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# Nonlinear Schrödinger Approximation for the Electron Euler-Poisson Equation*

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**Abstract** The nonlinear Schrödinger (NLS for short) equation plays an important role in describing slow modulations in time and space of an underlying spatially and temporarily oscillating wave packet. In this paper, the authors study the NLS approximation by providing rigorous error estimates in Sobolev spaces for the electron Euler-Poisson equation, an important model to describe Langmuir waves in a plasma. They derive an approximate wave packet-like solution to the evolution equations by the multiscale analysis, then they construct the modified energy functional based on the quadratic terms and use the rotating coordinate transform to obtain uniform estimates of the error between the true and approximate solutions.

**Keywords** Modulation approximation, Nonlinear Schrödinger equation, Electron Euler-Poisson equation
**2000 MR Subject Classification** 35M20, 35Q35

## 1 Introduction

In the current paper, we consider the NLS approximation for the amplitude of the electron oscillation in the following one-dimensional Euler-Poisson system

$$\begin{cases} n_t + (nv)_x = 0, & (1.1a)\\ v_t + vv_x + \dfrac{1}{m_e n}p(n)_x = \dfrac{e}{m_e}\psi_x, & (1.1b)\\ \psi_{xx} = 4\pi e(n - n_0), & (1.1c)\end{cases}$$

where $n$ denotes the density of electrons, $v$ denotes the velocity field of electrons, and electric field $\psi_x$ satisfies the linear Poisson equation (1.1c). These unknown functions are defined for $(t,x) \in \mathbb{R}\times\mathbb{R}$. Constants $e$, $m_e$ and $n_0$ represent the electrons of charge, mass and the average charge of an ion background, respectively. The electron Euler-Poisson system (1.1) is an important model for describing rich and complex dynamics of electrons in a plasma, in which the ions cannot follow the rapid fluctuation of the fluid due to the greater inertia and hence only provide a background of positive charge with uniform density $n_0$.

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The NLS type equation has been derived formally for the electron Euler-Poisson system (1.1) by multiple scaling analysis as early as 1974 (see [11]). As a modulation equation that describes slow modulation in time and space of the envelope of a temporally and spatially oscillating wave packet, the NLS equation is a completely integrable Hamiltonian system and can be explicitly solved with the help of inverse scattering schemes (see [1]). In this paper, we are dedicated to proving the NLS approximation of system (1.1) mathematically rigorously. For the sake of simplicity, we set all the physical constants $m_e = e = n_0 = 4\pi = 1$ and assume the pressure function $p(n) = \frac{1}{3}n^3$ in the following.

To obtain formally the NLS equation for describing the slow modulations in time and in space of the wave train $e^{i(k_0x+\omega_0t)}$ around the constant state $(1, 0)$, we set

$$\begin{pmatrix} n-1 \\ v \end{pmatrix} = \varepsilon\Psi_{NLS} + \mathcal{O}(\varepsilon^2) \tag{1.2}$$

with

$$\varepsilon\Psi_{NLS} = \varepsilon A(\varepsilon(x + c_gt), \varepsilon^2t)e^{i(k_0x+\omega_0t)}\phi(k_0) + c.c., \tag{1.3}$$

where $0 < \varepsilon \ll 1$ is a small perturbation parameter, $A$ is the complex-valued amplitude, $\phi(k_0) \in \mathbb{C}^2$ is chosen as the eigenvector, a wave packet of the form $e^{i(k_0x+\omega_0t)}$ is used in the approximation, $c_g$ is the group velocity and 'c.c.' stands for the complex conjugate, the basic temporal wave number $\omega_0 > 0$ is associated to the basic spatial wave number $k_0 > 0$ by the underlying temporally and spatially oscillating wave train $e^{i(k_0x+\omega_0t)}$. We obtain the following NLS equation for $A$ by inserting (1.2) with (1.3) to system (1.1),

