Chin. Ann. Math. Ser. B
44(3), 2023, 345–360
DOI: 10.1007/s11401-023-0019-8

Chinese Annals of Mathematics, Series B

Existence and Asymptotic Behavior of Ground State Solutions for Quasilinear Schrödinger Equations with Unbounded Potential*

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Abstract The authors study the existence of standing wave solutions for the quasilinear Schrödinger equation with the critical exponent and singular coefficients. By applying the mountain pass theorem and the concentration compactness principle, they get a ground state solution. Moreover, the asymptotic behavior of the ground state solution is also obtained.

Keywords Quasilinear Schrödinger equation, Ground state, Critical Hardy-Sobolev exponents, Coercive
2000 MR Subject Classification 35J62, 35J20

1 Introduction

In this paper, we consider the following quasilinear Schrödinger equation

$$\mathrm{i}\frac{\partial\psi}{\partial t}=-\triangle\psi+W(x)\psi-k(x,\psi)-\triangle\rho(|\psi|^2)\rho'(|\psi|^2)\psi, \tag{1.1}$$

where $W:\mathbb{R}^N\to\mathbb{R}$ is a given potential, and k, ρ are real functions. Seeking solutions of the type stationary waves, namely, the solutions of the form $\psi(t,x)=\exp(-\mathrm{i}Et)u(x)$, $E\in\mathbb{R}$ and u is a real function, (1.1) can be reduced to the corresponding equation of elliptic type

$$-\Delta u+V(x)u-\Delta\rho(u^2)\rho'(|u|^2)u=k(x,u),\quad x\in\mathbb{R}^N, \tag{1.2}$$

where $V(x)=W(x)-E$ is a new potential function. equation (1.2) has been derived as models of several physical phenomena and has been the subject of extensive study in recent years. If we take $\rho(s)=s$, we get the superfluid film equation in plasma physics

$$-\Delta u+V(x)u-\Delta(u^2)u=k(x,u),\quad x\in\mathbb{R}^N. \tag{1.3}$$

Manuscript received September 28, 2021. Revised September 29, 2022.
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*This work was supported by the National Natural Science Foundation of China (Nos. 11971393, 11901499, 11801465) and Nanhu Scholar Program for Young Scholars of XYNU (No. 201912).

If we set $\rho(s) = \sqrt{1+s}$, we get the equation

$$-\Delta u + V(x)u - \frac{\gamma u}{2\sqrt{1+u^2}}\Delta(\sqrt{1+u^2}) = k(x,u), \quad x \in \mathbb{R}^N, \tag{1.4}$$

which models the self-channeling of a high-power ultrashort laser in matter (see [7]). For more physical motivations, we may refer to [11–12] and references therein.

Recent studies have been focused on problem (1.3). The main mathematical difficulty with problem (1.3) is caused by the second order derivatives $\Delta(u^2)u$, the natural functional corresponding to problem (1.3) is not well defined for all $u \in H^1(\mathbb{R}^N)$ if $N \geq 2$. To overcome this difficulty, various arguments have been developed, such as a constrained minimization argument (see [12]), the perturbation method (see [10]) and a change of variables (see [6, 15]). The above method is also suitable for (1.4). Up to our knowledge, there are few results on problem (1.4). The greatest part of the literature focuses on the study of problem (1.4) when the potential V is assumed to be a potential well (see [4, 9, 14] and references therein) or radially symmetric potential (see [5]). Regretfully, the coercive potential and singular coefficients are not considered.

Motivated by references [4, 15], we study the existence of solutions for (1.4) with coercive potential and $k(x,u) = \lambda|u|^{p-2}u + \frac{|u|^{2^*(s)-2}u}{|x|^s}$. Namely, the following quasilinear Schrödinger equation

$$-\Delta u + V(x)u - \frac{\gamma u}{2\sqrt{1+u^2}}\Delta(\sqrt{1+u^2}) = \lambda|u|^{p-2}u + \frac{|u|^{2^*(s)-2}u}{|x|^s}, \quad x \in \mathbb{R}^N, \tag{1.5}$$

where $N \geq 3, \gamma, \lambda > 0, 0 \leq s < 2, 2 < p < 2^*(s), 2^*(s) = \frac{2(N-s)}{N-2}$, the potential $V(x)$ satisfies the following condition:

(V) $V(x) \in C(\mathbb{R}^N, \mathbb{R})$, $\inf\limits_{x\in\mathbb{R}^N} V(x) \geq a_0 > 0$, and there exists $d_0 > 0$, such that for every $M > 0$,

$$\lim_{|y|\to\infty} \text{meas}\{x \in \mathbb{R}^N : |x-y| \leq d_0, V(x) \leq M\} = 0.$$

The condition (V) was first introduced by Bartsch and Wang [2] to guarantee the compactness of embeddings of the working spaces. The limit in condition (V) can be replaced by one of the following simpler conditions:

(V$'$) $\lim\limits_{|x|\to\infty} V(x) = +\infty$ (see [17]).

(V$''$) For each $M > 0$, meas$\{x \in \mathbb{R}^N : V(x) \leq M\} < \infty$ (see [2]).

Now we state our main results.

Theorem 1.1 *Suppose that the condition* (V) *holds and* $N \geq 3$, $\lambda > 0$, $0 \leq s < 2$, $p_N < p < 2^*(s)$, *where*

$$p_N = \begin{cases} 4, & \text{if } N = 3, \\ 2, & \text{if } N \geq 4. \end{cases}$$

Then problem (1.5) *possesses a ground state solution if* $0 < \gamma < \gamma^*$, *where*

$$\gamma^* = \begin{cases} \dfrac{16(p-2)}{(p-4)^2}, & \text{if } p < 4, \\ +\infty, & \text{if } p \geq 4. \end{cases}$$

Theorem 1.2 *Suppose that the conditions of Theorem* 1.1 *hold,* u_{γ_n} *is the ground state solution of* (1.5) *obtained in Theorem* 1.1 *with* $\gamma = \gamma_n$. *Then for each sequence* $\{\gamma_n\}$ *with* $\gamma_n \to 0^+$ *as* $n \to \infty$, *there exists a subsequence, still denoted by* $\{\gamma_n\}$, *such that* $u_{\gamma_n} \to u_0$ *in* E, *where* u_0 *is the ground state solution of semilinear problem*

$$-\Delta u + V(x)u = \lambda |u|^{p-2}u + \frac{|u|^{2^*(s)-2}u}{|x|^s}, \quad x \in \mathbb{R}^N. \tag{1.6}$$

Remark 1.1 In [4], the nonlinearity is more general, but it does not include singular coefficients. Here we consider problem (1.5) with critical Hardy-Sobolev exponents. Moreover, the potential here is different from that in literature [4]. It is a complement of [4].

