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C^*-Isomorphisms Associated with Two Projections on a Hilbert C^*-Module*

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Abstract Motivated by two norm equations used to characterize the Friedrichs angle, this paper studies C^*-isomorphisms associated with two projections by introducing the matched triple and the semi-harmonious pair of projections. A triple (P,Q,H) is said to be matched if H is a Hilbert C^*-module, P and Q are projections on H such that their infimum $P\wedge Q$ exists as an element of $\mathcal{L}(H)$, where $\mathcal{L}(H)$ denotes the set of all adjointable operators on H. The C^*-subalgebras of $\mathcal{L}(H)$ generated by elements in $\{P-P\wedge Q, Q-P\wedge Q, I\}$ and $\{P,Q,P\wedge Q,I\}$ are denoted by $i(P,Q,H)$ and $o(P,Q,H)$, respectively. It is proved that each faithful representation (π,X) of $o(P,Q,H)$ can induce a faithful representation $(\widetilde{\pi},X)$ of $i(P,Q,H)$ such that

$$\begin{aligned}\widetilde{\pi}(P-P\wedge Q)&=\pi(P)-\pi(P)\wedge\pi(Q),\\ \widetilde{\pi}(Q-P\wedge Q)&=\pi(Q)-\pi(P)\wedge\pi(Q).\end{aligned}$$

When (P,Q) is semi-harmonious, that is, $\overline{\mathcal{R}(P+Q)}$ and $\overline{\mathcal{R}(2I-P-Q)}$ are both orthogonally complemented in H, it is shown that $i(P,Q,H)$ and $i(I-Q,I-P,H)$ are unitarily equivalent via a unitary operator in $\mathcal{L}(H)$. A counterexample is constructed, which shows that the same may be not true when (P,Q) fails to be semi-harmonious. Likewise, a counterexample is constructed such that (P,Q) is semi-harmonious, whereas $(P,I-Q)$ is not semi-harmonious. Some additional examples indicating new phenomena of adjointable operators acting on Hilbert C^*-modules are also provided.

Keywords Hilbert C^*-module, Projection, Orthogonal complementarity, C^*-Isomorphism
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1 Introduction

Let P and Q be two projections on a Hilbert space X. Their infimum $P\wedge Q$ is the projection from X onto $\mathcal{R}(P)\cap\mathcal{R}(Q)$, which can be obtained by taking the limit of $\{(PQP)^n\}_{n=1}^\infty$ in the strong operator topology (see [9, Lemma 22]). The cosine of the Friedrichs angle (see [4]) between $M=\mathcal{R}(P)$ and $N=\mathcal{R}(Q)$ is denoted by $c(M,N)$, and can be calculated as

$$c(M,N)=\|(P-P\wedge Q)(Q-P\wedge Q)\|.$$

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The characterization of $c(M,N)=c(N^\perp,M^\perp)$ given in [3, Section 2] yields

$$\begin{aligned}&\|(P-P\wedge Q)(Q-P\wedge Q)\|\\ =&\|[I-Q-(I-Q)\wedge(I-P)][I-P-(I-Q)\wedge(I-P)]\|.\end{aligned}\tag{1.1}$$

The Friedrichs angle has also been studied in the setting of C^*-algebras. To deal with the Friedrichs angle associated with two projections P and Q in a C^*-algebra $\mathfrak{A}$, one approach employed in [1] is to embed $\mathfrak{A}$ into its enveloping von Neumann algebra $\mathfrak{A}''$ via the universal representation (π_u,H_u) of $\mathfrak{A}$, and then to use the universal property of $\mathfrak{A}''$ (see [11, Theorem 3.7.7]). By identifying $\mathfrak{A}$ with $\pi_u(\mathfrak{A})$, $P\wedge Q$ can be obtained in $\mathfrak{A}''$, and it is proved in [1, Proposition 2.4] that for every faithful representation (π,X) of $\mathfrak{A}$,

$$\begin{aligned}&\|(P-P\wedge Q)(Q-P\wedge Q)\|\\ =&\|[\pi(P)-\pi(P)\wedge\pi(Q)][\pi(Q)-\pi(P)\wedge\pi(Q)]\|.\end{aligned}\tag{1.2}$$

Hilbert C^*-modules are natural generalizations of Hilbert spaces, and every C^*-algebra can be regarded as a Hilbert C^*-module over itself in a natural way. The purpose of this paper is, in the framework of Hilbert C^*-modules, to give a deeper understanding of (1.1)–(1.2) via algebraic systems rather than on products of finitely many operators.

It is notable that a closed submodule of a Hilbert C^*-module may fail to be orthogonally complemented. In this paper much attention has been paid on this aspect. Let $\mathcal{L}(H)$ be the set of all adjointable operators on a Hilbert C^*-module H. For two projections $P,Q\in\mathcal{L}(H)$, let

$$\mathcal{R}=\mathcal{R}(Q)\cap\mathcal{R}(P)\quad\text{and}\quad\mathcal{N}=\mathcal{N}(Q)\cap\mathcal{N}(P).\tag{1.3}$$

To check the validity of the Halmos' two projections theorem in the Hilbert C^*-module case, the term of the harmonious pair of projections is introduced in [7, Section 4], and it is shown later in [12, Theorem 3.3] that for every pair (P,Q) of projections, the Halmos' two projections theorem is valid if and only if (P,Q) is harmonious. In view of the conditions stated in [7, Lemma 5.4] and [12, Theorem 3.3], we make a definition as follows.

Definition 1.1 *A pair (P,Q) of projections on a Hilbert C^*-module H is said to be semi-harmonious if both $\overline{\mathcal{R}(P+Q)}$ and $\overline{\mathcal{R}(2I-P-Q)}$ are orthogonally complemented in H. If (P,Q) and $(P,I-Q)$ are both semi-harmonious, then (P,Q) is said to be harmonious.*

Let $\mathcal{R}$ and $\mathcal{N}$ be defined by (1.3) for projections P and Q on a Hilbert C^*-module H. Since $\overline{\mathcal{R}(2I-P-Q)}^\perp=\mathcal{R}$ and $\overline{\mathcal{R}(P+Q)}^\perp=\mathcal{N}$, a condition weaker than the semi-harmony of (P,Q) turns out to be the orthogonal complementarity of $\mathcal{R}$ and $\mathcal{N}$, which is necessary and sufficient to make use of the notations $P_\mathcal{R}$ and $P_\mathcal{N}$ (the projections from H onto $\mathcal{R}$ and $\mathcal{N}$, respectively). The example constructed in [7, Section 3] (see also the proof of Theorem 2.2) shows that there exist projections P and Q on certain Hilbert C^*-module such that $\mathcal{R}=\mathcal{N}=\{0\}$, whereas (P,Q) is not semi-harmonious. So, generally the meaningfulness of $P_\mathcal{R}$ and $P_\mathcal{N}$ does not imply the semi-harmony of (P,Q).

Next, we introduce the matched triple as follows.

Definition 1.2 *A triple (P,Q,H) is said to be matched if H is a Hilbert C^*-module, P and Q are projections on H such that $\mathcal{R}$ defined by (1.3) is orthogonally complemented in H. In this case, the projection $P_\mathcal{R}$ is denoted by $P\wedge Q$. The C^*-subalgebras of $\mathcal{L}(H)$ generated by elements in $\{P-P\wedge Q,Q-P\wedge Q,I\}$ and $\{P,Q,P\wedge Q,I\}$ are denoted by $i(P,Q,H)$ and $o(P,Q,H)$, and are called the inner algebra and the outer algebra, respectively.*

Definition 1.3 *Two matched triple (P_i, Q_i, H_i) $(i = 1, 2)$ are said to be innerly unitarily equivalent if there exists a unitary operator $U : H_1 \to H_2$ such that*

$$U(P_1 - P_1 \wedge Q_1)U^* = P_2 - P_2 \wedge Q_2,$$
$$U(Q_1 - P_1 \wedge Q_1)U^* = Q_2 - P_2 \wedge Q_2.$$

Recall that a pair (π, X) is said to be a representation of a C^*-algebra $\mathfrak{A}$ if X is a Hilbert space and $\pi : \mathfrak{A} \to \mathbb{B}(X)$ is a C^*-morphism, where $\mathbb{B}(X)$ denotes the set of all bounded linear operators on X.

Definition 1.4 *Let (P, Q, H) be a matched triple. A representation (π, X) of $o(P, Q, H)$ is called an outer representation of (P, Q, H). If furthermore a C^*-morphism $\widetilde{\pi} : i(P, Q, H) \to \mathbb{B}(X)$ can be induced such that $\widetilde{\pi}(I) = \pi(I)$, and*

$$\widetilde{\pi}(P - P \wedge Q) = \pi(P) - \pi(P) \wedge \pi(Q),$$
$$\widetilde{\pi}(Q - P \wedge Q) = \pi(Q) - \pi(P) \wedge \pi(Q),$$

then (π, X) is called an inner-outer representation of (P, Q, H). When both π and $\widetilde{\pi}$ are faithful, (π, X) is called a faithful inner-outer representation of (P, Q, H).

Remark 1.1 Let (π, X) be an inner-outer representation of (P, Q, H). It is notable that generally $\pi(P) \wedge \pi(Q)$ is taken in the von Neumann algebra $[\pi[o(P, Q, H)]]''$ rather than in the C^*-algebra $\pi[o(P, Q, H)]$, so it may happen that $\pi(P \wedge Q) \neq \pi(P) \wedge \pi(Q)$.

With the terms given as above, we list the main results of this paper as follows:

(1) There exist projections P and Q on certain Hilbert C^*-module H such that (P, Q) is semi-harmonious, whereas it fails to be harmonious (see Theorem 2.2).

(2) For every semi-harmonious pair (P, Q) of projections on a Hilbert C^*-module H, the matched triples (P, Q, H) and $(I - Q, I - P, H)$ are innerly unitarily equivalent (see Theorem 3.1).

