$$\cdot\sum_{j=2}^{N_1}(1-\varepsilon_j)P_j(z_j)+\gamma_i[q(w_1)-(\mathrm{e}^{-\mathrm{i}\beta}-\alpha\cos\beta)]\frac{1}{w_1}\sum_{j=2}^{N_2}(1-\eta_j)Q_j(w_j)\Big\}$$

$$+\triangle_2\Big\{|z_1|^2q(z_1)+[(q(z_1)-(\mathrm{e}^{-\mathrm{i}\beta}-\alpha\cos\beta))+\mathrm{e}^{-\mathrm{i}\beta}]\frac{|z_1|^2}{z_1}\sum_{j=2}^{N_1}(1-\varepsilon_j)P_j(z_j)+\sum_{j=2}^{N_1}|z_j|^2$$

$$\cdot\Big[\frac{1}{\varepsilon_j}q(z_1)+(\mathrm{e}^{-\mathrm{i}\beta}-\alpha\cos\beta)\Big(1-\frac{1}{\varepsilon_j}\Big)+\frac{1}{\varepsilon_j}(q(z_1)-(\mathrm{e}^{-\mathrm{i}\beta}-\alpha\cos\beta))\frac{1}{z_1}\sum_{j=2}^{N_1}(1-\varepsilon_j)P_j(z_j)\Big]\Big\}$$

$$+\triangle_3\Big\{|w_1|^2q(w_1)+[(q(w_1)-(\mathrm{e}^{-\mathrm{i}\beta}-\alpha\cos\beta))+\mathrm{e}^{-\mathrm{i}\beta}]\frac{|w_1|^2}{w_1}\sum_{j=2}^{N_2}(1-\eta_j)Q_j(w_j)+\sum_{j=2}^{N_2}|w_j|^2$$

$$\cdot\Big[\frac{1}{\eta_j}q(w_1)+(\mathrm{e}^{-\mathrm{i}\beta}-\alpha\cos\beta)\Big(1-\frac{1}{\eta_j}\Big)+\frac{1}{\eta_j}(q(w_1)-(\mathrm{e}^{-\mathrm{i}\beta}-\alpha\cos\beta))\frac{1}{w_1}\sum_{j=2}^{N_2}(1-\eta_j)Q_j(w_j)\Big]\Big\}.$$

Since $\delta_i+\gamma_i\in[0,1]$ and $\varepsilon_j,\eta_j\geq 4$, from (4.3) and (3.4)–(3.5) we obtain

$$(\triangle_1+\|z\|^2\triangle_2+\|w\|^2\triangle_3)\Re\Big[\mathrm{e}^{-\mathrm{i}\beta}\frac{2}{\rho(\xi,z,w)}\frac{\partial\rho(\xi,z,w)}{\partial(\xi,z,w)}(DF(\xi,z,w))^{-1}F(\xi,z,w)-\alpha\cos\beta\Big]$$

$$\geq\triangle_1\Big\{(1-\alpha)\cos\beta(1-\delta_i-\gamma_i)-\delta_i\frac{2(1-\alpha)\cos\beta}{1-|z_1|}\sum_{j=2}^{N_1}(\varepsilon_j-1)|P_j(z_j)|-\gamma_i\frac{2(1-\alpha)\cos\beta}{1-|w_1|}\sum_{j=2}^{N_2}(\eta_j$$

$$-1)|Q_j(w_j)|\Big\}+\triangle_2\Big\{-\Big[|z_1|^2\frac{2(1-\alpha)\cos\beta}{1-|z_1|}+|z_1|\Big]\sum_{j=2}^{N_1}(\varepsilon_j-1)|P_j(z_j)|+\sum_{j=2}^{N_1}|z_j|^2\Big[(1-\alpha)$$

$$\cdot\cos\beta\Big(1-\frac{1}{\varepsilon_j}\Big)-\frac{2(1-\alpha)\cos\beta}{\varepsilon_j(1-|z_1|)}\sum_{j=2}^{N_1}(\varepsilon_j-1)|P_j(z_j)|\Big]\Big\}+\triangle_3\Big\{-\Big[|w_1|^2\frac{2(1-\alpha)\cos\beta}{1-|w_1|}+|w_1|\Big]$$