Remark 1.2 For the coercive potential, Wang et al. studied the existence of ground state solutions of problem (1.3) with singular coefficients in [15]. They make an unknown variable $v := f^{-1}(u)$, where f is defined by the ordinary differential equation

$$f'(t) = \frac{1}{(1 + 2f^2(t))^{\frac{1}{2}}}, \quad t \in [0, +\infty); \quad f(t) = -f(-t), \quad t \in (-\infty, 0].$$

In this paper, we study problem (1.5). We make a change of variables $v := F(u) = \int_0^u f(t)\mathrm{d}t$, where f is defined by

$$f(t) = \sqrt{1 + \frac{\gamma t^2}{2(1 + t^2)}}.$$

Obviously, our results are different from those in [15]. Because our transformation is a little more complex, we need more precise estimation when using the concentration compactness principle (see Lemmas 3.5 and 2.3 below). Moreover, we regard $\gamma > 0$ as a parameter in (1.5) and analyse the convergence property of the ground state solution as $\gamma \to 0^+$.

Remark 1.3 We denote $\int_{\mathbb{R}^N} h(x)\mathrm{d}x$ as $\int_{\mathbb{R}^N} h(x)$ for simplicity.

Notation In this paper, we use the following notations:

- $E := \left\{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2 < +\infty\right\}$ is the Hilbert space endowed with the norm

$$\|u\|^2 = \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2).$$

- $L^s(\mathbb{R}^N)$ is the usual Banach space endowed with the norm

$$\|u\|_s^s = \int_{\mathbb{R}^N} |u|^s, \quad \forall s \in [1, +\infty).$$

- $B_r(y) := \{x \in \mathbb{R}^N : |x - y| < r\}$.
- $C, C_1, C_2, \cdots$ denote various positive (possibly different) constants.

2 Some Preliminary Results

We observe that formally problem (1.5) is the Euler-Lagrange equation associated with the energy functional

$$\begin{aligned} J(u) = \frac{1}{2}\int_{\mathbb{R}^N}\Big[\Big(1+\frac{\gamma u^2}{2(1+u^2)}\Big)|\nabla u|^2\Big] + \frac{1}{2}\int_{\mathbb{R}^N} V(x)u^2 \\ - \frac{\lambda}{p}\int_{\mathbb{R}^N}|u|^p - \frac{1}{2^*(s)}\int_{\mathbb{R}^N}\frac{|u|^{2^*(s)}}{|x|^s}. \end{aligned}$$

Variational methods cannot be applied directly to find weak solutions of problem (1.5), since the natural associated functional $J(u)$ is not defined for all u in the space $H^1(\mathbb{R}^N)$. Hence we employ an argument developed in [13] to introduce a variational framework associated with problem (1.5). We make a change of variables $v := F(u) = \int_0^u f(t)\mathrm{d}t$, where f is defined by

$$f(t) = \sqrt{1+\frac{\gamma t^2}{2(1+t^2)}}. \tag{2.1}$$

After the change of variables from J, we obtain the following functional

$$I(v) = \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla v|^2 + V(x)|F^{-1}(v)|^2) - \frac{\lambda}{p}\int_{\mathbb{R}^N}|F^{-1}(v)|^p - \frac{1}{2^*(s)}\int_{\mathbb{R}^N}\frac{|F^{-1}(v)|^{2^*(s)}}{|x|^s}.$$

Then $I(v) = J(u) = J(F^{-1}(v))$ and I is well defined in E, $I \in C^1(E,\mathbb{R})$. Moreover, we observe that if v is a critical point of the functional I, then the function $u = F^{-1}(v)$ is a solution of problem (1.5) (see [7, 13–14]).

Now, we summarize the properties of F^{-1}, f, which have been proved in [14].

Lemma 2.1 *The functions F^{-1}, f satisfy the following properties:*

(1) $1 \le f(t) \le \sqrt{\frac{2+\gamma}{2}}$ *for all* $t \in \mathbb{R}$;

(2) $1 \le \frac{F^{-1}(t)f(F^{-1}(t))}{t} \le \frac{4+2\gamma-2\sqrt{4+2\gamma}}{\gamma}$ *for all* $t \in \mathbb{R}, t \neq 0$;

(3) $\sqrt{\frac{2}{2+\gamma}}|t| \le |F^{-1}(t)| \le |t|$ *for all* $t \in \mathbb{R}$;

(4) $\frac{F^{-1}(t)}{t} \to 1$ *as* $t \to 0$;

(5) $\frac{F^{-1}(t)}{t} \to \sqrt{\frac{2}{2+\gamma}}$ *as* $t \to \infty$;

(6) $0 \le \frac{f'(t)t}{f(t)} \le 1 + \frac{4-2\sqrt{4+2\gamma}}{\gamma}$ *for all* $t \in \mathbb{R}$.

Lemma 2.2 (see [15]) *Let $0 \le s < 2$. The embedding $E \hookrightarrow L^\alpha(\mathbb{R}^N, |x|^{-s})$ is continuous for $2 \le \alpha \le 2^*(s)$ and compact for $2 \le \alpha < 2^*(s)$ when $V(x)$ satisfies the condition* (V).

Lemma 2.3 *Suppose that $\{v_n\} \subset E$ is a bounded sequence and $v_n \to 0$ in $L^\alpha(\mathbb{R}^N, |x|^{-s})$ for $\alpha \in (2, 2^*)$, $s \in [0, 2)$. Then we have*

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\frac{|F^{-1}(v_n)|^{p-1}}{f(F^{-1}(v_n))}v_n = \lim_{n\to\infty}\int_{\mathbb{R}^N}|F^{-1}(v_n)|^p = 0. \tag{2.2}$$

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} V(x)\Big(|F^{-1}(v_n)|^2 - \frac{F^{-1}(v_n)v_n}{f(F^{-1}(v_n))}\Big) = 0. \tag{2.3}$$

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} V(x)(|F^{-1}(v_n)|^2 - v_n^2) = 0. \tag{2.4}$$

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} \Big(\frac{|F^{-1}(v_n)|^{2^*(s)-1}}{f(F^{-1}(v_n))|x|^s}v_n - \Big(\sqrt{\frac{2}{2+\gamma}}\Big)^{2^*(s)}\frac{|v_n|^{2^*(s)}}{|x|^s}\Big) = 0. \tag{2.5}$$

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} \Big(\Big(\sqrt{\frac{2}{2+\gamma}}\Big)^{2^*(s)}\frac{|v_n|^{2^*(s)}}{|x|^s} - \frac{|F^{-1}(v_n)|^{2^*(s)}}{|x|^s}\Big) = 0. \tag{2.6}$$

Proof (i) Since $v_n \to 0$ in $L^\alpha(\mathbb{R}^N, |x|^{-s})(2 < \alpha < 2^*, 0 \le s < 2)$ and $2 < p < 2^*(s) \le 2^*$, we can get (2.2) from the properties of F^{-1} and f easily.