$$\partial_T A = i\nu_1\partial_X^2 A + i\nu_2 A|A|^2, \tag{1.4}$$

where $X = \varepsilon(x + c_gt) \in \mathbb{R}$ is the slow spatial scale, $T = \varepsilon^2t \in \mathbb{R}$ is the slow time scale, and coefficients $\nu_j = \nu_j(k_0) \in \mathbb{R}$ with $j \in \{1, 2\}$. The time and space scales of the modulations are $\mathcal{O}\big(\frac{1}{\varepsilon^2}\big)$ and $\mathcal{O}\big(\frac{1}{\varepsilon}\big)$, respectively. For the electron Euler-Poisson system (1.1), the basic spatial wave number $k = k_0$ and the basic temporal wave number $\omega = \omega_0$ satisfy the following linear dispersion relation

$$\omega(k) = \operatorname{sgn}(k)\sqrt{1 + k^2}, \tag{1.5}$$

where $\operatorname{sgn}(k)$ denotes the sign function. The group velocity $c_g = \frac{\partial w}{\partial k}(k_0)$ can be found for the wave packet. If we replace $\omega_0$ and $c_g$ with $-\omega_0$ and $-c_g$ in (1.3), our ansatz makes waves moving to the left becomes to one moving to the right.

Our main result of this paper is as follows.

**Theorem 1.1** *Fix $s_A \geq 6$. Then for all $k_0 \neq 0$ and for all $C_1$, $T_0 > 0$, there exist $C_2 > 0$, $\varepsilon_0 > 0$ such that for all solutions $A \in C([0, T_0], H^{s_A}(\mathbb{R}, \mathbb{C}))$ of the NLS equation* (1.4) *with*

$$\sup_{T\in[0,T_0]} \|A(\cdot, T)\|_{H^{s_A}(\mathbb{R},\mathbb{C})} \leq C_1,$$

*the following holds: For all $\varepsilon \in (0, \varepsilon_0)$, there are solutions*

$$\begin{pmatrix} n-1 \\ v \end{pmatrix} \in \Big(C\Big(\Big[0, \frac{T_0}{\varepsilon^2}\Big], H^{s_A}(\mathbb{R}, \mathbb{R})\Big)\Big)^2$$

*of system* (1.6) *which satisfy*

$$\sup_{t\in[0,\frac{T_0}{\varepsilon^2}]}\left\|\begin{pmatrix} n-1 \\ v \end{pmatrix} - \varepsilon\Psi_{NLS}(\cdot,t)\right\|_{H^{s_A}(\mathbb{R},\mathbb{R})^2} \le C_2\varepsilon^{\frac{3}{2}},$$

*where* $\phi(k_0) = \begin{pmatrix} \mathrm{i}k_0 \\ -\mathrm{i}\omega(k_0) \end{pmatrix}$.

**Remark 1.1** It is noted that the error of order $\mathcal{O}(\varepsilon^{\frac{3}{2}})$ is small enough compared with the solution $(n-1, v)$ and the approximation $\varepsilon\Psi_{NLS}$, which are both of order $\mathcal{O}(\varepsilon)$ in $L^\infty$ such that the dynamics of the NLS equation can be found in system (1.6). In addition, the smoothness of the error bound is the same as the smoothness of the amplitude $A$. This can be achieved by applying a modified approximation that has compact support in Fourier space but differs only slightly from $\varepsilon\Psi_{NLS}$, based on the fact that the Fourier transform of $\varepsilon\Psi_{NLS}$ is sufficiently strongly concentrated around the wave numbers $\pm k_0$.

Before progressing, we would like to draw attention to some literature on the global existence of the electron Euler-Poisson system (1.1). The global existence of solutions with small amplitude in all physical dimensions to the electron Euler-Poisson system (1.1) has been obtained in the past decades. Guo [5] firstly constructed global irrotational solutions with small velocity by the Klein-Gordon effect for the three-dimensional electron fluid. For the two-dimensional electron Euler-Poisson system, Ionescu and Pausader [8] proved that small smooth perturbations exist globally in the constant background. Jang [9] obtained the global existence of smooth solutions with small amplitude and spherical symmetry initial data. Moreover, Jang, Li and Zhang [10] constructed the global smooth solutions, and Li and Wu [16] solved the Cauchy problem by constructing the wave operators for the two-dimensional electron Euler-Poisson system. Finally, Guo, Han and Zhang [6] obtained the global existence of solutions with no shocks for the one-dimensional electron Euler-Poisson system (1.1) for $p(n) \sim n^3$.