$$\cdot\sum_{j=2}^{N_2}(\eta_j-1)|Q_j(w_j)|+\sum_{j=2}^{N_2}|w_j|^2\Big[(1-\alpha)\cos\beta\Big(1-\frac{1}{\eta_j}\Big)-\frac{2(1-\alpha)\cos\beta}{\eta_j(1-|w_1|)}\sum_{j=2}^{N_2}(\eta_j-1)|Q_j(w_j)|\Big]\Big\}$$

$$\geq(1-\alpha)\cos\beta\sum_{i=1}^{r}\|\xi_{(i)}\|^{2s_i}\Big\{s_i(1-\delta_i-\gamma_i)+\frac{l}{1-\|z\|^2}\sum_{j=2}^{N_1}(\varepsilon_j-1)|z_j|^2\Big[\frac{1}{\varepsilon_j}-\Big(\frac{s_i\delta_i}{l}(1-\|z\|^2)$$

$$+|z_1|^2+\frac{|z_1|(1-|z_1|)}{2(1-\alpha)\cos\beta}+\sum_{j=2}^{N_1}\frac{|z_j|^2}{\varepsilon_j}\Big)\frac{2|z_j|^2}{1-|z_1|}\|P_j\|\Big]+\frac{t}{1-\|w\|^2}\sum_{j=2}^{N_2}(\eta_j-1)|w_j|^2\Big[\frac{1}{\eta_j}$$

$$-\Big(\frac{s_i\gamma_i}{t}(1-\|w\|^2)+|w_1|^2+\frac{|w_1|(1-|w_1|)}{2(1-\alpha)\cos\beta}+\sum_{j=2}^{N_2}\frac{|w_j|^2}{\eta_j}\Big)\frac{2|w_j|^2}{1-|w_1|}\|Q_j\|\Big]\Big\}$$

$$\geq(1-\alpha)\cos\beta\sum_{i=1}^{r}\|\xi_{(i)}\|^{2s_i}\Big\{s_i(1-\delta_i-\gamma_i)+\frac{l}{1-\|z\|^2}\sum_{j=2}^{N_1}(\varepsilon_j-1)|z_j|^2\Big[\frac{1}{\varepsilon_j}$$

$$- \frac{1+8a(1-\alpha)\cos\beta}{2(1-\alpha)\cos\beta}\|P_j\|\Big] + \frac{t}{1-\|w\|^2}\sum_{j=2}^{N_2}(\eta_j-1)|w_j|^2\Big[\frac{1}{\eta_j} - \frac{1+8b(1-\alpha)\cos\beta}{2(1-\alpha)\cos\beta}\|Q_j\|\Big]\Big\}$$

$$\geq 0$$

provided that

$$\|P_j\| \leq \frac{2(1-\alpha)\cos\beta}{[1+8a(1-\alpha)\cos\beta]\varepsilon_j}, \quad \|Q_j\| \leq \frac{2(1-\alpha)\cos\beta}{[1+8b(1-\alpha)\cos\beta]\eta_j}.$$

Hence $F(\xi, z, w)$ is an almost spirallike mapping of type β and order α.

From Theorem 4.1, we draw the following conclusion on the unit ball B^n in $\mathbb{C}^n$.

Corollary 4.1 *Let $f(z_1)$ be an almost spirallike function of type β and order α on D with $\alpha \in [0,1)$ and $\beta \in \big(-\frac{\pi}{2}, \frac{\pi}{2}\big)$. Let*

$$F(z) = \Big(f(z_1) + \frac{f(z_1)}{z_1}\sum_{j=2}^{n} P_j(z_j), \Big(\frac{f(z_1)}{z_1}\Big)^{\frac{1}{\varepsilon_2}} z_2, \cdots, \Big(\frac{f(z_1)}{z_1}\Big)^{\frac{1}{\varepsilon_n}} z_n\Big)'$$

with $\varepsilon_j \geq 4$ $(j = 2, \cdots, n)$ and $\big(\frac{f(z_1)}{z_1}\big)^{\frac{1}{\varepsilon_j}}|_{z_1=0} = 1$. Then $F(z)$ is an almost spirallike mapping of type β and order α on B^n provided that

$$\|P_j\| \leq \frac{2(1-\alpha)\cos\beta}{[1+8(1-\alpha)\cos\beta]\varepsilon_j}.$$

Remark 4.1 Setting $\alpha = 0$ and $\beta = 0$ in Theorem 4.1 and Corollary 4.1, respectively, we get the corresponding results for spirallike mappings of type β, almost starlike mappings of order α.