(ii) By Lemma 2.1(4) and the properties of f, one has

$$\lim_{|v_n|\to 0}\frac{v_n}{f(F^{-1}(v_n))F^{-1}(v_n)} = 1.$$

Hence, for any $\varepsilon > 0$, there is $\delta > 0$, such that

$$\Big|\frac{v_n}{f(F^{-1}(v_n))F^{-1}(v_n)} - 1\Big| < \varepsilon,$$

when $|v_n(x)| < \delta$. Then we have

$$\begin{aligned}&\int_{\{x:|v_n(x)|<\delta\}} V(x)\Big|\frac{v_n}{f(F^{-1}(v_n))F^{-1}(v_n)} - 1\Big|\cdot|F^{-1}(v_n)|^2\\ &\le \varepsilon\int_{\mathbb{R}^N} V(x)v_n^2 \le C\varepsilon.\end{aligned} \tag{2.7}$$

By Lemma 2.1(2)–(3), one has

$$\begin{aligned}&\lim_{n\to\infty}\int_{\{x:|v_n(x)|\ge\delta\}} V(x)\Big|\frac{v_n}{f(F^{-1}(v_n))F^{-1}(v_n)} - 1\Big|\cdot|F^{-1}(v_n)|^2\\ &\le C\lim_{n\to\infty}\int_{\{x:|v_n(x)|\ge\delta\}} V(x)v_n^2 \le C\delta^{2-p}\lim_{n\to\infty}\int_{\mathbb{R}^N} V(x)|v_n|^p = 0.\end{aligned} \tag{2.8}$$

Combining (2.7) with (2.8), we can get (2.3). Similarly, we can verify (2.4).

(iii) On one hand, by Lemma 2.1(2)–(3), we have

$$\begin{aligned}\int_{\{x:|v_n(x)|\le R\}} \frac{|F^{-1}(v_n)|^{2^*(s)-1}}{f(F^{-1}(v_n))|x|^s}v_n &\le \int_{\{x:|v_n(x)|\le R\}} \frac{|F^{-1}(v_n)|^{2^*(s)}}{|x|^s}\\ &\le \int_{\{x:|v_n(x)|\le R\}} \frac{|v_n|^{2^*(s)}}{|x|^s}\\ &\le R^{2^*(s)-p}\int_{\{x:|v_n(x)|\le R\}} \frac{|v_n|^p}{|x|^s}\\ &\le R^{2^*(s)-p}\int_{\mathbb{R}^N} \frac{|v_n|^p}{|x|^s}\\ &\to 0 \quad \text{as } n\to\infty.\end{aligned} \tag{2.9}$$

As we have shown in the proof of (2.9), it holds that

$$\int_{\{x:|v_n(x)|\le R\}} \frac{|v_n|^{2^*(s)}}{|x|^s} \to 0 \quad \text{as } n\to\infty. \tag{2.10}$$

On the other hand, by Lemma 2.1(5) and the fact that

$$\lim_{|v_n|\to\infty} \frac{v_n}{f(F^{-1}(v_n))F^{-1}(v_n)} = 1,$$

one gets

$$\begin{aligned}
&\int_{\{x:|v_n(x)|\ge R\}} \Big(\frac{|F^{-1}(v_n)|^{2^*(s)-1}}{f(F^{-1}(v_n))|x|^s} v_n - \Big(\sqrt{\frac{2}{2+\gamma}}\Big)^{2^*(s)} \frac{|v_n|^{2^*(s)}}{|x|^s}\Big)\\
&= \int_{\{x:|v_n(x)|\ge R\}} \Big(\frac{|F^{-1}(v_n)|^{2^*(s)}}{|v_n|^{2^*(s)}} \frac{v_n}{f(F^{-1}(v_n))F^{-1}(v_n)} - \Big(\sqrt{\frac{2}{2+\gamma}}\Big)^{2^*(s)}\Big) \frac{|v_n|^{2^*(s)}}{|x|^s}\\
&\to 0 \quad \text{as } R>0 \text{ sufficiently large.}
\end{aligned} \tag{2.11}$$

Combining (2.9)–(2.10) with (2.11), we can get (2.5). The proof of (2.6) is similar to the proof of (2.5), so we omit it.

Lemma 2.4 (Mountain pass theorem see [16]). *Let E be a real Banach space with its dual space E^* and suppose that $I \in C^1(E,\mathbb{R})$ satisfies*

$$\max\{I(0), I(e)\} \le \mu < \eta \le \inf_{\|u\|=\rho} I(u)$$

for some $\mu < \eta$, $\rho > 0$ and $e \in E$ with $\|e\| > \rho$. Let $c \ge \eta$ be characterized by $c = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I(\gamma(t))$, where $\Gamma = \{\gamma \in C([0,1],E) : \gamma(0) = 0, \gamma(1) = e\}$ is the set of continuous paths joining 0 and e, then there exists a sequence $\{u_n\} \subset E$ such that

$$I(u_n) \to c \ge \eta, \quad (1+\|u_n\|)\|I'(u_n)\| \to 0.$$

3 Proofs of Theorem 1.1 and Theorem 1.2

Lemma 3.1 *Suppose that the condition* (V) *holds and $\lambda > 0$, $0 \le s < 2$, $2 \le p_N < p < 2^*(s)$. Then there exist $\rho > 0$, $\eta > 0$ such that $\inf_{\|v\|=\rho} I(v) > \eta$.*

Proof From Lemmas 2.1(3) and 2.2, we have

$$\begin{aligned}
I(v) &= \frac12\int_{\mathbb{R}^N} |\nabla v|^2 + \frac12\int_{\mathbb{R}^N} V(x)|F^{-1}(v)|^2 - \frac{\lambda}{p}\int_{\mathbb{R}^N} |F^{-1}(v)|^p\\
&\quad - \frac{1}{2^*(s)}\int_{\mathbb{R}^N} \frac{|F^{-1}(v)|^{2^*(s)}}{|x|^s}\\
&\ge \frac12\int_{\mathbb{R}^N} |\nabla v|^2 + \frac{2}{2+\gamma}\int_{\mathbb{R}^N} V(x)|v|^2 - \frac{\lambda}{p}\int_{\mathbb{R}^N} |v|^p - \frac{1}{2^*(s)}\int_{\mathbb{R}^N} \frac{|v|^{2^*(s)}}{|x|^s}\\
&\ge C_1\|v\|^2 - C_2\|v\|^p - C_3\|v\|^{2^*(s)}.
\end{aligned}$$

Therefore, we conclude that there is $\rho > 0$ small enough, such that $I(v) > 0$ whenever $\|v\| \le \rho$, $v \neq 0$. And there exists $\eta > 0$ such that for any $\|v\| = \rho$, one has $I(v) \ge \eta > 0$.

Lemma 3.2 *Suppose that the condition* (V) *is satisfied and* $\lambda > 0,\ 0 \le s < 2,\ 2 \le p_N < p < 2^*(s)$. *Then there exists* $e \in E$ *with* $\|e\| > \rho$, *such that* $I(e) < 0$, *where* ρ *is given by Lemma* 3.1.