(3) There exist projections P and Q on certain Hilbert C^*-module H such that the triples (P, Q, H) and $(I - Q, I - P, H)$ are both matched, whereas they are not innerly unitarily equivalent (see Theorem 3.2).

(4) Every faithful outer representation of a matched triple is a faithful inner-outer representation (see Theorem 4.1).

An application of Theorems 3.1 and 4.1 will be illustrated in Corollary 4.3. Another application, as has been mentioned earlier, concerns a new insight into (1.1)–(1.2). Let P and Q be two projections on a Hilbert C^*-module H such that $\mathcal{R}$ and $\mathcal{N}$ defined by (1.3) are orthogonally complemented in H. By Theorem 4.1, we will see that each faithful unital representation (π, X) of $\mathcal{L}(H)$ can induce unital C^*-isomorphisms $\widetilde{\pi}_1 : i(P, Q, H) \to i\big(\pi(P), \pi(Q), X\big)$ and $\widetilde{\pi}_2 : i(I - Q, I - P, H) \to i\big(I - \pi(Q), I - \pi(P), X\big)$ such that

$$\widetilde{\pi}_1(P - P \wedge Q) = \pi(P) - \pi(P) \wedge \pi(Q),$$
$$\widetilde{\pi}_1(Q - P \wedge Q) = \pi(Q) - \pi(P) \wedge \pi(Q),$$
$$\widetilde{\pi}_2\big[I - Q - (I - Q) \wedge (I - P)\big] = I - \pi(Q) - \pi(I - Q) \wedge \pi(I - P),$$
$$\widetilde{\pi}_2\big[I - P - (I - Q) \wedge (I - P)\big] = I - \pi(P) - \pi(I - Q) \wedge \pi(I - P).$$

Thus, $\|\widetilde{\pi}_1(x)\| = \|x\|$ for every $x \in i(P, Q, H)$. Specifically, if we put $x = (P - P \wedge Q)(Q - P \wedge Q)$, then (1.2) is obtained.

Note that $(\pi(P), \pi(Q))$ is a pair of projections acting on a Hilbert space, so it is harmonious. Hence, by Theorem 3.1 there exists a unitary operator $U \in \mathbb{B}(X)$ such that

$$U[\pi(P) - \pi(P) \wedge \pi(Q)]U^* = I - \pi(Q) - \pi(I-Q) \wedge \pi(I-P),$$
$$U[\pi(Q) - \pi(P) \wedge \pi(Q)]U^* = I - \pi(P) - \pi(I-Q) \wedge \pi(I-P).$$

Thus, a C^*-isomorphism $\rho : i(P, Q, H) \to i(I-Q, I-P, H)$ can be constructed as

$$\rho(x) = (\widetilde{\pi}_2)^{-1} U \widetilde{\pi}_1(x) U^*, \quad \forall x \in i(P, Q, H).$$

Therefore, $\|\rho(x)\| = \|x\|$ for every $x \in i(P, Q, H)$. Likewise, if we take $x = (P - P \wedge Q)(Q - P \wedge Q)$, then (1.1) is obtained. So, a substantive generalization of (1.1)–(1.2) has been made.

The paper is organized as follows. The main purpose of Section 2 is to construct two projections P and Q such that (P, Q) is semi-harmonious, whereas it fails to be harmonious. Section 3 focuses on the construction of the unitary operator U satisfying (3.3)–(3.4). Section 4 is devoted to the study of the faithful inner-outer representation of a matched triple.

2 Semi-Harmonious Pairs of Projections

Throughout the rest of this paper, $\mathbb{N}$, $\mathbb{Z}_+$ and $\mathbb{C}$ are the sets of all positive integers, non-negative integers and complex numbers, respectively. Unless otherwise specified, $\mathfrak{A}$ is a C^*-algebra, E, H and K are Hilbert $\mathfrak{A}$-modules (see [5, 10]). The set of all adjointable operators from H to K is denoted by $\mathcal{L}(H, K)$. Given $A \in \mathcal{L}(H, K)$, the adjoint operator, the range and the null space of A are denoted by A^*, $\mathcal{R}(A)$ and $\mathcal{N}(A)$, respectively. Let $|A|$ designate the square root of A^*A. In case $H = K$, $\mathcal{L}(H, K)$ is abbreviated to $\mathcal{L}(H)$, whose subset consisting of all positive elements is denoted by $\mathcal{L}(H)_+$. The unit of $\mathcal{L}(H)$ (namely, the identity operator on H) is denoted by I_H, or simply by I when no ambiguity arises. An operator $P \in \mathcal{L}(H)$ is said to be a projection if $P = P^* = P^2$. The set of all projections on H is denoted by $\mathcal{P}(H)$.

Let M be a closed submodule of H. Clearly, there exists at most a projection in $\mathcal{L}(H)$, written P_M, such that $\mathcal{R}(P_M) = M$. It can be easily verified that P_M exists if and only if $H = M + M^\perp$, where

$$M^\perp = \{x \in H : \langle x, y\rangle = 0, \forall y \in M\}.$$

In this case, M is said to be orthogonally complemented in H.

To construct semi-harmonious pairs of projections, we need a couple of lemmas.

Lemma 2.1 (see [6, Proposition 2.9]) *For every $T \in \mathcal{L}(H)_+$ and $\alpha > 0$, we have $\overline{\mathcal{R}(T)} = \overline{\mathcal{R}(T^\alpha)}$.*

Lemma 2.2 (see [5, Proposition 3.7]) *For every $T \in \mathcal{L}(H, K)$, we have $\overline{\mathcal{R}(T)} = \overline{\mathcal{R}(TT^*)}$.*

Lemma 2.3 (see [6, Proposition 2.7]) *Let $B, C \in \mathcal{L}(E, H)$ be such that $\overline{\mathcal{R}(B)} = \overline{\mathcal{R}(C)}$. Then for every $A \in \mathcal{L}(H, K)$, we have $\overline{\mathcal{R}(AB)} = \overline{\mathcal{R}(AC)}$.*

An approach to construct semi-harmonious pairs of projections reads as follows.

Theorem 2.1 *For every $P, Q \in \mathcal{P}(H)$, let $M \subseteq H$ be defined by*

$$M = \overline{\mathcal{R}\big[(P+Q)(2I - P - Q)\big]}. \tag{2.1}$$

Then $P|_M$ and $Q|_M$ are projections on M such that $(P|_M, Q|_M)$ is semi-harmonious, where $P|_M$ and $Q|_M$ are the restrictions of P and Q on M, respectively.

Proof To simplify the notation, we put

$$A = P + Q \quad \text{and} \quad G = \mathcal{R}\big[A(2I - A)\big]. \tag{2.2}$$

Since G defined as above is the range of an adjointable operator, its closure M is a Hilbert $\mathfrak{A}$-module.

For every $T \in \mathcal{L}(H)$ and $X \subseteq H$, let $TX = \big\{Tx : x \in X\big\}$. Then $TX = \mathcal{R}(T|_X)$, and the boundedness of T gives $\overline{TX} = \overline{T\overline{X}}$. Hence

$$\overline{AM} = \overline{AG} \quad \text{and} \quad \overline{(2I - A)M} = \overline{(2I - A)G}. \tag{2.3}$$

Since A and $2I - A$ are positive and commutative, we may combine (2.2)–(2.3) with Lemmas 2.1 and 2.3 to get

$$\begin{aligned} \overline{\mathcal{R}(P|_M + Q|_M)} &= \overline{AM} = \overline{AG} = \overline{\mathcal{R}\big[(2I - A)A^2\big]} \\ &= \overline{\mathcal{R}\big[(2I - A)A\big]} = M. \end{aligned} \tag{2.4}$$

Similarly,

$$\begin{aligned} \overline{\mathcal{R}(2I_M - P|_M - Q|_M)} &= \overline{(2I - A)M} = \overline{(2I - A)G} = \overline{\mathcal{R}\big[A(2I - A)^2\big]} \\ &= \overline{\mathcal{R}\big[A(2I - A)\big]} = M. \end{aligned} \tag{2.5}$$

Furthermore, direct computations yield

$$\begin{aligned} P(2I - A)A &= P(I - Q)P = A(2I - A)P, \\ Q(2I - A)A &= Q(I - P)Q = A(2I - A)Q, \end{aligned}$$

which lead clearly to $PM \subseteq M$ and $QM \subseteq M$. Consequently, $P|_M$ and $Q|_M$ are projections on M. In view of (2.4)–(2.5), we conclude that $(P|_M, Q|_M)$ is semi-harmonious.