5 The Invariance of Parabolic and Spirallike Mappings of Type β and Order ρ

Theorem 5.1 *Let $f(z_1)$ be a parabolic and spirallike function of type β and order ρ on D with $\rho \in [0,1)$ and $\beta \in \big(-\frac{\pi}{2}, \frac{\pi}{2}\big)$. Let $F(\xi, z, w)$ be the function denoted by (1.1) with $\delta_i, \gamma_i, \delta_i + \gamma_i \in [0,1]$ $(i = 1, \cdots, r)$, $\varepsilon_j \geq 4$ $(j = 2, \cdots, N_1)$ and $\eta_j \geq 4$ $(j = 2, \cdots, N_2)$. Let the power functions take the principal value branches. Then $F(\xi, z, w)$ is a parabolic and spirallike mappings of type β and order ρ on Ω_N provided that $\rho < \cos\beta$ and*

$$\|P_j\| \leq \frac{2(\cos\beta - \rho)}{[1+8a(\cos\beta-\rho)]\varepsilon_j}, \quad \|Q_j\| \leq \frac{2(\cos\beta - \rho)}{[1+8b(\cos\beta-\rho)]\eta_j},$$

where

$$a = \max_{1\leq i\leq r}\Big\{1, \frac{s_i\delta_i}{l}\Big\}, \quad b = \max_{1\leq i\leq r}\Big\{1, \frac{s_i\gamma_i}{t}\Big\}.$$

Proof By Definition 2.3, we need only to prove

$$\Big|\mathrm{e}^{-\mathrm{i}\beta}\frac{2}{\rho(\xi,z,w)}\frac{\partial\rho(\xi,z,w)}{\partial(\xi,z,w)}(DF(\xi,z,w))^{-1}F(\xi,z,w) - (1-\mathrm{i}\sin\beta)\Big|$$

$$< (1-2\rho) + \Re\Big[\mathrm{e}^{-\mathrm{i}\beta}\frac{2}{\rho(\xi,z,w)}\frac{\partial\rho(\xi,z,w)}{\partial(\xi,z,w)}(DF(\xi,z,w))^{-1}F(\xi,z,w)\Big]. \tag{5.1}$$

Similar to Theorem 3.1,

$$\mathrm{e}^{-\mathrm{i}\beta}\frac{2}{\rho(\xi,z,w)}\frac{\partial\rho(\xi,z,w)}{\partial(\xi,z,w)}(DF(\xi,z,w))^{-1}F(\xi,z,w) - (1-\mathrm{i}\sin\beta)$$

is a holomorphic function, so $\Re\big[\mathrm{e}^{-\mathrm{i}\beta}\frac{2}{\rho(\xi,z,w)}\frac{\partial\rho(\xi,z,w)}{\partial(\xi,z,w)}(DF(\xi,z,w))^{-1}F(\xi,z,w)\big]$ is harmonic. From the maximum modulus principle of holomorphic functions and the minimum principle of harmonic functions, we need only to prove that (5.1) holds for $z\in\partial\Omega_N$. Thus $\rho(\xi,z,w)=1$.

Let

$$q(z_1) = \mathrm{e}^{-\mathrm{i}\beta}\frac{f}{z_1 f'} - (1-\mathrm{i}\sin\beta). \tag{5.2}$$

Then

$$|q(z_1)| < 2(1-\rho) + \Re q(z_1). \tag{5.3}$$

From the geometric properties of $q(z_1)$, we get $\Re[q(z_1)+(1-\rho)]>0$. Let $g(z_1)=q(z_1)+(1-\rho)$ then $\Re g(z_1)>0$ and $g(0)=\cos\beta-\rho$. Therefore

$$\Big|\frac{g(z_1)-(\cos\beta-\rho)}{g(z_1)+(\cos\beta-\rho)}\Big| < 1,$$

which follows

$$\Big|\frac{g(z_1)-(\cos\beta-\rho)}{g(z_1)+(\cos\beta-\rho)}\Big| \le |z_1|$$

by Schwarz lemma. So we obtain $|g(z_1)-(\cos\beta-\rho)|\le 2(\cos\beta-\rho)\frac{|z_1|}{1-|z_1|}$, that is