Proof We choose some $\varphi \in C_0^\infty(\mathbb{R}^N, [0,1])$, with $\operatorname{supp}\varphi = \overline{B}_1$, where $\overline{B}_1$ is the closed unit ball in $\mathbb{R}^N$. We will prove that $I(t\varphi) \to -\infty$ as $t \to \infty$, which will prove the result if we take $e = t\varphi$ with t large enough. In fact, by Lemma 2.1(3), one has

$$\begin{aligned} I(t\varphi) &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla(t\varphi)|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|F^{-1}(t\varphi)|^2 - \frac{\lambda}{p}\int_{\mathbb{R}^N} |F^{-1}(t\varphi)|^p \\ &\quad - \frac{1}{2^*(s)}\int_{\mathbb{R}^N} \frac{|F^{-1}(t\varphi)|^{2^*(s)}}{|x|^s} \\ &\le \frac{t^2}{2}\int_{\mathbb{R}^N} |\nabla\varphi|^2 + \frac{t^2}{2}\int_{\mathbb{R}^N} V(x)|\varphi|^2 - \frac{t^p\lambda}{p}\Big(\sqrt{\frac{2}{2+\gamma}}\Big)^p \int_{\mathbb{R}^N} |\varphi|^p \\ &\quad - \frac{t^{2^*(s)}}{2^*(s)}\Big(\sqrt{\frac{2}{2+\gamma}}\Big)^{2^*(s)} \int_{\mathbb{R}^N} \frac{|\varphi|^{2^*(s)}}{|x|^s} \\ &\to -\infty \quad \text{as } t \to \infty. \end{aligned}$$

Lemma 3.3 *Suppose that* (V) *holds,* $\lambda > 0,\ 0 \le s < 2,\ 2 \le p_N < p < 2^*(s), 0 < \gamma < \gamma^*$. *Then there exists* $\{v_n\} \subset E$ *such that* $I(v_n) \to c, (1 + \|v_n\|)\|I'(v_n)\| \to 0$ *and* $\{v_n\}$ *is bounded in* E.

Proof It follows from Lemmas 3.1–3.2 and 2.4 that, there exists a Cerami sequence $\{v_n\}$ for I. We only need to prove that $\{v_n\}$ is bounded. Let $\{v_n\} \subset E$ be an arbitrary Cerami sequence for I at level $c > 0$, namely

$$\begin{aligned} I(v_n) &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v_n|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|F^{-1}(v_n)|^2 - \frac{\lambda}{p}\int_{\mathbb{R}^N} |F^{-1}(v_n)|^p \\ &\quad - \frac{1}{2^*(s)}\int_{\mathbb{R}^N} \frac{|F^{-1}(v_n)|^{2^*(s)}}{|x|^s} = c + o_n(1), \end{aligned} \tag{3.1}$$

and for any $\varphi \in E$,

$$\begin{aligned} \langle I'(v_n), \varphi\rangle &= \int_{\mathbb{R}^N} \Big(\nabla v_n \cdot \nabla\varphi + V(x)\frac{F^{-1}(v_n)}{f(F^{-1}(v_n))}\varphi\Big) - \lambda\int_{\mathbb{R}^N} \frac{|F^{-1}(v_n)|^{p-1}}{f(F^{-1}(v_n))}\varphi \\ &\quad - \int_{\mathbb{R}^N} \frac{|F^{-1}(v_n)|^{2^*(s)-1}}{|x|^s f(F^{-1}(v_n))}\varphi = o_n(1)\|\varphi\|. \end{aligned}$$

Choosing $\varphi = \varphi_n = F^{-1}(v_n)f(F^{-1}(v_n))$, from Lemma 2.1(1)(3)(6), we get $|\varphi_n| \le C|v_n|$ and

$$|\nabla\varphi_n| = \Big|\Big(1 + \frac{F^{-1}(v_n)f'(F^{-1}(v_n))}{f(F^{-1}(v_n))}\Big)\nabla v_n\Big| \le C|\nabla v_n|.$$

Recalling that $\{v_n\} \subset E$ is a (C) sequence, we get

$$\begin{aligned} pc + o_n(1)\|v_n\| &= pI(v_n) - \langle I'(v_n), \varphi_n \rangle \\ &= \int_{\mathbb{R}^N} \Big(\frac{p-2}{2} - \frac{F^{-1}(v_n) f'(F^{-1}(v_n))}{f(F^{-1}(v_n))} \Big) |\nabla v_n|^2 \\ &\quad + \frac{p-2}{2} \int_{\mathbb{R}^N} V(x) |F^{-1}(v_n)|^2 + \frac{2^*(s) - p}{2^*(s)} \int_{\mathbb{R}^N} \frac{|F^{-1}(v_n)|^{2^*(s)}}{|x|^s} \\ &\geq \int_{\mathbb{R}^N} \Big(\frac{p-2}{2} - \frac{F^{-1}(v_n) f'(F^{-1}(v_n))}{f(F^{-1}(v_n))} \Big) |\nabla v_n|^2 \\ &\quad + \frac{p-2}{2+\gamma} \int_{\mathbb{R}^N} V(x) v_n^2. \end{aligned} \tag{3.2}$$

By Lemma 2.1(6), one has

$$\begin{aligned} \frac{p-2}{2} - \frac{F^{-1}(v_n) f'(F^{-1}(v_n))}{f(F^{-1}(v_n))} &\geq \frac{p-2}{2} - 1 - \frac{4 - 2\sqrt{4+2\gamma}}{\gamma} \\ &:= \frac{p-4}{2} + h(\gamma). \end{aligned}$$

If $p \geq 4, \gamma > 0$, we get $\frac{p-4}{2} \geq 0, h(\gamma) > 0$. Then

$$\frac{p-2}{2} - \frac{F^{-1}(v_n) f'(F^{-1}(v_n))}{f(F^{-1}(v_n))} \geq \frac{p-4}{2} + h(\gamma) \geq 0. \tag{3.3}$$

If $2 < p < 4, 0 < \gamma < \frac{16(p-2)}{(p-4)^2}$, we obtain $\inf\limits_{\gamma > 0} h(\gamma) = \frac{4-p}{2}$. Then

$$\frac{p-2}{2} - \frac{F^{-1}(v_n) f'(F^{-1}(v_n))}{f(F^{-1}(v_n))} \geq \frac{p-4}{2} + h(\gamma) \geq 0. \tag{3.4}$$

Combining (3.2)–(3.3) with (3.4), one gets that $\|v_n\|$ is bounded.

It follows from [8] that the minimization problem

$$S = \inf \Big\{ \int_{\mathbb{R}^N} |\nabla v|^2 : \int_{\mathbb{R}^N} \frac{|v|^{2^*(s)}}{|x|^s} = 1, \ v \in D^{1,2}(\mathbb{R}^N) \Big\}$$

has a solution given by

$$w_\varepsilon(x) = \frac{[(N-s)(N-2)\varepsilon]^{\frac{N-2}{2(2-s)}}}{(\varepsilon + |x|^{2-s})^{\frac{N-2}{2-s}}}.$$

Let $\phi \in C_0^\infty(\mathbb{R}^N, [0,1])$ be a cut-off function satisfying $\phi \equiv 1$ in $B_R(0)$, $\phi \equiv 0$ in $\mathbb{R}^N \backslash B_{2R}(0)$, $v_\varepsilon = \phi w_\varepsilon$. We follow the strategy used in [1, 3, 8] to get the following estimates:

$$\int_{\mathbb{R}^N} |\nabla v_\varepsilon|^2 = S^{\frac{N-s}{2-s}} + O(\varepsilon^{\frac{N-2}{2-s}}), \tag{3.5}$$

$$\int_{\mathbb{R}^N} \frac{|v_\varepsilon|^{2^*(s)}}{|x|^s} = S^{\frac{N-s}{2-s}} + O(\varepsilon^{\frac{N-s}{2-s}}), \tag{3.6}$$

$$\int_{\mathbb{R}^N} |v_\varepsilon|^2 \leq O(\varepsilon^{\frac{N-2}{2-s}} |\ln \varepsilon|), \tag{3.7}$$

$$K_1 \varepsilon^{\frac{2N - p(N-2)}{2(2-s)}} \leq \int_{\mathbb{R}^N} |v_\varepsilon|^p \leq K_2 \varepsilon^{\frac{2N - p(N-2)}{2(2-s)}}, \quad \frac{N}{N-2} < p < 2^*, \tag{3.8}$$

where $K_1, \ K_2 > 0$ are constants.