Theorem 2.2 *There exist projections P and Q on certain Hilbert C^*-module such that (P, Q) is semi-harmonious, whereas it fails to be harmonious.*

Proof We follow the line initiated in [8, Section 3] and modified in [7, Section 3]. Let $M_2(\mathbb{C})$ and I_2 be the set of all 2×2 complex matrices and the identity matrix in $M_2(\mathbb{C})$, respectively. Denote by $\|\cdot\|$ the operator norm on $M_2(\mathbb{C})$. Let $\mathfrak{A} = C([0, 1]; M_2(\mathbb{C}))$ be the set of all continuous matrix-valued functions from $[0, 1]$ to $M_2(\mathbb{C})$. For $x \in \mathfrak{A}$ and $t \in [0, 1]$, we put

$$x^*(t) = (x(t))^* \quad \text{and} \quad \|x\| = \max_{0 \le s \le 1} \|x(s)\|.$$

With the $*$-operation as above and the usual algebraic operations, $\mathfrak{A}$ is a unital C^*-algebra, which is also a Hilbert $\mathfrak{A}$-module with the inner-product given by

$$\langle x, y \rangle = x^* y \quad \text{for } x, y \in \mathfrak{A}.$$

Let e be the unit of $\mathfrak{A}$, that is, $e(t) = I_2$ for every $t \in [0, 1]$. It is known that $\mathfrak{A} \cong \mathcal{L}(\mathfrak{A})$ via $a \to L_a$ (see [7, Section 3]), where $L_a(x) = ax$ for $a, x \in \mathfrak{A}$. For simplicity, we identify $\mathcal{L}(\mathfrak{A})$ with $\mathfrak{A}$ and set

$$c_t = \cos \frac{\pi}{2} t \quad \text{and} \quad s_t = \sin \frac{\pi}{2} t \quad \text{for } t \in [0, 1]. \tag{2.6}$$

Let $P, Q \in \mathfrak{A}$ be projections determined by the matrix-valued functions

$$P(t) \equiv \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad Q(t) = \begin{pmatrix} c_t^2 & s_t c_t \\ s_t c_t & s_t^2 \end{pmatrix} \quad \text{for } t \in [0,1]. \tag{2.7}$$

It is shown in [7, Section 3] that

$$\mathcal{R}(P) \cap \mathcal{R}(Q) = \mathcal{R}(P) \cap \mathcal{N}(Q) = \mathcal{N}(P) \cap \mathcal{R}(Q) = \mathcal{N}(P) \cap \mathcal{N}(Q) = \{0\}. \tag{2.8}$$

Now, let $H = \mathfrak{A}$ and M be defined by (2.1). According to Theorem 2.1, $(P|_M, Q|_M)$ is semi-harmonious. In what follows, we prove that $(P|_M, Q|_M)$ is not harmonious.

Direct computation yields

$$(P+Q)(2I-P-Q) = P+Q-PQ-QP = (P-Q)(P-Q)^*,$$

which leads by (2.1) and Lemma 2.2 to

$$M = \overline{\mathcal{R}(P-Q)}. \tag{2.9}$$

Utilizing (2.7) we obtain $P-Q = au$, where $a, u \in H$ are determined by

$$a(t) = \begin{pmatrix} s_t & 0 \\ 0 & s_t \end{pmatrix}, \quad u(t) = \begin{pmatrix} s_t & -c_t \\ -c_t & -s_t \end{pmatrix}, \quad t \in [0,1].$$

Since u is a unitary and M can be represented by (2.9), we have $M = \overline{\mathcal{R}(a)}$. Specifically,

$$a = ae \in M, \quad \overline{\mathcal{R}(P|_M + I_M - Q|_M)} = \overline{\mathcal{R}(T)},$$

where $T = (P+I-Q)a \in H$. For any $x \in H$ with $x(t) = (x_{ij}(t))_{1\le i,j\le 2}$, it is easy to verify that

$$(Tx)(1) = \begin{pmatrix} 2x_{11}(1) & 2x_{12}(1) \\ 0 & 0 \end{pmatrix},$$

hence

$$\|Tx - a\| \ge \|(Tx)(1) - a(1)\| = \left\| \begin{pmatrix} 2x_{11}(1) - 1 & 2x_{12}(1) \\ 0 & -1 \end{pmatrix} \right\| \ge 1,$$

which implies that $a \notin \overline{\mathcal{R}(T)}$. Furthermore, by (2.8) we have

$$\overline{\mathcal{R}(P|_M + I_M - Q|_M)}^{\perp} = \mathcal{N}(P|_M) \cap \mathcal{R}(Q|_M) \subseteq \mathcal{N}(P) \cap \mathcal{R}(Q) = \{0\}.$$

This shows

$$a \notin \overline{\mathcal{R}(P|_M + I_M - Q|_M)} + \overline{\mathcal{R}(P|_M + I_M - Q|_M)}^{\perp},$$

whereas $a \in M$. So $\overline{\mathcal{R}(P|_M + I_M - Q|_M)}$ is not orthogonally complemented in M.

Remark 2.1 It is notable that there exist projections P and Q such that $\overline{\mathcal{R}(P+Q)}$ is orthogonally complemented, whereas $\overline{\mathcal{R}(2I-P-Q)}$ fails to be orthogonally complemented. We provide such an example as follows.

Example 2.1 Let $\mathfrak{A} = H = C\big([0,1]; M_2(\mathbb{C})\big)$ and $P, Q \in \mathcal{P}(H)$ be as in the proof of Theorem 2.2. Put $H_0 = \overline{\mathcal{R}(P+Q)}$, $P_0 = P|_{H_0}$ and $Q_0 = Q|_{H_0}$. From [7, Lemma 2.3] we have

$H_0 = \overline{\mathcal{R}(P)+\mathcal{R}(Q)}$, which means that $\mathcal{R}(P)\subseteq H_0$, hence $P_0H_0\subseteq H_0$. Consequently, P_0 is a projection on H_0. Similarly, we have $Q_0\in\mathcal{P}(H_0)$. In view of (2.8), we get

$$\mathcal{R}(P_0)\cap\mathcal{R}(Q_0)=\mathcal{N}(P_0)\cap\mathcal{N}(Q_0)=\{0\}. \tag{2.10}$$

According to Lemma 2.1, we have

$$H_0=\overline{\mathcal{R}\big[(P+Q)^2\big]}\subseteq\overline{\mathcal{R}(P_0+Q_0)}\subseteq H_0.$$

As a result, we arrive at

$$H_0=\overline{\mathcal{R}(P_0+Q_0)}=\overline{\mathcal{R}(P_0+Q_0)}+\mathcal{N}(P_0)\cap\mathcal{N}(Q_0).$$

This shows the orthogonal complementarity of $\overline{\mathcal{R}(P_0+Q_0)}$ in H_0.

Let $F=(2I-P-Q)(P+Q)$. Clearly, $F=(I-P)Q+(I-Q)P$, so by (2.7) we have

$$F(t)=\begin{pmatrix} s_t^2 & \\ & s_t^2\end{pmatrix},\quad \forall t\in[0,1],$$

which implies that for every $x\in\mathfrak{A}$, $(Fx)(0)=F(0)x(0)=\begin{pmatrix} 0 & \\ & 0\end{pmatrix}$. Hence

$$\|P-Fx\|\geq\|P(0)-F(0)x(0)\|=\left\|\begin{pmatrix} 1 & 0\\ 0 & 0\end{pmatrix}\right\|=1.$$

Due to the definition of F and the observation of (2.10), we conclude that

$$P\notin\overline{\mathcal{R}(F)}=\overline{\mathcal{R}(2I_{H_0}-P_0-Q_0)}+\mathcal{R}(P_0)\cap\mathcal{R}(Q_0).$$

On the other hand, $P=PI\in\mathcal{R}(P)\subseteq H_0$. Therefore, $\overline{\mathcal{R}(2I_{H_0}-P_0-Q_0)}$ is not orthogonally complemented in H_0.

3 Unitary Equivalences Associated with Two Projections

In this section, we deal with unitary equivalences associated with two projections. We begin with a known result as follows.

Lemma 3.1 (see [7, Lemma 4.1]) *Let $P,Q\in\mathcal{P}(H)$ be such that $\overline{\mathcal{R}(I-Q+P)}$ is orthogonally complemented in H. Then $\overline{\mathcal{R}(QP)}$ is also orthogonally complemented in H such that $P_{\overline{\mathcal{R}(QP)}}=Q-P_{\mathcal{R}(Q)\cap\mathcal{N}(P)}$.*

Next, we provide a useful lemma as follows.

Lemma 3.2 *For every $P,Q\in\mathcal{P}(H)$, we have*

$$|P(I-Q)|+|(I-P)Q|=|(I-Q)P|+|Q(I-P)|. \tag{3.1}$$

Proof For simplicity, we put

$$T_1=P(I-Q)\quad\text{and}\quad T_2=(I-P)Q. \tag{3.2}$$

It is clear that

$$T_1^*T_1=(I-Q)T_1^*T_1(I-Q)\quad\text{and}\quad T_2^*T_2=QT_2^*T_2Q,$$

so

$$|T_1|=(I-Q)|T_1|(I-Q)\quad\text{and}\quad |T_2|=Q|T_2|Q,$$

hence $|T_1|\cdot|T_2|=0$. Similarly, $|T_1^*|\cdot|T_2^*|=0$. As a result,

$$\begin{aligned}\big(|T_1|+|T_2|\big)^2&=T_1^*T_1+T_2^*T_2=P+Q-PQ-QP=T_1T_1^*+T_2T_2^*\\&=\big(|T_1^*|+|T_2^*|\big)^2,\end{aligned}$$

which gives (3.1) by taking the square roots of positive operators.

Now, we are in the position to provide the main result of this section.

Theorem 3.1 *Let $P,Q\in\mathcal{P}(H)$ be such that (P,Q) is semi-harmonious. Then there exists a unitary $U\in\mathcal{L}(H)$ such that*

$$U(Q-P_{\mathcal{R}})U^*=I-P-P_{\mathcal{N}},\tag{3.3}$$
$$U(P-P_{\mathcal{R}})U^*=I-Q-P_{\mathcal{N}},\tag{3.4}$$

where $\mathcal{R}$ and $\mathcal{N}$ are defined by (1.3).