$$|q(z_1)+1-\cos\beta| \le 2(\cos\beta-\rho)\frac{|z_1|}{1-|z_1|}. \tag{5.4}$$

From Lemma 2.2 and (5.2), we get

$$\begin{aligned}
&(\triangle_1+\|z\|^2\triangle_2+\|w\|^2\triangle_3)\Big[\mathrm{e}^{-\mathrm{i}\beta}\frac{2\partial\rho(\xi,z,w)}{\partial(\xi,z,w)}(DF(\xi,z,w))^{-1}F(\xi,z,w)-(1-\mathrm{i}\sin\beta)\Big]\\
&=\mathrm{e}^{-\mathrm{i}\beta}H-(1-\mathrm{i}\sin\beta)(\triangle_1+\|z\|^2\triangle_2+\|w\|^2\triangle_3)\\
&=\sum_{i=1}^{r}s_i\|\xi_{(i)}\|^{2s_i}\Big\{\mathrm{e}^{-\mathrm{i}\beta}+\delta_i[q(z_1)+1-\cos\beta]\Big[1+\frac{1}{z_1}\sum_{j=2}^{N_1}(\varepsilon_j-1)P_j(z_j)\Big]+\gamma_i[q(w_1)+1-\cos\beta]\Big[1+\frac{1}{w_1}\\
&\quad\cdot\sum_{j=2}^{N_2}(\eta_j-1)Q_j(w_j)\Big]\Big\}+\triangle_2\Big\{|z_1|^2[q(z_1)+1-\mathrm{i}\sin\beta]\Big[1+\frac{1}{z_1}\sum_{j=2}^{N_1}(\varepsilon_j-1)P_j(z_j)\Big]+\sum_{j=2}^{N_1}|z_j|^2\Big[\mathrm{e}^{-\mathrm{i}\beta}\\
&\quad+\frac{1}{\varepsilon_j}(q(z_1)+1-\cos\beta)\Big(1+\frac{1}{z_1}\sum_{k=2}^{N_1}(1-\varepsilon_k)P_k(z_k)\Big)\Big]\Big\}+\triangle_3\Big\{|w_1|^2[q(w_1)+1-\mathrm{i}\sin\beta]\Big[1+\frac{1}{w_1}\\
&\quad\cdot\sum_{j=2}^{N_2}(\eta_j-1)Q_j(z_j)\Big]+\sum_{j=2}^{N_2}|w_j|^2\Big[\mathrm{e}^{-\mathrm{i}\beta}+\frac{1}{\eta_j}(q(z_1)+1-\cos\beta)\Big(1+\frac{1}{w_1}\sum_{k=2}^{N_2}(1-\eta_k)Q_k(w_k)\Big)\Big]\Big\}\\
&\quad-(1-\mathrm{i}\sin\beta)(\triangle_1+\|z\|^2\triangle_2+\|w\|^2\triangle_3)
\end{aligned}$$

$$
\begin{aligned}
&=\sum_{i=1}^{r}s_i\|\xi_{(i)}\|^{2s_i}\Big\{(\cos\beta-1)(1-\delta_i-\gamma_i)+\delta_i q(z_1)+\gamma_i q(w_1)+\delta_i[q(z_1)+1-\cos\beta]\frac{1}{z_1}\sum_{j=2}^{N_1}(\varepsilon_j-1)P_j(z_j)\\
&\quad+\gamma_i[q(w_1)+1-\cos\beta]\frac{1}{w_1}\sum_{j=2}^{N_2}(\eta_j-1)Q_j(w_j)\Big\}+\triangle_2\Big\{|z_1|^2q(z_1)+[(q(z_1)+1-\cos\beta)+\mathrm{e}^{-\mathrm{i}\beta}]\\
&\quad\cdot\frac{|z_1|^2}{z_1}\sum_{j=2}^{N_1}(\varepsilon_j-1)P_j(z_j)+\sum_{j=2}^{N_1}|z_j|^2\Big[(\cos\beta-1)\Big(1-\frac{1}{\varepsilon_j}\Big)+\frac{1}{\varepsilon_j}q(z_1)+\frac{1}{\varepsilon_j}(q(z_1)+1-\cos\beta)\frac{1}{z_1}\\
&\quad\cdot\sum_{k=2}^{N_1}(1-\varepsilon_k)P_k(z_k)\Big]\Big\}+\triangle_3\Big\{|w_1|^2q(w_1)+[(q(w_1)+1-\cos\beta)+\mathrm{e}^{-\mathrm{i}\beta}]\frac{|w_1|^2}{w_1}\sum_{j=2}^{N_2}(\eta_j-1)Q_j(w_j)\\
&\quad+\sum_{j=2}^{N_2}|w_j|^2\Big[(\cos\beta-1)\Big(1-\frac{1}{\eta_j}\Big)+\frac{1}{\eta_j}q(w_1)+\frac{1}{\eta_j}(q(w_1)+1-\cos\beta)\frac{1}{w_1}\sum_{k=2}^{N_2}(1-\eta_k)Q_k(w_k)\Big]\Big\}.
\end{aligned}
$$