Lemma 3.4 *The minimax level c satisfies*

$$c < \frac{2-s}{2(N-s)}\Big(\frac{2+\gamma}{2}\Big)^{\frac{N-s}{2-s}} S^{\frac{N-s}{2-s}}.$$

Proof It suffices to show that there exists $v_0 \in E, v_0 \neq 0$ such that

$$\max_{t\geq 0} I(tv_0) < \frac{2-s}{2(N-s)}\Big(\frac{2+\gamma}{2}\Big)^{\frac{N-s}{2-s}} S^{\frac{N-s}{2-s}}.$$

Since $\lim\limits_{t\to\infty} I(tv_\varepsilon) = -\infty$ and $I(tv_\varepsilon) > 0$ for $t > 0$ small enough, there exists $t_\varepsilon > 0$ such that $I(t_\varepsilon v_\varepsilon) = \max\limits_{t\geq 0} I(tv_\varepsilon)$. We claim that there are constants $T_1,\ T_2$ such that $0 < T_1 \leq t_\varepsilon \leq T_2$. First, we prove that t_ε is bounded from below by a positive constant. Otherwise, if $t_\varepsilon \to 0$ as $\varepsilon \to 0$, we have $t_\varepsilon v_\varepsilon \to 0$. Therefore, $0 < c \leq \max\limits_{t\geq 0} I(tv_\varepsilon) \to 0$, which is a contradiction. On the other hand, if $t_\varepsilon \to +\infty$ as $\varepsilon \to 0$, it follows from Lemma 2.1(3) that

$$\begin{aligned} I(t_\varepsilon v_\varepsilon) &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla(t_\varepsilon v_\varepsilon)|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|F^{-1}(t_\varepsilon v_\varepsilon)|^2 - \frac{\lambda}{p}\int_{\mathbb{R}^N} |F^{-1}(t_\varepsilon v_\varepsilon)|^p \\ &\quad - \frac{1}{2^*(s)}\int_{\mathbb{R}^N} \frac{|F^{-1}(t_\varepsilon v_\varepsilon)|^{2^*(s)}}{|x|^s} \\ &\leq \frac{t_\varepsilon^2}{2}\int_{\mathbb{R}^N} |\nabla v_\varepsilon|^2 + \frac{t_\varepsilon^2}{2}\int_{\mathbb{R}^N} V(x)|v_\varepsilon|^2 - \frac{t_\varepsilon^p\lambda}{p}\Big(\sqrt{\frac{2}{2+\gamma}}\Big)^p \int_{\mathbb{R}^N} |v_\varepsilon|^p \\ &\quad - \frac{t_\varepsilon^{2^*(s)}}{2^*(s)}\Big(\sqrt{\frac{2}{2+\gamma}}\Big)^{2^*(s)} \int_{\mathbb{R}^N} \frac{|v_\varepsilon|^{2^*(s)}}{|x|^s} \\ &\to -\infty \quad \text{as } t_\varepsilon \to +\infty. \end{aligned}$$

Then we have $0 < c \leq \max\limits_{t\geq 0} I(tv_\varepsilon) = I(t_\varepsilon v_\varepsilon) \to -\infty$ as $t_\varepsilon \to +\infty$, which is a contradiction. Hence there is $T_2 > 0$ such that $t_\varepsilon \leq T_2$ for ε small enough.

Now, by Lemma 2.1(3) and (3.5)–(3.8), we observe that

$$\begin{aligned} I(t_\varepsilon v_\varepsilon) &\leq \frac{t_\varepsilon^2}{2}\int_{\mathbb{R}^N} |\nabla v_\varepsilon|^2 + \frac{t_\varepsilon^2}{2}\int_{\mathbb{R}^N} V(x)|v_\varepsilon|^2 - \frac{t_\varepsilon^p\lambda}{p}\Big(\sqrt{\frac{2}{2+\gamma}}\Big)^p \int_{\mathbb{R}^N} |v_\varepsilon|^p \\ &\quad - \frac{t_\varepsilon^{2^*(s)}}{2^*(s)}\Big(\sqrt{\frac{2}{2+\gamma}}\Big)^{2^*(s)} \int_{\mathbb{R}^N} \frac{|v_\varepsilon|^{2^*(s)}}{|x|^s} \\ &\leq \frac{t_\varepsilon^2}{2}[S^{\frac{N-s}{2-s}} + O(\varepsilon^{\frac{N-2}{2-s}})] + O(\varepsilon^{\frac{N-2}{2-s}}|\ln\varepsilon|) - O(\varepsilon^{\frac{2N-p(N-2)}{2(2-s)}}) \\ &\quad - \frac{t_\varepsilon^{2^*(s)}}{2^*(s)}\Big(\sqrt{\frac{2}{2+\gamma}}\Big)^{2^*(s)}[S^{\frac{N-s}{2-s}} + O(\varepsilon^{\frac{N-s}{2-s}})] \\ &= \Big[\frac{t_\varepsilon^2}{2} - \frac{t_\varepsilon^{2^*(s)}}{2^*(s)}\Big(\sqrt{\frac{2}{2+\gamma}}\Big)^{2^*(s)}\Big] S^{\frac{N-s}{2-s}} + O(\varepsilon^{\frac{N-2}{2-s}}) \\ &\quad + O(\varepsilon^{\frac{N-2}{2-s}}|\ln\varepsilon|) - O(\varepsilon^{\frac{2N-p(N-2)}{2(2-s)}}) - O(\varepsilon^{\frac{N-s}{2-s}}). \end{aligned}$$

Denote

$$h(t) := \frac{t^2}{2} - \frac{t^{2^*(s)}}{2^*(s)}\Big(\sqrt{\frac{2}{2+\gamma}}\Big)^{2^*(s)}.$$

It is very standard to get that $h(t)$ achieves its maximum at

$$t_0 = \Big(\sqrt{\frac{2+\gamma}{2}}\Big)^{\frac{2^*(s)}{2^*(s)-2}}$$

and

$$h(t_0) = \frac{2-s}{2(N-s)}\Big(\frac{2+\gamma}{2}\Big)^{\frac{N-s}{2-s}}. \tag{3.9}$$