We would also like to draw attention to some recent results on the NLS approximation for nonlinear dispersive systems. The first NLS approximation result for extended systems with cubic nonlinearities was shown by using the Gronwall's inequality directly (see [14]), in which the quadratic term does not appear. In the case of semilinear quadratic terms and the eigenvalue of the linearized problem satisfies a non-resonance condition, the NLS approximation can be obtained by applying a normal-form transform (see [12]). When the quasilinear quadratic terms occur in the original dispersive system, it is a highly nontrivial problem to prove rigorously the NLS approximation due to the emergence of resonances and the loss of derivatives. In the process of the NLS approximation of the long time scale, the quasilinear quadratic terms can be eliminated in the following two cases at present. One case is that some special transforms are used to eliminate the quadratic terms in the process of modulation approximation such as the water wave problem without surface tension and infinite depth by finding special transforms adapted to the special structure of these problems (see [21–22]), and the Korteweg-de Vries equation by applying a Miura transformation (see [17]). The other case is to use the normal-form transform to eliminate the quadratic terms directly or construct the energy functional. If the quadratic term loses only half a derivative in quasilinear terms of the dispersive system, then the transformed system loses only one derivative and the NLS approximation can be handled with the help of the normal-form transform and Cauchy-Kowalevskaya theorem, such as in [4, 18–19]. If the quasilinear quadratic term loses one derivative and hence makes the transformed system lose two derivatives, the quadratic terms are removed by constructing a new modified energy functional with normal-form transforms, such as the NLS approximation for a quasilinear Klein-Gordon equation without resonances (see [2]) and a quasilinear dispersive scalar equation (see [3]). Very recently, by using the normal-form transform to eliminate the low-frequency parts and defining new energy to handle the high-frequency parts, the authors of [15] obtained the NLS approximation for the ion Euler-Poisson system, where the quadratic nonlinearity loses one derivative and resonances occur.

The NLS approximation for the electron Euler-Poisson equation (1.1) studied in this paper is different from the results mentioned above. Firstly, the quadratic term of (1.1) loses one derivative but not half a derivative, thus the Cauchy-Kowalevskaya theorem used in the water wave system (see [4, 18–19]) is no longer suitable for this situation. Secondly, the electron Euler-Poisson system (1.1) is drastically different from the ion Euler-Poisson system studied in [15]: They have different dispersive relations, different Poisson equations and many others. Most vitally to our present problem, the operator $\partial_x^2$ is irreversible in the linear Poisson equation (1.1c) for electrons, so we can not diagonalize the linearised system of (1.1) directly, drastically different as done for the ion Euler-Poisson system (see [15]), where the linearized operator $1-\partial_x^2$ is revertible. Finally, the method applied in this paper is slightly different from the classical Normal-Form method of Shatah [20] to close the energy estimate for the error as done in the paper [2–3, 15].

To prove Theorem 1.1, we will apply the form of the quadratic terms to construct the modified energy functional, and then use the rotating coordinate transform to obtain a uniform error estimate. We first transform (1.1) into a diagonalized system of $(E, v)$ by $E := \psi_x$,

$$\begin{cases} E_t + v + vE_x = 0, & (1.6a) \\ v_t - E + E_{xx} + vv_x + E_x E_{xx} = 0. & (1.6b) \end{cases}$$

By the following transform, we can diagonalize the linear part of system (1.6),

$$S = \begin{pmatrix} 1 & 1 \\ -\Omega & \Omega \end{pmatrix}, \quad \begin{pmatrix} E \\ v \end{pmatrix} = S \begin{pmatrix} U_1 \\ U_{-1} \end{pmatrix}, \tag{1.7}$$

where $\Omega$ is an operator defined by the dispersive relation (1.5) such that $\widehat{\Omega u}(k) = \mathrm{i}\omega(k)\widehat{u}(k)$ for function $u$, and every column vector of the invertible matrix $S$ is the eigenfunction for system (1.6). By the relation $E := \psi_x$ and (1.7), we have

$$\begin{pmatrix} n-1 \\ v \end{pmatrix} = \widetilde{S} \begin{pmatrix} U_1 \\ U_{-1} \end{pmatrix}, \quad \widetilde{S} = \begin{pmatrix} \partial_x & \partial_x \\ -\Omega & \Omega \end{pmatrix}. \tag{1.8}$$