Therefore by (5.3)–(5.4) and (3.5) we have

$$
\begin{aligned}
&(\triangle_1+\|z\|^2\triangle_2+\|w\|^2\triangle_3)\Big|\mathrm{e}^{-\mathrm{i}\beta}\frac{2\partial\rho(\xi,z,w)}{\partial(\xi,z,w)}(DF(\xi,z,w))^{-1}F(\xi,z,w)-(1-\mathrm{i}\sin\beta)\Big|\\
&<\sum_{i=1}^{r}\|\xi_{(i)}\|^{2s_i}A,
\end{aligned}
$$

$$
\begin{aligned}
A&=(1-\cos\beta)\Big[s_i(1-\delta_i-\gamma_i)+\frac{l}{1-\|z\|^2}\sum_{j=2}^{N_1}|z_j|^2\Big(1-\frac{1}{\varepsilon_j}\Big)+\frac{t}{1-\|w\|^2}\sum_{j=2}^{N_2}|w_j|^2\Big(1-\frac{1}{\eta_j}\Big)\Big]\\
&\quad+[2(1-\rho)+\Re q(z_1)]\Big[s_i\delta_i+\frac{l}{1-\|z\|^2}\Big(|z_1|^2+\sum_{j=2}^{N_1}\frac{|z_j|^2}{\varepsilon_j}\Big)\Big]+[2(1-\rho)+\Re q(w_1)]\Big[s_i\gamma_i\\
&\quad+\frac{t}{1-\|w\|^2}\Big(|w_1|^2+\sum_{j=2}^{N_2}\frac{|w_j|^2}{\eta_j}\Big)\Big]+\frac{2(\cos\beta-\rho)}{1-|z_1|}\sum_{j=2}^{N_1}(\varepsilon_j-1)|P_j(z_j)|\Big[\frac{s_i\delta_i}{l}(1-\|z\|^2)+|z_1|^2\\
&\quad+\frac{|z_1|(1-|z_1|)}{2(\cos\beta-\rho)}+\sum_{j=2}^{N_1}\frac{|z_j|^2}{\varepsilon_j}\Big]\frac{l}{1-\|z\|^2}+\frac{2(\cos\beta-\rho)}{1-|w_1|}\sum_{j=2}^{N_2}(\eta_j-1)|Q_j(w_j)|\Big[\frac{s_i\gamma_i}{t}(1-\|w\|^2)+|w_1|^2\\
&\quad+\frac{|w_1|(1-|w_1|)}{2(\cos\beta-\rho)}+\sum_{j=2}^{N_2}\frac{|w_j|^2}{\eta_j}\Big]\frac{t}{1-\|w\|^2}.
\end{aligned}
$$