It follows from (3.9) that

$$\begin{aligned} I(t_\varepsilon v_\varepsilon) &\le \Big[\frac{t_\varepsilon^2}{2} - \frac{t_\varepsilon^{2^*(s)}}{2^*(s)}\Big(\sqrt{\frac{2}{2+\gamma}}\Big)^{2^*(s)}\Big] S^{\frac{N-s}{2-s}} + O(\varepsilon^{\frac{N-2}{2-s}}) \\ &\quad + O(\varepsilon^{\frac{N-2}{2-s}}|\ln\varepsilon|) - O(\varepsilon^{\frac{2N-p(N-2)}{2(2-s)}}) - O(\varepsilon^{\frac{N-s}{2-s}}) \\ &\le \frac{2-s}{2(N-s)}\Big(\frac{2+\gamma}{2}\Big)^{\frac{N-s}{2-s}} S^{\frac{N-s}{2-s}} + O(\varepsilon^{\frac{N-2}{2-s}}) \\ &\quad + O(\varepsilon^{\frac{N-2}{2-s}}|\ln\varepsilon|) - O(\varepsilon^{\frac{2N-p(N-2)}{2(2-s)}}) - O(\varepsilon^{\frac{N-s}{2-s}}). \end{aligned}$$

Since $\frac{2N-p(N-2)}{2(2-s)} < \frac{N-2}{2-s} < \frac{N-s}{2-s}$ when $p_N < p < 2^*(s)$, we can get our result.

In order to complete the proof of Theorem 1.1, we must show that v is non-trivial. To prove this, we need the following result.

Lemma 3.5 *Assume that $\{v_n\}$ is a $(C)_c$ sequence for I with*

$$c < \frac{2-s}{2(N-s)}\Big(\frac{2+\gamma}{2}\Big)^{\frac{N-s}{2-s}} S^{\frac{N-s}{2-s}},$$

and $v_n \rightharpoonup 0$ in E. Then there exists a sequence $\{y_n\} \subset \mathbb{R}^N$ and $r, \eta > 0$ such that $|y_n| \to \infty$ and

$$\limsup_{n\to\infty} \int_{B_r(y_n)} v_n^2 \mathrm{d}x \ge \eta > 0.$$

Proof Suppose that the conclusion is not true, then it follows from the Lions lemma (see [16, Lemma 1.21]) that $v_n \to 0$ in $L^q(\mathbb{R}^N)$ for all $2 < q < 2^*$. By Lemma 2.2, one has $v_n \to 0$ in $L^\alpha(\mathbb{R}^N, |x|^{-s})(2 \le \alpha < 2^*(s))$. Since $\{v_n\} \subset E$ is a Cerami sequence for I at level $c > 0$, we have

$$\begin{aligned} I(v_n) &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v_n|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|F^{-1}(v_n)|^2 \\ &\quad - \frac{\lambda}{p}\int_{\mathbb{R}^N} |F^{-1}(v_n)|^p - \frac{1}{2^*(s)}\int_{\mathbb{R}^N} \frac{|F^{-1}(v_n)|^{2^*(s)}}{|x|^s} \\ &= c + o_n(1) \end{aligned} \tag{3.10}$$

and

$$\langle I'(v_n), v_n\rangle = \int_{\mathbb{R}^N} \Big(|\nabla v_n|^2 + V(x)\frac{F^{-1}(v_n)}{f(F^{-1}(v_n))}v_n\Big)$$

$$
\begin{aligned}
&- \int_{\mathbb{R}^N} \lambda \frac{|F^{-1}(v_n)|^{p-1}}{f(F^{-1}(v_n))} v_n - \int_{\mathbb{R}^N} \frac{|F^{-1}(v_n)|^{2^*(s)-1}}{|x|^s f(F^{-1}(v_n))} v_n \\
&= o_n(1). \qquad (3.11)
\end{aligned}
$$

Setting

$$l = \lim_{n\to\infty} \int_{\mathbb{R}^N} (|\nabla v_n|^2 + V(x)|F^{-1}(v_n)|^2)$$

and

$$m = \lim_{n\to\infty} \int_{\mathbb{R}^N} \frac{|v_n|^{2^*(s)}}{|x|^s}.$$

Then by Lemma 2.3 and (3.10)–(3.11), we get

$$\frac{l}{2} - \Big(\sqrt{\frac{2}{2+\gamma}}\Big)^{2^*(s)} \frac{m}{2^*(s)} = c, \quad l = \Big(\sqrt{\frac{2}{2+\gamma}}\Big)^{2^*(s)} m. \qquad (3.12)$$

Observing that

$$S\Big(\int_{\mathbb{R}^N} \frac{|v_n|^{2^*(s)}}{|x|^s}\Big)^{\frac{2}{2^*(s)}} \le \int_{\mathbb{R}^N} |\nabla v_n|^2 \le \int_{\mathbb{R}^N} (|\nabla v_n|^2 + V(x)|F^{-1}(v_n)|^2),$$

we get $Sm^{\frac{2}{2^*(s)}} \le l$. Therefore, by (3.12), we get

$$c \ge \frac{2-s}{2(N-s)}\Big(\frac{2+\gamma}{2}\Big)^{\frac{N-s}{2-s}} S^{\frac{N-s}{2-s}},$$

which contradicts the assumption

$$c < \frac{2-s}{2(N-s)}\Big(\frac{2+\gamma}{2}\Big)^{\frac{N-s}{2-s}} S^{\frac{N-s}{2-s}}.$$

Proof of Theorem 1.1 From Lemma 3.3, I has a bounded $(C)_c$ sequence $\{v_n\} \subset E$. We may get, up to a subsequence, $v_n \rightharpoonup v$ in E. Then, by Lemma 2.2, $v_n \to v$ in $L^q(\mathbb{R}^N)$ $(2 \le q < 2^*)$, $v_n \to v$ in $L^\alpha(\mathbb{R}^N, |x|^{-s})$ $(0 \le s < 2,\ 2 \le \alpha < 2^*(s))$ and $v_n(x) \to v(x)$ a.e. in $\mathbb{R}^N$. For any $\phi \in C_0^\infty(\mathbb{R}^N)$, one has

$$0 = \langle I'(v_n), \phi\rangle + o(1) = \langle I'(v), \phi\rangle,$$

i.e., v is a critical point of I. Moreover, by Lemma 3.5, there exists a constant $\eta > 0$ such that

$$\int_{\mathbb{R}^N} v^2 \mathrm{d}x = \lim_{n\to\infty} \int_{\mathbb{R}^N} v_n^2 \mathrm{d}x \ge \eta > 0,$$

which implies that $v \ne 0$.