Proof Let T_1 and T_2 be defined by (3.2). Since $\overline{\mathcal{R}(P+Q)}$ is orthogonally complemented in H, by Lemma 3.1 both $\overline{\mathcal{R}(T_1^*)}$ and $\overline{\mathcal{R}(T_2)}$ are orthogonally complemented in H. Similarly, the orthogonal complementarity of $\overline{\mathcal{R}(2I-P-Q)}$ leads to that of $\overline{\mathcal{R}(T_1)}$ and $\overline{\mathcal{R}(T_2^*)}$. So for $i=1,2$, the notations $P_{\overline{\mathcal{R}(T_i)}}$ and $P_{\overline{\mathcal{R}(T_i^*)}}$ are meaningful. The point is, these projections can be used to obtain the canonical forms of T_1 and T_2 (see [2]). In fact, in view of (3.2) we have

$$\overline{\mathcal{R}(T_1)}\subseteq\mathcal{R}(P)\quad\text{and}\quad\overline{\mathcal{R}(T_1^*)}\subseteq\mathcal{R}(I-Q),$$

hence

$$P_{\overline{\mathcal{R}(T_1)}}P=P_{\overline{\mathcal{R}(T_1)}}\quad\text{and}\quad(I-Q)P_{\overline{\mathcal{R}(T_1^*)}}=P_{\overline{\mathcal{R}(T_1^*)}},$$

which lead to

$$T_1=P_{\overline{\mathcal{R}(T_1)}}T_1P_{\overline{\mathcal{R}(T_1^*)}}=P_{\overline{\mathcal{R}(T_1)}}P_{\overline{\mathcal{R}(T_1^*)}}.\tag{3.5}$$

Replacing P and Q with $I-P$ and $I-Q$, respectively, we obtain

$$T_2=P_{\overline{\mathcal{R}(T_2)}}P_{\overline{\mathcal{R}(T_2^*)}}.\tag{3.6}$$

Taking $*$-operation, from (3.5)–(3.6) we arrive at

$$T_1^*=P_{\overline{\mathcal{R}(T_1^*)}}P_{\overline{\mathcal{R}(T_1)}}\quad\text{and}\quad T_2^*=P_{\overline{\mathcal{R}(T_2^*)}}P_{\overline{\mathcal{R}(T_2)}}.\tag{3.7}$$

To make use of (3.1), we need the polar decompositions of T_1 and T_2. For $i=1,2$, as $\overline{\mathcal{R}(T_i)}$ and $\overline{\mathcal{R}(T_i^*)}$ are both orthogonally complemented in H, by [6, Lemma 3.6 and Theorem 3.8] there exists a unique partial isometry $V_i\in\mathcal{L}(H)$ such that

$$T_i=V_i|T_i|,\quad T_i^*=V_i^*|T_i^*|,\quad V_i^*V_i=P_{\overline{\mathcal{R}(T_i^*)}},\quad V_iV_i^*=P_{\overline{\mathcal{R}(T_i)}}.\tag{3.8}$$

Combining the last two equations in (3.8) with (3.5)–(3.7), we obtain

$$T_i=V_iV_i^*V_i^*V_i\quad\text{and}\quad T_i^*=V_i^*V_iV_iV_i^*.$$

These two equations together with (3.8), Lemmas 2.1–2.2 yield

$$(V_i^*)^2V_i = V_i^*(V_iV_i^*V_i^*V_i) = V_i^*T_i = V_i^*V_i|T_i| = |T_i|,$$

which gives

$$(V_i^*)^2V_i = |T_i| = V_i^*V_i^2$$

by taking $*$-operation. Similarly, we have

$$V_i^2V_i^* = |T_i^*| = V_i(V_i^*)^2.$$

Consequently, (3.1) turns out to be

$$V_1^*V_1^2 + V_2^*V_2^2 = V_1^2V_1^* + V_2^2V_2^*. \tag{3.9}$$

Now, we are ready to construct the desired unitary operator. Let

$$U_1 = V_1 + P_{\mathcal{R}}, \quad U_2 = V_2 + P_{\mathcal{N}}, \quad U = U_1 - U_2, \tag{3.10}$$

where $\mathcal{R}$ and $\mathcal{N}$ are defined by (1.3). Then by Lemma 3.1, (3.2) and (3.8), we have

$$P = V_1V_1^* + P_{\mathcal{R}}, \quad I - P = V_2V_2^* + P_{\mathcal{N}}, \tag{3.11}$$

$$Q = V_2^*V_2 + P_{\mathcal{R}}, \quad I - Q = V_1^*V_1 + P_{\mathcal{N}}. \tag{3.12}$$

It follows from (3.11) that

$$V_1V_1^*P_{\mathcal{R}} = V_2V_2^*P_{\mathcal{N}} = V_1V_1^*V_2V_2^* = V_1V_1^*P_{\mathcal{N}} = P_{\mathcal{R}}V_2V_2^* = P_{\mathcal{R}}P_{\mathcal{N}} = 0,$$

or equivalently,

$$V_1^*P_{\mathcal{R}} = V_2^*P_{\mathcal{N}} = V_1^*V_2 = V_1^*P_{\mathcal{N}} = P_{\mathcal{R}}V_2 = P_{\mathcal{R}}P_{\mathcal{N}} = 0. \tag{3.13}$$

Similarly, it can be inferred from (3.12) that

$$V_2P_{\mathcal{R}} = V_1P_{\mathcal{N}} = V_2V_1^* = V_2P_{\mathcal{N}} = P_{\mathcal{R}}V_1^* = 0. \tag{3.14}$$

It follows from (3.10)–(3.14) that

$$U_1U_2^* = U_1^*U_2 = 0, \quad U_1U_1^* = P, \quad U_2U_2^* = I - P,$$
$$U_1^*U_1 + U_2^*U_2 = V_1^*V_1 + P_{\mathcal{R}} + V_2^*V_2 + P_{\mathcal{N}} = Q + I - Q = I.$$

Therefore $UU^* = U^*U = I$, so the operator U defined by (3.10) is a unitary.

Finally, we check the validity of (3.3)–(3.4). According to (3.11)–(3.12), we have

$$Q - P_{\mathcal{R}} = V_2^*V_2, \quad I - P - P_{\mathcal{N}} = V_2V_2^*,$$
$$P - P_{\mathcal{R}} = V_1V_1^*, \quad I - Q - P_{\mathcal{N}} = V_1^*V_1,$$

and thus

$$V_1V_1^* + V_2V_2^* = I - P_{\mathcal{R}} - P_{\mathcal{N}} = V_1^*V_1 + V_2^*V_2.$$

The above equations together with (3.9)–(3.10) and (3.13)–(3.14) yield

$$U(Q - P_{\mathcal{R}}) = (U_1 - U_2)V_2^*V_2 = (V_1 + P_{\mathcal{R}} - V_2 - P_{\mathcal{N}})V_2^*V_2 = -V_2$$

$$\begin{aligned}
&= V_2V_2^*(V_1+P_{\mathcal{R}}-V_2-P_{\mathcal{N}}) = (I-P-P_{\mathcal{N}})U,\\
U(P-P_{\mathcal{R}}) &= (V_1+P_{\mathcal{R}}-V_2-P_{\mathcal{N}})V_1V_1^* = V_1^2V_1^*-V_2V_1V_1^*\\
&= V_1^2V_1^*-V_2(I-P_{\mathcal{R}}-P_{\mathcal{N}}-V_2V_2^*) = V_1^2V_1^*+V_2^2V_2^*-V_2\\
&= V_1^*V_1^2+V_2^*V_2^2-V_2 = V_1^*V_1^2+(I-P_{\mathcal{R}}-P_{\mathcal{N}}-V_1^*V_1)V_2-V_2\\
&= V_1^*V_1^2-V_1^*V_1V_2 = V_1^*V_1(V_1+P_{\mathcal{R}}-V_2-P_{\mathcal{N}})\\
&= (I-Q-P_{\mathcal{N}})U.
\end{aligned}$$

Therefore, (3.3)–(3.4) are satisfied.

Remark 3.1 Let $P,Q\in\mathcal{P}(H)$ be such that (P,Q) is harmonious. In this case, the Halmos' two projections theorem (see [12, Theorem 3.3]) indicates that up to unitary equivalence, P and Q have the block matrix forms

$$P=\begin{pmatrix} I_{H_1} &&&&& \\ & I_{H_2} &&&& \\ && 0 &&& \\ &&& 0 && \\ &&&& I_{H_5} & \\ &&&&& 0\end{pmatrix},\quad Q=\begin{pmatrix} I_{H_1} &&&& \\ & 0 &&& \\ && I_{H_3} && \\ &&& 0 & \\ &&&& Q_0\end{pmatrix},$$

where

$$\begin{aligned}
&H_1=\mathcal{R},\quad H_2=\mathcal{R}(P)\cap\mathcal{N}(Q),\quad H_3=\mathcal{N}(P)\cap\mathcal{R}(Q),\quad H_4=\mathcal{N},\\
&H_5=\mathcal{R}(P)\ominus(H_1\oplus H_2),\quad H_6=\mathcal{N}(P)\ominus(H_3\oplus H_4),\\
&Q_0=\begin{pmatrix} A & A^{\frac12}\big(I_{H_5}-A\big)^{\frac12}U_0^* \\ U_0A^{\frac12}\big(I_{H_5}-A\big)^{\frac12} & U_0\big(I_{H_5}-A\big)U_0^*\end{pmatrix}\in\mathcal{L}(H_5\oplus H_6),
\end{aligned}\tag{3.15}$$

in which $U_0\in\mathcal{L}(H_5,H_6)$ is a unitary, $A\in\mathcal{L}(H_5)$ is a positive contraction such that both A and $I_{H_5}-A$ are injective and $\overline{\mathcal{R}(A-A^2)}=H_5$, which implies that

$$\overline{\mathcal{R}(A)}=\overline{\mathcal{R}(I_{H_5}-A)}=H_5.$$

With the notations as above and that in the proof of Theorem 3.1, we have

$$\begin{aligned}
&P_{\mathcal{R}}=I_{H_1}\oplus 0\oplus 0\oplus 0\oplus 0\oplus 0,\\
&P_{\mathcal{N}}=0\oplus 0\oplus 0\oplus I_{H_4}\oplus 0\oplus 0,\\
&T_1=0\oplus I_{H_2}\oplus 0\oplus 0\oplus S_1,\quad V_1=0\oplus I_{H_2}\oplus 0\oplus 0\oplus S_2,\\
&T_2=0\oplus 0\oplus I_{H_3}\oplus 0\oplus S_3,\quad V_2=0\oplus 0\oplus I_{H_3}\oplus 0\oplus S_4,
\end{aligned}$$

where

$$\begin{aligned}
S_1&=\begin{pmatrix} I_{H_5}-A & -A^{\frac12}\big(I_{H_5}-A\big)^{\frac12}U_0^* \\ 0 & 0\end{pmatrix},\\
S_2&=\begin{pmatrix} \big(I_{H_5}-A\big)^{\frac12} & -A^{\frac12}U_0^* \\ 0 & 0\end{pmatrix},
\end{aligned}$$

$$S_3 = \begin{pmatrix} 0 & 0 \\ U_0 A^{\frac{1}{2}}\big(I_{H_5} - A\big)^{\frac{1}{2}} & U_0\big(I_{H_5} - A\big)U_0^* \end{pmatrix},$$
$$S_4 = \begin{pmatrix} 0 & 0 \\ U_0 A^{\frac{1}{2}} & U_0\big(I_{H_5} - A\big)^{\frac{1}{2}}U_0^* \end{pmatrix}.$$