On the other hand, from (5.3)–(5.4) and (3.5) we get

$$
\begin{aligned}
&(\triangle_1+\|z\|^2\triangle_2+\|w\|^2\triangle_3)\Big\{1-2\rho+\Re\Big[\mathrm{e}^{-\mathrm{i}\beta}\frac{2\partial\rho(\xi,z,w)}{\partial(\xi,z,w)}(DF(\xi,z,w))^{-1}F(\xi,z,w)\Big]\Big\}\\
&=(1-2\rho)(\triangle_1+\|z\|^2\triangle_2+\|w\|^2\triangle_3)+\Re(\mathrm{e}^{-\mathrm{i}\beta}H)\\
&=\sum_{i=1}^{r}s_i\|\xi_{(i)}\|^{2s_i}\Big\{\cos\beta+(\delta_i+\gamma_i)(1-\cos\beta)+(1-2\rho)+\delta_i\Re q(z_1)+\gamma_i\Re q(w_1)+\delta_i\Re\Big[(q(z_1)+1\\
&\quad-\cos\beta)\frac{1}{z_1}\sum_{j=2}^{N_1}(\varepsilon_j-1)P_j(z_j)\Big]+\gamma_i\Re\Big[(q(w_1)+1-\cos\beta)\frac{1}{w_1}\sum_{j=2}^{N_2}(\eta_j-1)Q_j(w_j)\Big]\Big\}\\
&\quad+\triangle_2\Big\{|z_1|^2[\Re q(z_1)+2(1-\rho)]+|z_1|^2\Re\Big[((q(z_1)+1-\cos\beta)+\mathrm{e}^{-\mathrm{i}\beta})\frac{1}{z_1}\sum_{j=2}^{N_1}(\varepsilon_j-1)P_j(z_j)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{j=2}^{N_1}|z_j|^2\Big[1-2\rho+\cos\beta+\frac{1}{\varepsilon_j}(1-\cos\beta)+\frac{1}{\varepsilon_j}\Re q(z_1)+\frac{1}{\varepsilon_j}\Re\Big((q(z_1)+1-\cos\beta)\frac{1}{z_1}\\
&\cdot\sum_{k=2}^{N_1}(1-\varepsilon_k)P_k(z_k)\Big)\Big]\Big\}+\triangle_3\Big\{|w_1|^2[\Re q(w_1)+2(1-\rho)]+|w_1|^2\Re\Big[((q(w_1)+1-\cos\beta)+\mathrm{e}^{-\mathrm{i}\beta})\\
&\cdot\frac{1}{w_1}\sum_{j=2}^{N_2}(\eta_j-1)Q_j(w_j)\Big]+\sum_{j=2}^{N_2}|w_j|^2\Big[1-2\rho+\cos\beta+\frac{1}{\eta_j}(1-\cos\beta)+\frac{1}{\eta_j}\Re q(w_1)\\
&+\frac{1}{\eta_j}\Re\Big((q(w_1)+1-\cos\beta)\frac{1}{w_1}\sum_{k=2}^{N_2}(1-\eta_k)Q_k(w_k)\Big)\Big]\Big\}\\
\geq&\sum_{i=1}^{r}s_i\|\xi_{(i)}\|^{2s_i}\Big\{(\cos\beta-1)(1-\delta_i-\gamma_i)+2(1-\rho)+\delta_i\Re q(z_1)+\gamma_i\Re q(w_1)-\delta_i\frac{2(\cos\beta-\rho)}{1-|z_1|}\\
&\cdot\sum_{j=2}^{N_1}(\varepsilon_j-1)|P_j(z_j)|-\gamma_i\frac{2(\cos\beta-\rho)}{1-|w_1|}\sum_{j=2}^{N_2}(\eta_j-1)|Q_j(w_j)|\Big\}+\triangle_2\Big\{|z_1|^2[\Re q(z_1)+2(1-\rho)]\\
&-\Big(|z_1|^2\frac{2(\cos\beta-\rho)}{1-|z_1|}+|z_1|\Big)\sum_{j=2}^{N_1}(\varepsilon_j-1)|P_j(z_j)|+\sum_{j=2}^{N_1}|z_j|^2\Big[(\cos\beta-1)\Big(1-\frac{1}{\varepsilon_j}\Big)+2(1-\rho)\\
&+\frac{1}{\varepsilon_j}\Re q(z_1)-\frac{1}{\varepsilon_j}\frac{2(\cos\beta-\rho)}{1-|z_1|}\sum_{k=2}^{N_1}(\varepsilon_k-1)|P_k(z_k)|\Big]\Big\}+\triangle_3\Big\{|w_1|^2[\Re q(w_1)+2(1-\rho)]\\
&-\Big(|w_1|^2\frac{2(\cos\beta-\rho)}{1-|w_1|}+|w_1|\Big)\sum_{j=2}^{N_2}(\eta_j-1)|Q_j(w_j)|+\sum_{j=2}^{N_2}|w_j|^2\Big[(\cos\beta-1)\Big(1-\frac{1}{\eta_j}\Big)+2(1-\rho)\\
&+\frac{1}{\eta_j}\Re q(w_1)-\frac{1}{\eta_j}\frac{2(\cos\beta-\rho)}{1-|w_1|}\sum_{k=2}^{N_2}(\eta_k-1)|Q_k(w_k)|\Big]\Big\}\\
=&\sum_{i=1}^{r}\|\xi_{(i)}\|^{2s_i}B,
\end{aligned}
$$