Finally, we try to find the ground state solution. We give another equation

$$-\mathrm{div}\Big[\Big(1 + \frac{\gamma u^2}{2(1+u^2)}\Big)\nabla u\Big] + V(x)u + \frac{\gamma u|\nabla u|^2}{2(1+u^2)^2} = \lambda|u|^{p-2}u + \frac{|u|^{2^*(s)-2}u}{|x|^s}. \qquad (3.13)$$

The calculation shows that (1.5) and (3.13) are equivalent. If u is a solution of problem (1.5), then it is also a solution of (3.13), and it should satisfy

$$\begin{aligned}\int_{\mathbb{R}^N}\Big[\Big(1+\frac{\gamma u^2}{2(1+u^2)}\Big)\nabla u\cdot\nabla\varphi+\frac{\gamma u}{2(1+u^2)^2}|\nabla u|^2\varphi\\+V(x)u\varphi-\lambda|u|^{p-1}\varphi-\frac{|u|^{2^*(s)-1}}{|x|^s}\varphi\Big]=0\end{aligned}\tag{3.14}$$

for all $\varphi\in C_0^\infty(\mathbb{R}^N)$. We claim that (3.14) is equivalent to the following equality

$$\begin{aligned}\langle I'(v),\psi\rangle=\int_{\mathbb{R}^N}\Big(\nabla v\cdot\nabla\psi+V(x)\frac{F^{-1}(v)}{f(F^{-1}(v))}\psi-\lambda\frac{|F^{-1}(v)|^{p-1}}{f(F^{-1}(v))}\psi\\-\frac{|F^{-1}(v)|^{2^*(s)-1}}{|x|^sf(F^{-1}(v))}\psi\Big)=0,\quad\psi\in C_0^\infty(\mathbb{R}^N).\end{aligned}\tag{3.15}$$

In fact, if we let $\varphi=\frac{\psi}{f(u)}$ in (3.14), we get (3.15). Since $u=F^{-1}(v)$, if we choose $\psi=f(u)\varphi$ in (3.15), we obtain (3.14). Moreover, from Lemma 2.1, we can deduce that if $\psi\in C_0^\infty(\mathbb{R}^N),u\in H^1(\mathbb{R}^N)$, then $\varphi=\frac{\psi}{f(u)}\in H^1(\mathbb{R}^N)$. Similarly, if $\varphi\in C_0^\infty(\mathbb{R}^N),u\in H^1(\mathbb{R}^N)$, then $\psi=f(u)\varphi\in H^1(\mathbb{R}^N)$. Hence, the equivalence of (3.14)–(3.15) can be obtained by the denseness. Therefore, in order to find the ground state solutions of problem (1.5), it suffices to study the existence of ground state solutions of the following equation

$$-\triangle v+V(x)\frac{F^{-1}(v)}{f(F^{-1}(v))}=\lambda\frac{|F^{-1}(v)|^{p-1}}{f(F^{-1}(v))}+\frac{|F^{-1}(v)|^{2^*(s)-1}}{|x|^sf(F^{-1}(v))},\quad x\in\mathbb{R}^N.$$

Namely, we only need to find the critical point of I with the least energy. Let

$$m=\inf\{I(u):u\in E,u\neq0,I'(u)=0\}.$$

By the definition of m, there exists $\{w_n\}\subset E$ such that $w_n\neq0,I(w_n)\to m$ and $I'(w_n)=0$. Similar to the proof of Lemma 3.3, we have $\|w_n\|\le C$. Following the same lines as the proof of Theorem 1.1, we have $w_n\rightharpoonup w$ in E, $w\neq0$ and $I'(w)=0$. Hence, by the Fatou lemma, we get

$$\begin{aligned}pm&\le pI(w)-\langle I'(w),F^{-1}(w)f(F^{-1}(w))\rangle\\&=\int_{\mathbb{R}^N}\Big(\frac{p-2}{2}-\frac{F^{-1}(w)f'(F^{-1}(w))}{f(F^{-1}(w))}\Big)|\nabla w|^2\\&\quad+\frac{p-2}{2}\int_{\mathbb{R}^N}V(x)|F^{-1}(w)|^2+\frac{2^*(s)-p}{2^*(s)}\int_{\mathbb{R}^N}\frac{|F^{-1}(w)|^{2^*(s)}}{|x|^s}\\&\le\liminf_{n\to\infty}(pI(w_n)-\langle I'(w_n),F^{-1}(w_n)f(F^{-1}(w_n))\rangle)\\&=pm.\end{aligned}$$

Therefore $w\neq0$ satisfies $I(w)=m$ and $I'(w)=0$. The proof is complete.

Now, we are going to prove Theorem 1.2. In the following, we regard $\gamma>0$ as a parameter in (1.5). We shall analyse the convergence property of u_γ as $\gamma\to0^+$.

Proof of Theorem 1.2 Suppose that v_{γ_n} is the ground state solution of (1.5) with $\gamma = \gamma_n$ in Theorem 1.1, and $\gamma_n \to 0^+$ as $n \to \infty$. Then, $I(v_{\gamma_n}) = J(u_{\gamma_n}) = m_{\gamma_n}$ and $I'(v_{\gamma_n}) = 0$, where $u_{\gamma_n} = F^{-1}(v_{\gamma_n})$,

$$\begin{aligned} m_{\gamma_n} &= \inf\{J_{\gamma_n}(u) : u \in E, u \neq 0, u \text{ satisfies (3.14) with } \gamma = \gamma_n\}, \\ J_{\gamma_n}(u) &= \frac{1}{2}\int_{\mathbb{R}^N} \Big[\Big(1 + \frac{\gamma_n u^2}{2(1+u^2)}\Big)|\nabla u|^2\Big] + \frac{1}{2}\int_{\mathbb{R}^N} V(x)u^2 \\ &\quad - \frac{\lambda}{p}\int_{\mathbb{R}^N} |u|^p - \frac{1}{2^*(s)}\int_{\mathbb{R}^N} \frac{|u|^{2^*(s)}}{|x|^s}. \end{aligned}$$

The variational function associated with (1.6) is defined by

$$J_0(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)|u|^2) - \frac{\lambda}{p}\int_{\mathbb{R}^N} |u|^p - \frac{1}{2^*(s)}\int_{\mathbb{R}^N} \frac{|u|^{2^*(s)}}{|x|^s}.$$

Define

$$m_0 = \inf\{J_0(u) : u \in E, u \neq 0, J_0'(u) = 0\}.$$

Then, as $\gamma_n \to 0$, we have

$$J_{\gamma_n}(u) \to J_0(u), \quad I(v_{\gamma_n}) = J(u_{\gamma_n}) = m_{\gamma_n} \to m_0,$$

and for any $\varphi \in C_0^\infty(\mathbb{R}^N)$, one has

$$\begin{aligned} 0 &= \langle I'(v_{\gamma_n}), f(F^{-1}(v_{\gamma_n}))\varphi\rangle \\ &= \int_{\mathbb{R}^N} (\nabla u_{\gamma_n} \cdot \nabla\varphi + V(x)u_{\gamma_n}\varphi) \\ &\quad - \gamma_n \int_{\mathbb{R}^N} \Big(\frac{u_{\gamma_n}}{2(1+u_{\gamma_n}^2)^2}|\nabla u_{\gamma_n}|^2\varphi + \frac{u_{\gamma_n}^2}{2(1+u_{\gamma_n}^2)}\nabla u_{\gamma_n}\nabla\varphi\Big) \\ &\quad - \int_{\mathbb{R}^N} \Big(\lambda|u_{\gamma_n}|^{p-2}u_{\gamma_n}\varphi + \frac{|u_{\gamma_n}|^{2^*(s)-2}}{|x|^s}u_{\gamma_n}\varphi\Big) \\ &\to \int_{\mathbb{R}^N} (\nabla u_{\gamma_n} \cdot \nabla\varphi + V(x)u_{\gamma_n}\varphi) \\ &\quad - \int_{\mathbb{R}^N} \Big(\lambda|u_{\gamma_n}|^{p-2}u_{\gamma_n}\varphi + \frac{|u_{\gamma_n}|^{2^*(s)-2}}{|x|^s}u_{\gamma_n}\varphi\Big) \\ &= \langle J_0'(u_{\gamma_n}), \varphi\rangle. \end{aligned}$$