In virtue of (3.10), we have

$$U = V_1 + P_{\mathcal{R}} - V_2 - P_{\mathcal{N}} = \operatorname{diag}(X, Y),$$

where $X = \operatorname{diag}(I_{H_1}, I_{H_2}, -I_{H_3}, -I_{H_4})$ and

$$Y = S_2 - S_4 = \begin{pmatrix} \big(I_{H_5} - A\big)^{\frac{1}{2}} & -A^{\frac{1}{2}}U_0^* \\ -U_0 A^{\frac{1}{2}} & -U_0\big(I_{H_5} - A\big)^{\frac{1}{2}}U_0^* \end{pmatrix}.$$

This gives the block matrix form of the unitary operator U satisfying (3.3)–(3.4).

Remark 3.2 Since every closed linear subspace of a Hilbert space is orthogonally complemented, Theorem 3.1 is therefore always applicable to every pair of projections on a Hilbert space.

Theorem 3.2 *There exist projections P and Q on certain Hilbert C^*-module such that $\mathcal{R} = \mathcal{N} = \{0\}$, whereas* (3.3)–(3.4) *have no common unitary operator solution.*

Proof Following [8, Section 3], we put $\mathfrak{B} = C\big([0,1]; M_2(\mathbb{C})\big)$ and set

$$\mathfrak{A} = \{f \in \mathfrak{B} : f(0) \text{ and } f(1) \text{ are both diagonal}\}. \tag{3.16}$$

As is shown in the proof of Theorem 2.2, $\mathfrak{A}$ itself is a Hilbert $\mathfrak{A}$-module, and we can identify $\mathcal{L}(\mathfrak{A})$ with $\mathfrak{A}$. Let $H = \mathfrak{A}$ and $Q \in \mathcal{P}(H)$ be determined by (2.7), and let $P \in \mathcal{P}(H)$ be changed to

$$P(t) = \begin{pmatrix} c_t^2 & -s_t c_t \\ -s_t c_t & s_t^2 \end{pmatrix} \quad \text{for } t \in [0,1],$$

where c_t and s_t are defined by (2.6).

Let $\mathcal{R}$ and $\mathcal{N}$ be defined by (1.3), and suppose that $x \in \mathcal{N}$ is determined by $x(t) = (x_{ij}(t))_{1\le i,j\le 2}$ for $t \in [0,1]$. Utilizing

$$P(t) + Q(t) = \begin{pmatrix} 2c_t^2 & 0 \\ 0 & 2s_t^2 \end{pmatrix} \quad \text{and} \quad [P(t) + Q(t)]x(t) = 0 \tag{3.17}$$

for $t \in [0,1]$, we obtain $x_{ij}(t) = 0$ for $i, j \in \{1,2\}$ and every $t \in (0,1)$, which imply that $x_{ij} = 0$ for $1 \le i, j \le 2$, since all functions considered are continuous on $[0,1]$. This shows that $\mathcal{N} = \{0\}$. In view of $\mathcal{R} = \mathcal{N}(I - P) \cap \mathcal{N}(I - Q)$, the proof of $\mathcal{R} = \{0\}$ is similar.

Suppose that U determined by $U(t) = (U_{ij}(t))_{1\le i,j\le 2}$ is a unitary in H, which satisfies both (3.3) and (3.4). Due to $P_{\mathcal{R}} = 0$ and $P_{\mathcal{N}} = 0$, (3.3)–(3.4) are simplified as

$$UP = (I - Q)U \quad \text{and} \quad UQ = (I - P)U,$$

or equivalently,

$$U(Q - P) = (Q - P)U \quad \text{and} \quad U(P + Q) = (2I - P - Q)U. \tag{3.18}$$

Substituting

$$(Q-P)(t)=\begin{pmatrix} 0 & 2s_tc_t \\ 2s_tc_t & 0 \end{pmatrix}, \quad t\in[0,1]$$

into the first equation in (3.18) yields

$$U_{12}(t)=U_{21}(t), \quad U_{11}(t)=U_{22}(t), \quad t\in[0,1].$$

Combining the above equations with the expression of $P(t)+Q(t)$ given in (3.17) and the second equation in (3.18), we arrive at

$$U_{11}(t)\left(c_t^2-s_t^2\right)=0, \quad \forall t\in[0,1],$$

hence $U_{11}(t)\equiv 0$ for $t\in[0,1]$ by the continuity of U_{11}. Consequently,

$$U(t)=\begin{pmatrix} 0 & U_{12}(t) \\ U_{12}(t) & 0 \end{pmatrix}.$$

This together with (3.16) yields $U(0)=U(1)=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. It is a contradiction, since U is a unitary in H which ensures that all the 2×2 matrices $U(t)(t\in[0,1])$ are unitary.

4 C^*-Isomorphisms Associated with Two Projections

Unless otherwise specified, throughout this section (P,Q,H) is a matched triple, $i(P,Q,H)$ and $o(P,Q,H)$ are its inner algebra and outer algebra (see Definitions 1.2). It is clear that

$$i(P,Q,H)=\overline{\operatorname{span}\left\{X^{(P,Q,k)}: X\in\{A,B,C,D\}, k\in\mathbb{Z}_+\right\}}, \tag{4.1}$$

where $\mathcal{R}$ (also $\mathcal{N}$) is defined by (1.3), $P_{\mathcal{R}}=P\wedge Q$ and

$$A^{(P,Q,k)}=\left[(P-P_{\mathcal{R}})(Q-P_{\mathcal{R}})\right]^k, \tag{4.2}$$

$$B^{(P,Q,k)}=A^{(P,Q,k)}(P-P_{\mathcal{R}}), \tag{4.3}$$

$$C^{(P,Q,k)}=(A^{(P,Q,k)})^*=A^{(Q,P,k)}, \tag{4.4}$$

$$D^{(P,Q,k)}=C^{(P,Q,k)}(Q-P_{\mathcal{R}})=B^{(Q,P,k)} \tag{4.5}$$

with the convention that $A^{(P,Q,0)}=I$. For each $k\geq 1$, by utilizing $P_{\mathcal{R}}\leq P$ and $P_{\mathcal{R}}\leq Q$ we obtain

$$A^{(P,Q,k)}=(PQ-P_{\mathcal{R}})^k=(PQ)^k-P_{\mathcal{R}}, \tag{4.6}$$

which gives

$$\begin{aligned} A^{(P,Q,k)}(A^{(P,Q,k)})^* &= [(PQ)^k-P_{\mathcal{R}}][(QP)^k-P_{\mathcal{R}}] \\ &= (PQ)^k(QP)^k-P_{\mathcal{R}}=(PQP)^{2k-1}-P_{\mathcal{R}} \\ &= (PQP-P_{\mathcal{R}})^{2k-1}. \end{aligned} \tag{4.7}$$

It follows that

$$A^{(P,Q,k)}(A^{(P,Q,k)})^*=[A^{(P,Q,1)}(A^{(P,Q,1)})^*]^{2k-1}, \quad \forall k\geq 1. \tag{4.8}$$

Similarly,

$$\begin{aligned} B^{(P,Q,k)} &= [(PQ)^k - P_{\mathcal{R}}](P - P_{\mathcal{R}}) = (PQP)^k - P_{\mathcal{R}} = (PQP - P_{\mathcal{R}})^k \\ &= [A^{(P,Q,1)}(A^{(P,Q,1)})^*]^k, \quad \forall k \geq 1. \end{aligned} \tag{4.9}$$

Now, let (π, X) be a faithful representation of $o(P, Q, H)$. Replacing X with $\pi(I)X$ if necessary, in what follows we always assume that π is unital. The infimum of $\pi(P)$ and $\pi(Q)$, and its subtraction by $\pi(P_{\mathcal{R}})$ are denoted simply by $P_{\mathcal{R}\pi}$ and $\widetilde{P_{\mathcal{R}}}$, respectively, that is,

$$P_{\mathcal{R}\pi} = P_{\mathcal{R}(\pi(P)) \cap \mathcal{R}(\pi(Q))}, \quad \widetilde{P_{\mathcal{R}}} = P_{\mathcal{R}\pi} - \pi(P_{\mathcal{R}}). \tag{4.10}$$

It is clear that $\pi(P_{\mathcal{R}}) \leq P_{\mathcal{R}\pi}$, so $\widetilde{P_{\mathcal{R}}}$ defined as above is a projection. By (4.2) we have $A^{(\pi(P),\pi(Q),0)} = I$ and when $k \geq 1$,

$$A^{(\pi(P),\pi(Q),k)} = [(\pi(P) - P_{\mathcal{R}\pi})(\pi(Q) - P_{\mathcal{R}\pi})]^k = [\pi(P)\pi(Q)]^k - P_{\mathcal{R}\pi}.$$