where

$$
\begin{aligned}
B=&(\cos\beta-1)\Big[s_i(1-\delta_i-\gamma_i)+\frac{l}{1-\|z\|^2}\sum_{j=2}^{N_1}|z_j|^2\Big(1-\frac{1}{\varepsilon_j}\Big)+\frac{t}{1-\|w\|^2}\sum_{j=2}^{N_2}|w_j|^2\Big(1-\frac{1}{\eta_j}\Big)\Big]\\
&+[\Re q(z_1)+2(1-\rho)]\Big[s_i\delta_i+\frac{l}{1-\|z\|^2}\Big(|z_1|^2+\sum_{j=2}^{N_1}\frac{|z_j|^2}{\varepsilon_j}\Big)\Big]+2(1-\rho)\Big[s_i(1-\delta_i)\\
&+\frac{l}{1-\|z\|^2}\sum_{j=2}^{N_1}|z_j|^2\Big(1-\frac{1}{\varepsilon_j}\Big)\Big]+[\Re q(w_1)+2(1-\rho)]\Big[s_i\gamma_i+\frac{t}{1-\|w\|^2}\Big(|w_1|^2+\sum_{j=2}^{N_2}\frac{|w_j|^2}{\eta_j}\Big)\Big]\\
&+2(1-\rho)\Big[-s_i\gamma_i+\frac{t}{1-\|w\|^2}\sum_{j=2}^{N_2}|w_j|^2\Big(1-\frac{1}{\eta_j}\Big)\Big]\\
&-\frac{l}{1-\|z\|^2}\frac{2(\cos\beta-\rho)}{1-|z_1|}\sum_{j=2}^{N_1}(\varepsilon_j-1)|P_j(z_j)|\Big[\frac{s_i\delta_i}{l}(1-\|z\|^2)+|z_1|^2+\frac{|z_1|(1-|z_1|)}{2(\cos\beta-\rho)}+\sum_{j=2}^{N_1}\frac{|z_j|^2}{\varepsilon_j}\Big]
\end{aligned}
$$

$$-\frac{t}{1-\|w\|^2}\frac{2(\cos\beta-\rho)}{1-|w_1|}\sum_{j=2}^{N_2}(\eta_j-1)|Q_j(w_j)|\Big[\frac{s_i\gamma_i}{t}(1-\|w\|^2)+|w_1|^2+\frac{|w_1|(1-|w_1|)}{2(\cos\beta-\rho)}+\sum_{j=2}^{N_2}\frac{|w_j|^2}{\eta_j}\Big].$$