From Lemma 3.4 with $\gamma = 0$, one gets

$$m_0 < \frac{2-s}{2(N-s)} S^{\frac{N-s}{2-s}} := \widehat{c}. \tag{3.16}$$

Setting $u_n := u_{\gamma_n}$, then we have $J_0(u_n) \to m_0 < \widehat{c}$ and $J_0'(u_n) \to 0$ as $n \to \infty$. Next, we only need to prove that J_0 satisfies the $(PS)_{m_0}$ condition.

Similarly to the proof of Lemma 3.3, we can get that the sequence $\{v_{\gamma_n}\}$ is bounded. From Lemma 2.1(3), we have

$$\|u_n\| = \|F^{-1}(v_{\gamma_n})\| \le C\|v_{\gamma_n}\| \le C_1, \tag{3.17}$$

which implies that $\{u_n\}$ is bounded in E. Passing to a subsequence, we may assume that $u_n \rightharpoonup u_0$ in E. Then, by Lemma 2.2, $u_n \to u_0$ in $L^{\alpha}(\mathbb{R}^N, |x|^{-s})$ $(0 \le s < 2,\ 2 \le \alpha < 2^*(s))$ and $u_n(x) \to u_0(x)$ a.e. in $\operatorname{supp}\varphi$, $\varphi \in C_0^{\infty}(\mathbb{R}^N)$. Since $\{u_n\}$ is bounded in $L^{2^*(s)}(\mathbb{R}^N, |x|^{-s})$, $\{|u_n|^{2^*(s)-1}\}$ is bounded in $L^{\frac{2^*(s)}{2^*(s)-1}}(\mathbb{R}^N, |x|^{-s})$. It follows that

$$u_n^{2^*(s)-1} \rightharpoonup u_0^{2^*(s)-1} \quad \text{in } L^{\frac{2^*(s)}{2^*(s)-1}}(\mathbb{R}^N, |x|^{-s}),$$

and so

$$\begin{aligned}
-\triangle u_0 + V(x)u_0 = \lambda |u_0|^{p-2}u_0 + \frac{|u_0|^{2^*(s)-2}}{|x|^s}u_0,& \\
J_0(u_0) &= \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u_0|^2 + V(x)u_0^2) - \frac{\lambda}{p}\int_{\mathbb{R}^N}|u_0|^p - \frac{1}{2^*(s)}\int_{\mathbb{R}^N}\frac{|u_0|^{2^*(s)}}{|x|^s} \\
&= \Big(\frac{1}{2} - \frac{1}{p}\Big)\lambda\int_{\mathbb{R}^N}|u_0|^p + \Big(\frac{1}{2} - \frac{1}{2^*(s)}\Big)\int_{\mathbb{R}^N}\frac{|u_0|^{2^*(s)}}{|x|^s} \\
&\ge 0.
\end{aligned} \tag{3.18}$$

We write $w_n := u_n - u_0$, the Brezis-Lieb lemma leads to

$$\|u_n\|^2 = \|w_n\|^2 + \|u_0\|^2 + o_n(1) \tag{3.19}$$

and

$$\int_{\mathbb{R}^N}\frac{|u_n|^{2^*(s)}}{|x|^s} = \int_{\mathbb{R}^N}\frac{|w_n|^{2^*(s)}}{|x|^s} + \int_{\mathbb{R}^N}\frac{|u_0|^{2^*(s)}}{|x|^s} + o_n(1). \tag{3.20}$$

Since $u_n \to u_0$ in $L^p(\mathbb{R}^N)$ $(2 \le p < 2^*)$, we obtain

$$\int_{\mathbb{R}^N}(|u_n|^p - |u_0|^p) = o_n(1). \tag{3.21}$$

Then from (3.19)–(3.21), we get

$$\begin{aligned}
J_0(u_n) - J_0(u_0) &= \frac{1}{2}(\|u_n\|^2 - \|u_0\|^2) - \frac{\lambda}{p}\int_{\mathbb{R}^N}(|u_n|^p - |u_0|^p) \\
&\quad - \frac{1}{2^*(s)}\int_{\mathbb{R}^N}\Big(\frac{|u_n|^{2^*(s)}}{|x|^s} - \frac{|u_0|^{2^*(s)}}{|x|^s}\Big) \\
&= \frac{1}{2}\|w_n\|^2 - \frac{1}{2^*(s)}\int_{\mathbb{R}^N}\frac{|w_n|^{2^*(s)}}{|x|^s} + o_n(1)
\end{aligned} \tag{3.22}$$

and

$$\begin{aligned}
o_n(1) &= \langle J_0'(u_n), u_n\rangle - \langle J_0'(u_0), u_0\rangle \\
&= \|u_n\|^2 - \|u_0\|^2 - \lambda\int_{\mathbb{R}^N}(|u_n|^p - |u_0|^p) \\
&\quad - \int_{\mathbb{R}^N}\Big(\frac{|u_n|^{2^*(s)}}{|x|^s} - \frac{|u_0|^{2^*(s)}}{|x|^s}\Big)
\end{aligned}$$

$$= \|w_n\|^2 - \int_{\mathbb{R}^N} \frac{|w_n|^{2^*(s)}}{|x|^s} + o_n(1).$$

We may therefore assume that

$$\|w_n\|^2 \to b, \quad \int_{\mathbb{R}^N} \frac{|w_n|^{2^*(s)}}{|x|^s} \to b.$$

Then it follows from (3.18) and (3.22) that

$$m_0 \geq \Big(\frac{1}{2} - \frac{1}{2^*(s)}\Big) b. \tag{3.23}$$

By the definition of S, we can get that

$$S\Big(\int_{\mathbb{R}^N} \frac{|w_n|^{2^*(s)}}{|x|^s}\Big)^{\frac{2}{2^*(s)}} \leq \int_{\mathbb{R}^N} |\nabla w_n|^2 \leq \|w_n\|^2,$$

namely, $Sb^{\frac{2}{2^*(s)}} \leq b$. Either $b = 0$ or $b \geq S^{\frac{N-s}{2-s}}$. If $b = 0$, the proof is complete. Assume $b \geq S^{\frac{N-s}{2-s}}$, it follows from (3.23) that

$$m_0 \geq \Big(\frac{1}{2} - \frac{1}{2^*(s)}\Big) b \geq \frac{2-s}{2(N-s)} S^{\frac{N-s}{2-s}} = \widehat{c},$$

which is contrary to (3.16). This completes the proof of Theorem 1.2.

Acknowledgement The authors would like to express sincere thanks to the referees and the handling editor.

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