This together with (4.6) and (4.10) gives

$$\pi(A^{(P,Q,k)}) = A^{(\pi(P),\pi(Q),k)} + \widetilde{P_{\mathcal{R}}}.$$

The derivation above shows that

$$\pi(X^{(P,Q,k)}) = X^{(\pi(P),\pi(Q),k)} + \widetilde{P_{\mathcal{R}}} \quad \text{whenever } X^{(P,Q,k)} \neq I. \tag{4.11}$$

Note that $\widetilde{P_{\mathcal{R}}}$ is a projection, and

$$X^{(\pi(P),\pi(Q),k)} \cdot \widetilde{P_{\mathcal{R}}} = \widetilde{P_{\mathcal{R}}} \cdot X^{(\pi(P),\pi(Q),k)} = 0 \quad \text{whenever } X^{(P,Q,k)} \neq I, \tag{4.12}$$

so by (4.11) together with the observation $\|I\| = 1 \geq \|\widetilde{P_{\mathcal{R}}}\|$, we arrive at

$$\begin{aligned} \|X^{(P,Q,k)}\| &= \|\pi(X^{(P,Q,k)})\| \\ &= \max\{\|X^{(\pi(P),\pi(Q),k)}\|, \|\widetilde{P_{\mathcal{R}}}\|\}, \quad \forall k \geq 0. \end{aligned} \tag{4.13}$$

We are now ready to derive a couple of norm equations. The first one reads as follows.

Lemma 4.1 *Let (π, X) be a faithful representation of $o(P, Q, H)$. Then for every $X \in \{A, B, C, D\}$ and $k \in \mathbb{Z}_+$, we have*

$$\|X^{(P,Q,k)}\| = \|X^{(\pi(P),\pi(Q),k)}\|. \tag{4.14}$$

Proof First, we prove that

$$\|A^{(P,Q,k)}\| = \|A^{(\pi(P),\pi(Q),k)}\|, \quad \forall k \in \mathbb{Z}_+. \tag{4.15}$$

The case of $k = 0$ is trivial, so we start with $k = 1$. From (4.11), we have

$$\begin{aligned} \|A^{(P,Q,1)}\|^2 &= \|\pi(A^{(P,Q,1)})\|^2 = \|\pi(A^{(P,Q,1)})[\pi(A^{(P,Q,1)})]^*\| \\ &= \|(A^{(\pi(P),\pi(Q),1)} + \widetilde{P_{\mathcal{R}}})[(A^{(\pi(P),\pi(Q),1)})^* + \widetilde{P_{\mathcal{R}}}]\| \\ &= \|A^{(\pi(P),\pi(Q),1)}(A^{(\pi(P),\pi(Q),1)})^* + \widetilde{P_{\mathcal{R}}}\| \\ &= \max\{\|A^{(\pi(P),\pi(Q),1)}(A^{(\pi(P),\pi(Q),1)})^*\|, \|\widetilde{P_{\mathcal{R}}}\|\} \\ &= \max\{\|A^{(\pi(P),\pi(Q),1)}\|^2, \|\widetilde{P_{\mathcal{R}}}\|\}, \end{aligned} \tag{4.16}$$

which implies that

$$\|A^{(P,Q,1)}\| = \|A^{(\pi(P),\pi(Q),1)}\| \tag{4.17}$$

whenever $\|A^{(\pi(P),\pi(Q),1)}\| = 1$. Suppose that $\|A^{(\pi(P),\pi(Q),1)}\| < 1$. Then according to (4.7)–(4.8), we have

$$\begin{aligned}\|(\pi(P)\pi(Q)\pi(P))^{2k-1} - P_{\mathcal{R}\pi}\| &= \|[A^{(\pi(P),\pi(Q),1)}(A^{(\pi(P),\pi(Q),1)})^*]^{2k-1}\| \\ &= \|A^{(\pi(P),\pi(Q),1)}(A^{(\pi(P),\pi(Q),1)})^*\|^{2k-1} \\ &= \|A^{(\pi(P),\pi(Q),1)}\|^{4k-2} \to 0 \quad \text{as } k \to \infty.\end{aligned}$$

It follows that $P_{\mathcal{R}\pi}$ is contained in the C^*-algebra $\pi(C^*_{(P,Q)})$, so $\pi^{-1}(P_{\mathcal{R}\pi}) \le \pi^{-1}(\pi(P)) = P$ and $\pi^{-1}(P_{\mathcal{R}\pi}) \le Q$ as well. Therefore, $\pi^{-1}(P_{\mathcal{R}\pi}) \le P_{\mathcal{R}}$, and thus $P_{\mathcal{R}\pi} \le \pi(P_{\mathcal{R}}) \le P_{\mathcal{R}\pi}$. Plugging $\widetilde{P_{\mathcal{R}}} = 0$ into (4.16) gives (4.17) immediately.

Now, we consider the case that $k \ge 2$. Due to (4.8) and (4.17), we have

$$\|A^{(P,Q,k)}\| = \|A^{(P,Q,1)}\|^{2k-1} = \|A^{(\pi(P),\pi(Q),1)}\|^{2k-1} = \|A^{(\pi(P),\pi(Q),k)}\|.$$

This completes the proof of (4.15).

Next, we prove that

$$\|B^{(P,Q,k)}\| = \|B^{(\pi(P),\pi(Q),k)}\|, \quad \forall k \in \mathbb{Z}_+. \tag{4.18}$$

By (4.3), both $B^{(P,Q,0)}$ and $B^{(\pi(P),\pi(Q),0)}$ are projections, and

$$\|B^{(P,Q,0)}\| = 0 \Longleftrightarrow P \le Q \Longleftrightarrow \pi(P) \le \pi(Q) \Longleftrightarrow \|B^{(\pi(P),\pi(Q),0)}\| = 0.$$

This shows the validity of (4.18) for $k = 0$. Suppose that $k \ge 1$. Then by (4.9), we can get

$$\|B^{(P,Q,k)}\| = \|A^{(P,Q,1)}\|^{2k} = \|A^{(\pi(P),\pi(Q),1)}\|^{2k} = \|B^{(\pi(P),\pi(Q),k)}\|.$$

Exchanging P with Q, we conclude that for every $k \in \mathbb{Z}_+$,

$$\begin{aligned}\|C^{(P,Q,k)}\| &= \|A^{(Q,P,k)}\| = \|A^{(\pi(Q),\pi(P),k)}\| = \|C^{(\pi(P),\pi(Q),k)}\|, \\ \|D^{(P,Q,k)}\| &= \|B^{(Q,P,k)}\| = \|B^{(\pi(Q),\pi(P),k)}\| = \|D^{(\pi(P),\pi(Q),k)}\|.\end{aligned}$$

This completes the proof of (4.14).

Corollary 4.1 *Let (π, X) be a faithful representation of $o(P,Q,H)$. Then for every $n \in \mathbb{N}$, $X_i \in \{A, B, C, D\}$ and $k_i \in \mathbb{Z}_+$ $(1 \le i \le n)$, we have*

$$\Big\|\prod_{i=1}^{n} X_i^{(P,Q,k_i)}\Big\| = \Big\|\prod_{i=1}^{n} X_i^{(\pi(P),\pi(Q),k_i)}\Big\|. \tag{4.19}$$

Proof From the definition of $X^{(P,Q,k)}$ given by (4.2)–(4.5), it is clear that

$$\prod_{i=1}^{n} X_i^{(P,Q,k_i)} = Z^{(P,Q,k)} \quad \text{and} \quad \prod_{i=1}^{n} X_i^{(\pi(P),\pi(Q),k_i)} = Z^{(\pi(P),\pi(Q),k)}$$

for some $Z \in \{A, B, C, D\}$ and $k \in \mathbb{Z}_+$. Due to (4.14), the desired norm equation follows.

We provide a technical lemma as follows.

Lemma 4.2 *Let (P,Q) be a harmonious pair of projections on H. Suppose that $n \in \mathbb{N}$, $X_i \in \{A,B,C,D\}$ and $k_i \in \mathbb{Z}_+$ $(1 \le i \le n)$ are given such that*

$$\Big\| \prod_{i=1}^{n} X_i^{(P,Q,k_i)} \Big\| = 1. \tag{4.20}$$

Then for every $\lambda_i \in \mathbb{C}$ $(1 \le i \le n)$, we have

$$\Big| \sum_{i=1}^{n} \lambda_i \Big| \le \Big\| \sum_{i=1}^{n} \lambda_i X_i^{(P,Q,k_i)} \Big\|. \tag{4.21}$$

Proof Denote by $\lambda = \sum\limits_{i=1}^{n} \lambda_i$. The verification of

$$|\lambda| \le \Big\| \sum_{i=1}^{n} \lambda_i X_i^{(P,Q,k_i)} \Big\|$$

will be carried out by taking several cases into consideration.

Case 1 $X_i^{(P,Q,k_i)} \in \{I, P-P_{\mathcal{R}}\}$ for all $i \in \{1,2,\cdots,n\}$. If $X_i^{(P,Q,k_i)} \equiv I$, then (4.21) is obviously satisfied. Otherwise, we have

$$\prod_{i=1}^{n} X_i^{(P,Q,k_i)} = P - P_{\mathcal{R}},$$

so according to (4.20) we obtain $\|P-P_{\mathcal{R}}\| = 1$. It follows that

$$\Big\| \sum_{i=1}^{n} \lambda_i X_i^{(P,Q,k_i)} \Big\| \ge \Big\| (P-P_{\mathcal{R}}) \sum_{i=1}^{n} \lambda_i X_i^{(P,Q,k_i)} \Big\| = \|\lambda(P-P_{\mathcal{R}})\| = |\lambda|.$$

Case 2 $X_i^{(P,Q,k_i)} \in \{I, Q-P_{\mathcal{R}}\}$ for all $i \in \{1,2,\cdots,n\}$. The same verification gives (4.21).