Thus, by $\varepsilon_j, \eta_j \geq 4$ and (3.4), we obtain

$$\begin{aligned}
\frac{A-B}{2(\cos\beta-\rho)} &= s_i(\delta_i+\gamma_i-1)-\frac{l}{1-\|z\|^2}\sum_{j=2}^{N_1}|z_j|^2\Big(1-\frac{1}{\varepsilon_j}\Big)-\frac{t}{1-\|w\|^2}\sum_{j=2}^{N_2}|w_j|^2\Big(1-\frac{1}{\eta_j}\Big)\\
&\quad+\frac{2}{1-|z_1|}\sum_{j=2}^{N_1}(\varepsilon_j-1)|P_j(z_j)|\Big[\frac{s_i\delta_i}{l}(1-\|z\|^2)+|z_1|^2+\frac{|z_1|(1-|z_1|)}{2(\cos\beta-\rho)}\\
&\quad+\sum_{j=2}^{N_1}\frac{|z_j|^2}{\varepsilon_j}\Big]\frac{l}{1-\|z\|^2}+\frac{2}{1-|w_1|}\sum_{j=2}^{N_2}(\eta_j-1)|Q_j(w_j)|\Big[\frac{s_i\gamma_i}{t}(1-\|w\|^2)+|w_1|^2\\
&\quad+\frac{|w_1|(1-|w_1|)}{2(\cos\beta-\rho)}+\sum_{j=2}^{N_2}\frac{|w_j|^2}{\eta_j}\Big]\frac{t}{1-\|w\|^2}\\
&\leq s_i(\delta_i+\gamma_i-1)-\frac{l}{1-\|z\|^2}\sum_{j=2}^{N_1}|z_j|^2\Big(1-\frac{1}{\varepsilon_j}\Big)-\frac{t}{1-\|w\|^2}\sum_{j=2}^{N_2}|w_j|^2\Big(1-\frac{1}{\eta_j}\Big)\\
&\quad+\frac{l}{1-\|z\|^2}\frac{1+8a(\cos\beta-\rho)}{4(\cos\beta-\rho)}\frac{1}{1-|z_1|}\sum_{j=2}^{N_1}(\varepsilon_j-1)\|P_j\||z_j|^4\\
&\quad+\frac{t}{1-\|w\|^2}\frac{1+8b(\cos\beta-\rho)}{4(\cos\beta-\rho)}\frac{1}{1-|w_1|}\sum_{j=2}^{N_2}(\eta_j-1)\|Q_j\||w_j|^4\\
&\leq s_i(\delta_i+\gamma_i-1)+\frac{l}{1-\|z\|^2}\sum_{j=2}^{N_1}(\varepsilon_j-1)|z_j|^2\Big[\frac{-1}{\varepsilon_j}+\frac{1+8a(\cos\beta-\rho)}{2(\cos\beta-\rho)}\|P_j\|\Big]\\
&\quad+\frac{t}{1-\|w\|^2}\sum_{j=2}^{N_2}(\eta_j-1)|w_j|^2\Big[\frac{-1}{\eta_j}+\frac{1+8b(\cos\beta-\rho)}{2(\cos\beta-\rho)}\|Q_j\|\Big]\\
&\leq 0
\end{aligned}$$

provided that

$$\|P_j\|\leq\frac{2(\cos\beta-\rho)}{[1+8a(\cos\beta-\rho)]\varepsilon_j},\quad \|Q_j\|\leq\frac{2(\cos\beta-\rho)}{[1+8b(\cos\beta-\rho)]\eta_j}.$$

Therefore (5.1) holds, so $F(\xi,z,w)$ is a parabolic and spirallike mappings of type β and order ρ.

From Theorem 5.1, we draw the following conclusion on the unit ball B^n in $\mathbb{C}^n$.

Corollary 5.1 *Let $f(z_1)$ be a parabolic and spirallike function of type β and order ρ on D with $\rho\in[0,1)$ and $\beta\in\big(-\frac{\pi}{2},\frac{\pi}{2}\big)$. Let*

$$F(z)=\Big(f(z_1)+\frac{f(z_1)}{z_1}\sum_{j=2}^{n}P_j(z_j),\Big(\frac{f(z_1)}{z_1}\Big)^{\frac{1}{\varepsilon_2}}z_2,\cdots,\Big(\frac{f(z_1)}{z_1}\Big)^{\frac{1}{\varepsilon_n}}z_n\Big)'$$

with $\varepsilon_j \geq 4$ $(j = 2, \cdots, n)$ *and* $\left(\frac{f(z_1)}{z_1}\right)^{\frac{1}{\varepsilon_j}}|_{z_1=0} = 1$. *Then* $F(z)$ *is a parabolic and spirallike mappings of type* β *and order* ρ *on* B^n *provided that* $\rho < \cos\beta$ *and*

$$\|P_j\| \leq \frac{2(\cos\beta - \rho)}{[1 + 8(\cos\beta - \rho)]\varepsilon_j}.$$

Remark 5.1 Setting $\rho = 0$ and $\beta = 0$ in Theorem 5.1 and Corollary 5.1, respectively, we get the corresponding results for parabolic and spirallike mappings of type β, parabolic and starlike mappings of order ρ.

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