Case 3 There exist $i_1, i_2 \in \{1,2,\cdots,n\}$ such that $X_{i_1}^{(P,Q,k_{i_1})} \notin \{I, P-P_{\mathcal{R}}\}$ and $X_{i_2}^{(P,Q,k_{i_2})} \notin \{I, Q-P_{\mathcal{R}}\}$. In this case, firstly we show that

$$\|A^{(P,Q,1)}\| = 1. \tag{4.22}$$

Subcase 1 $k_{i_1} \ne 0$ or $k_{i_2} \ne 0$. Without loss of generality, we may assume that $k_{i_1} \ne 0$. In this subcase,

$$1 \ge \|A^{(P,Q,1)}\| = \|C^{(P,Q,1)}\| \ge \|X_{i_1}^{(P,Q,k_{i_1})}\| \ge \Big\| \prod_{i=1}^{n} X_i^{(P,Q,k_i)} \Big\| = 1.$$

Thus, (4.22) is satisfied.

Subcase 2 $k_{i_1} = k_{i_2} = 0$. In this subcase, we have $X_{i_1}^{(P,Q,k_{i_1})} = Q - P_{\mathcal{R}}$ and $X_{i_2}^{(P,Q,k_{i_2})} = P - P_{\mathcal{R}}$, which mean that

$$\prod_{i=1}^{n} X_i^{(P,Q,k_i)} = W_1 A^{(P,Q,1)} W_2 \quad \text{or} \quad \prod_{i=1}^{n} X_i^{(P,Q,k_i)} = W_1 C^{(P,Q,1)} W_2$$

for some contractions $W_1, W_2 \in \mathcal{L}(H)$. As is shown in Subcase 1, (4.22) is also satisfied.

Since (P,Q) is harmonious, the Halmos' two projections theorem is applicable. Following the notations as in Remark 3.1, we have[1]

$$\begin{aligned} P-P_{\mathcal{R}}&=0\oplus I_{H_2}\oplus 0\oplus 0\oplus I_{H_5}\oplus 0,\\ Q-P_{\mathcal{R}}&=0\oplus 0\oplus I_{H_3}\oplus 0\oplus 0\oplus Q_0,\\ A^{(P,Q,1)}&=0\oplus 0\oplus 0\oplus 0\oplus 0\oplus S,\end{aligned}\tag{4.23}$$

where Q_0 is defined by (3.15) and S is given by

$$S=\begin{pmatrix} A & A^{\frac12}\left(I_{H_5}-A\right)^{\frac12}U_0^* \\ 0 & 0\end{pmatrix}.$$

Based on the above block matrices, we have

$$\|A\|=\left\|\begin{pmatrix} A & 0\\ 0 & 0\end{pmatrix}\right\|=\|SS^*\|=\|S\|^2=\|A^{(P,Q,1)}\|^2=1.\tag{4.24}$$

This together with the positivity and contraction of A implies that $1\in \mathrm{sp}(A)$, where $\mathrm{sp}(A)$ denotes the spectrum of A.

It is easy to verify that for every $X\in\{A,B,C,D\}$ and $k\in\mathbb{Z}_+$, there exists $r\in\mathbb{N}$ depending on X and k such that

$$A^{(P,Q,1)}X^{(P,Q,k)}(P-P_{\mathcal{R}})=0\oplus 0\oplus 0\oplus 0\oplus 0\oplus\begin{pmatrix} A^r & \\ & 0\end{pmatrix}.$$

Consequently,

$$\begin{aligned}\Big\|\sum_{i=1}^n\lambda_i X_i^{(P,Q,k_i)}\Big\|&\geq \Big\|A^{(P,Q,1)}\Big(\sum_{i=1}^n\lambda_i X_i^{(P,Q,k_i)}\Big)(P-P_{\mathcal{R}})\Big\|\\ &=\left\|\begin{pmatrix}\sum\limits_{i=1}^n\lambda_iA^{r_i} & \\ & 0\end{pmatrix}\right\|=\Big\|\sum_{i=1}^n\lambda_iA^{r_i}\Big\|\end{aligned}\tag{4.25}$$

for some $r_i\in\mathbb{N}$ $(1\leq i\leq n)$. Let $f(t)=\sum\limits_{i=1}^n\lambda_it^{r_i}$ for $t\geq 0$. Then

$$\Big\|\sum_{i=1}^n\lambda_iA^{r_i}\Big\|=\max\{|f(t)|:t\in\mathrm{sp}(A)\}\geq |f(1)|=|\lambda|.$$

Combining the above inequality with (4.25) gives (4.21).

Along the same line, another technical lemma can be provided as follows.

Lemma 4.3 *Let (P,Q) be a harmonious pair of projections on H such that $PQ\neq QP$. Suppose that $n\in\mathbb{N}$, $X_i\in\{A,B,C,D\}$ and $k_i\in\mathbb{Z}_+$ are given such that* (4.20) *is satisfied and $X_i^{(P,Q,k_i)}\neq I$ for all $i\in\{1,2,\cdots,n\}$. Then for every $\lambda_i\in\mathbb{C}$ $(0\leq i\leq n)$, we have*

$$|\lambda_0|\leq\Big\|\lambda_0(I-P_{\mathcal{R}})+\sum_{i=1}^n\lambda_iX_i^{(P,Q,k_i)}\Big\|.\tag{4.26}$$

[1] In some cases, it may happen that the closed subspaces H_5 and H_6 constructed for the Halmos decomposition are trivial, that is, $H_5=H_6=\{0\}$. Due to (4.22), both H_5 and H_6 are non-trivial, hence $A^{(P,Q,1)}$ has the form (4.23).

Proof As in the proof of Lemma 4.2, the verification of (4.26) will be dealt with via several cases. Let $\lambda \in \mathbb{C}$ and $W \in \mathcal{L}(H)$ be defined by

$$\lambda = \sum_{i=0}^{n} \lambda_i, \quad W = \lambda_0(I - P_{\mathcal{R}}) + \sum_{i=1}^{n} \lambda_i X_i^{(P,Q,k_i)}. \tag{4.27}$$

Case 1 $X_i^{(P,Q,k_i)} = P - P_{\mathcal{R}}$ for all $i \in \{1, 2, \cdots, n\}$. In this case,

$$W = \lambda_0(I - P) + \lambda(P - P_{\mathcal{R}}).$$

Since $PQ \neq QP$, we have $I - P \neq 0$, hence

$$\|W\| \geq \|\lambda_0(I - P)\| = |\lambda_0|.$$

Case 2 $X_i^{(P,Q,k_i)} = Q - P_{\mathcal{R}}$ for all $i \in \{1, 2, \cdots, n\}$. As is shown in the above case, we have $\|W\| \geq |\lambda_0|$.

Case 3 There exist $i_1, i_2 \in \{1, 2, \cdots, n\}$ such that $X_{i_1}^{(P,Q,k_{i_1})} \neq P - P_{\mathcal{R}}$ and $X_{i_2}^{(P,Q,k_{i_2})} \neq Q - P_{\mathcal{R}}$. Following the notations as in Remark 3.1, we have $H = \bigoplus\limits_{i=1}^{6} H_i$ and up to unitary equivalence, every operator $Y \in \mathcal{L}(H)$ has the matrix form $Y = (Y_{ij})_{1 \leq i,j \leq 6}$ with $Y_{ij} \in \mathcal{L}(H_j, H_i)$. Let the linear map $\phi : \mathcal{L}(H) \to \mathcal{L}(H_6)$ be defined by $\phi(Y) = Y_{66}$. According to the definition of $X^{(P,Q,k)}$ given by (4.2)–(4.5), we have $\phi(B^{(P,Q,0)}) = 0$ and

$$\phi(A^{(P,Q,k)}) = \phi(B^{(P,Q,k)}) = \phi(C^{(P,Q,k)}) = 0$$

for every $k \geq 1$. Furthermore, direct computations yield

$$\phi(D^{(P,Q,k)}) = U_0(I - A)A^k U_0^*, \quad \forall k \in \mathbb{Z}_+.$$

It follows from (4.27) that[2]

$$\phi(W) = U_0 \Big[\lambda_0 I + \sum_j \lambda_{i_j} (I - A) A^{k_{i_j}}\Big] U_0^*,$$

where i_j is chosen in $\{1, 2, \cdots, n\}$ whenever $X_{i_j}^{(P,Q,k_{i_j})} = D^{(P,Q,k_{i_j})}$. Let

$$f(t) = \lambda_0 + \sum_j \lambda_{i_j} (1 - t) t^{k_{i_j}}, \quad t \geq 0.$$

Since $0 \leq A \leq I$ and $1 \in \mathrm{sp}(A)$ (see (4.24)), we have

$$\|W\| \geq \|\phi(W)\| = \max\{|f(t)| : t \in \mathrm{sp}(A)\} \geq |f(1)| = |\lambda_0|.$$

This completes the proof of (4.26).

Now, we provide the main result of this section as follows.

Theorem 4.1 *For each faithful representation (π, X) of $o(P, Q, H)$, a faithful representation $(\widetilde{\pi}, X)$ of $i(P, Q, H)$ can be induced such that $\widetilde{\pi}(I) = \pi(I)$, and*

$$\widetilde{\pi}(P - P_{\mathcal{R}}) = \pi(P) - \pi(P) \wedge \pi(Q), \quad \widetilde{\pi}(Q - P_{\mathcal{R}}) = \pi(Q) - \pi(P) \wedge \pi(Q). \tag{4.28}$$

[2] If $X_i \neq D$ for all $i \in \{1, 2, \cdots, n\}$, then $\phi(W) = \lambda_0 I$, so in this case $\|\phi(W)\| = |\lambda_0|$.

Proof According to (4.1), we need only to prove that $\|T\| = \|\underline{T}\|$ for every $n \in \mathbb{N}$, $X_i \in \{A, B, C, D\}$, $k_i \in \mathbb{Z}_+$ and $\lambda_i \in \mathbb{C}$ $(0 \le i \le n)$ such that $X_i^{(P,Q,k_i)} \neq I$ for all $i \ge 1$, where[3]

$$T = \lambda_0 I + \sum_{i=1}^{n} \lambda_i X_i^{(P,Q,k_i)}, \quad \underline{T} = \lambda_0 I + \sum_{i=1}^{n} \lambda_i X_i^{(\pi(P),\pi(Q),k_i)}. \tag{4.29}$$

If $\widetilde{P_{\mathcal{R}}} = 0$, then we see from (4.11) that $\pi(T) = \underline{T}$, which gives $\|T\| = \|\underline{T}\|$ as π is faithful. In what follows, we assume that $\widetilde{P_{\mathcal{R}}} \neq 0$. In this case, we have $PQ \neq QP$. Otherwise, $P_{\mathcal{R}} = PQ$ and $P_{\mathcal{R}\pi} = \pi(P)\pi(Q) = \pi(P_{\mathcal{R}})$, which contradicts the assumption of $\widetilde{P_{\mathcal{R}}} \neq 0$. Let $\lambda = \sum\limits_{i=0}^{n} \lambda_i$. By (4.11) we have

$$\pi(T) = \underline{T} + (\lambda - \lambda_0)\widetilde{P_{\mathcal{R}}} = L + \lambda \widetilde{P_{\mathcal{R}}}, \tag{4.30}$$

where

$$L = \lambda_0 (I - \widetilde{P_{\mathcal{R}}}) + \sum_{i=1}^{n} \lambda_i X_i^{(\pi(P),\pi(Q),k_i)} = Z + \lambda_0 \pi(P_{\mathcal{R}}),$$

in which

$$Z = \lambda_0 (I - P_{\mathcal{R}\pi}) + \sum_{i=1}^{n} \lambda_i X_i^{(\pi(P),\pi(Q),k_i)}.$$

Hence

$$\|L\| = \max\{\|Z\|, |\lambda_0|\, \|\pi(P_{\mathcal{R}})\|\}, \tag{4.31}$$

$$\|T\| = \|\pi(T)\| = \max\{\|L\|, |\lambda|\, \|\widetilde{P_{\mathcal{R}}}\|\} = \max\{\|L\|, |\lambda|\}, \tag{4.32}$$

$$\|\underline{T}\| = \|L + \lambda_0 \widetilde{P_{\mathcal{R}}}\| = \max\{\|L\|, |\lambda_0|\, \|\widetilde{P_{\mathcal{R}}}\|\} = \max\{\|L\|, |\lambda_0|\|\}. \tag{4.33}$$

Let

$$\alpha = \Big\| \prod_{i=1}^{n} X_i^{(\pi(P),\pi(Q),k_i)} \Big\|.$$

By (4.19) and (4.11)–(4.12), we have

$$\begin{aligned} \alpha &= \Big\| \pi\Big(\prod_{i=1}^{n} X_i^{(P,Q,k_i)} \Big) \Big\| = \Big\| \prod_{i=1}^{n} \pi(X_i^{(P,Q,k_i)}) \Big\| \\ &= \Big\| \prod_{i=1}^{n} X_i^{(\pi(P),\pi(Q),k_i)} + \widetilde{P_{\mathcal{R}}} \Big\| = \max\{\alpha, \|\widetilde{P_{\mathcal{R}}}\|\} = \max\{\alpha, 1\}. \end{aligned}$$

Hence $\alpha = 1$, which apparently gives

$$\Big\| \prod_{i=0}^{n} X_i^{(\pi(P),\pi(Q),k_i)} \Big\| = 1$$

by setting $X_0^{(\pi(P),\pi(Q),k_0)} = I$. It is notable that $\pi(P)$ and $\pi(Q)$ are projections acting on a Hilbert space, so $(\pi(P), \pi(Q))$ is harmonious. It follows from Lemmas 4.2–4.3 that

$$\|\underline{T}\| \ge |\lambda| \quad \text{and} \quad \|Z\| \ge |\lambda_0|.$$

[3] When $\lambda_0 = 0$, the first term in T and $\underline{T}$ will disappear.

So by (4.31) we have $\|L\| \geq \|Z\| \geq |\lambda_0|$, which leads by (4.33) to $\|\underline{T}\| = \|L\|$. Combining this equality with $\|\underline{T}\| \geq |\lambda|$ and (4.32), we arrive at $\|T\| = \|\underline{T}\| = \|L\|$.

Under the restriction of $\lambda_0 = 0$ in (4.29), a corollary can be derived immediately as follows.

Corollary 4.2 *Let $C^*(P, Q, P_{\mathcal{R}})$ and $C^*(P - P_{\mathcal{R}}, Q - P_{\mathcal{R}})$ denote the C^*-subalgebras of $\mathcal{L}(H)$ generated by elements in $\{P, Q, P_{\mathcal{R}}\}$ and $\{P - P_{\mathcal{R}}, Q - P_{\mathcal{R}}\}$, respectively. Then each faithful representation (π, X) of $C^*(P, Q, P_{\mathcal{R}})$ can induce a faithful representation $(\widetilde{\pi}, X)$ of $C^*(P - P_{\mathcal{R}}, Q - P_{\mathcal{R}})$ such that (4.28) is satisfied.*

In the derivations given as above, we merely consider the meaningfulness of $P_{\mathcal{R}}$. At this moment, we do not know whether there exist projections P and Q such that $\mathcal{R}$ is orthogonally complemented, whereas $\mathcal{N}$ fails to be orthogonally complemented[4]. To give a partial answer, we need an auxiliary lemma, whose proof is given for the sake of completeness.

Lemma 4.4 *Let $P, Q \in \mathcal{P}(H)$ be such that the sequence $\{(PQP)^n\}_{n=1}^{\infty}$ converges to $T \in \mathcal{L}(H)$ in norm-topology. Then T is a projection such that $\mathcal{R}(T) = \mathcal{R}$, where $\mathcal{R}$ is defined by (1.3).*

Proof Clearly, T is a projection such that $\mathcal{R} \subseteq \mathcal{R}(T)$ and $PT = T$. So it needs only to show that $QT = T$, or equivalently, $[(I - Q)T]^*(I - Q)T = 0$, which can be derived from the equations

$$(PQP)^n(I - Q)(PQP)^n = (PQP)^{2n} - (PQP)^{2n+1}, \quad \forall n \in \mathbb{N}.$$

Corollary 4.3 *Let (P, Q, H) be a matched triple such that $\|(P - P_{\mathcal{R}})(Q - P_{\mathcal{R}})\| < 1$. Then $(I - Q, I - P, H)$ is also a matched triple.*

Proof Choose any faithful unital representation (π, X) of $\mathcal{L}(H)$. Let $P_{\mathcal{R}\pi}$ be defined by (4.10), and put

$$\begin{aligned}
&P_{\mathcal{N}\pi} = (\pi(I - Q)) \wedge (\pi(I - P)),\\
&S = [I - \pi(Q) - P_{\mathcal{N}\pi}][I - \pi(P) - P_{\mathcal{N}\pi}],\\
&T = [\pi(P) - P_{\mathcal{R}\pi}][\pi(Q) - P_{\mathcal{R}\pi}],\\
&W = (I - P)(I - Q)(I - P).
\end{aligned} \tag{4.34}$$

Then

$$\begin{aligned}
P_{\mathcal{N}\pi} &= P_{\mathcal{R}(\pi(I-Q)) \cap \mathcal{R}(\pi(I-P))} = P_{\mathcal{R}(I-\pi(Q)) \cap \mathcal{R}(I-\pi(P))}\\
&= P_{\mathcal{N}(\pi(P)) \cap \mathcal{N}(\pi(Q))}.
\end{aligned}$$

By Theorems 3.1 and 4.1, there exists a unitary $U \in \mathbb{B}(X)$ such that

$$\|S\| = \|U^*TU\| = \|T\| = \|(P - P_{\mathcal{R}})(Q - P_{\mathcal{R}})\| < 1,$$

hence $\|(S^*S)^n\| = \|S^*S\|^n = \|S\|^{2n} \to 0$ as $n \to \infty$. For each $n \in \mathbb{N}$, it is clear that

$$(S^*S)^n = (\pi(W))^n - P_{\mathcal{N}\pi},$$

so $\{(\pi(W))^n\}_{n=1}^{\infty}$ is norm-convergent and thus is a Cauchy sequence, and so does for $\{W^n\}_{n=1}^{\infty}$ by the faithfulness of π. Due to the completeness of $\mathcal{L}(H)$, $\{W^n\}_{n=1}^{\infty}$ is norm-convergent. The assertion then follows from Lemma 4.4.

[4] Alternatively, is it possible to find a matched triple (P, Q, H) such that $(I - Q, I - P, H)$ is not a matched triple?

Remark 4.1 Our next example shows that there exists a matched triple (P, Q, H) such that the sequence $\{(PQP)^n\}_{n=1}^{\infty}$ does not converge strongly to $P_{\mathcal{R}}$.

Example 4.1 Let $\mathfrak{A}, H, P$ and Q be as in Example 2.1. According to (2.8), we have $P_{\mathcal{R}} = 0$. From (2.7) we obtain

$$[(PQP)^n](0) = \begin{pmatrix} 1 & \\ & 0 \end{pmatrix}$$

for every $n \geq 1$. It follows that

$$\|(PQP)^n P\| \geq \|[(PQP)^n P](0)\| = 1, \quad \forall n \geq 1.$$

Thus, $\{(PQP)^n\}_{n=1}^{\infty}$ does not converge strongly to zero.

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