Then we can obtain the NLS approximation for the transformed variables $(E, v)$ and hence for $(n-1, v)$ by the relation (1.7) and (1.8), as long as we prove rigorously the NLS approximation for variables $(U_1, U_{-1})$. Plugging $E = U_1 + U_{-1}$ and $v = -\Omega(U_1 - U_{-1})$ into (1.6), we obtain the following linearised system

$$\partial_t U_j = j\Omega U_j + Q_j(U, U) \tag{1.9}$$

with

$$Q_j(U, U) = \frac{1}{2}\partial_x(U_1 + U_{-1})\Omega(U_1 - U_{-1}) + \frac{j\partial_x}{4\Omega}[(\Omega(U_1 - U_{-1}))^2 + (\partial_x(U_1 + U_{-1}))^2], \tag{1.10}$$

where $j \in \{1, -1\}$ and $Q_j$ denotes the quadratic term of the evolution equation for $U_j$. Then the NLS equation can be formally derived by applying a modified ansatz $U = \varepsilon\Psi$.

To justify the NLS approximation for system (1.9) on its natural timescale in some Sobolev space, we have to estimate the error

$$\varepsilon^{\beta} R := U - \varepsilon\Psi$$

to be of order $\mathcal{O}(\varepsilon^{\beta})$ for all $t \in \left[0, \frac{T_0}{\varepsilon^2}\right]$ for some $\beta > 1$. To make the time interval to be $\mathcal{O}(\varepsilon^{-2})$, the quadratic terms need to be transformed into cubic terms and a closed energy estimate for the error $R$ needs to be obtained. Thus we adopt the modified energy method based on the quadratic terms to define

$$\mathcal{E}_s = \sum_{\ell=0}^{s} \Big[ \int (\partial_x^{\ell} R)^2 \mathrm{d}x + (2\ell - 1)\varepsilon \int (\partial_x \Psi + \Omega\Psi)(\partial_x^{\ell} R)^2 \mathrm{d}x \Big] \tag{1.11}$$

for $s = s_A \geq 6$. Obviously, $\sqrt{\mathcal{E}_s}$ is equivalent to $\|R\|_{H^s}$ since $\varepsilon\|(\partial_x + \Omega)\Psi\|_{L^\infty} = \mathcal{O}(\varepsilon)$ due to the compact support of the modified approximation $\varepsilon\Psi$ in Fourier space. Our energy functional contains some modified terms of order $\mathcal{O}(\varepsilon)$, which is used to eliminate the highest derivatives terms from the quadratic terms. In addition, we use the rotating coordinate transform for the approximation solution $\varepsilon\Psi$ and the error $R$ to translate the quadratic terms of order $\mathcal{O}(\varepsilon)$ into cubic terms of order $\mathcal{O}(\varepsilon^2)$, and take advantage of the properties of no-resonance and quadratic terms for system (1.9) to bound the time derivative of the energy in Fourier spaces. For more on the modified energy method, see also Hunter et al. [7]. To close the error estimates, the energy will be further modified into

$$\widetilde{\mathcal{E}}_s = \mathcal{E}_s + \varepsilon^2 h,$$

where $h = \mathcal{O}(\|R\|_{H^s}^2)$ as long as $\|R\|_{H^s} = \mathcal{O}(1)$. Consequently, we obtain

$$\partial_t \widetilde{\mathcal{E}}_s \leq C\varepsilon^2(\widetilde{\mathcal{E}}_s + 1)$$

as long as $\|R\|_{H^s} = \mathcal{O}(1)$. Gronwall's inequality then yields the $\mathcal{O}(1)$ boundedness of $\widetilde{\mathcal{E}}_s$ and hence of $R$ for all $t \in \left[0, \frac{T_0}{\varepsilon^2}\right]$. So the NLS approximation for system (1.9) is achieved by combing the estimates of the residual terms and the error. The details are given in Section 3.

In order to switch back into the $(n-1, v)$ variables, we use the relation $(n-1, v) = \widetilde{S}(U_1, U_{-1})$ and the fact that the Fourier transform of the approximation solution $\varepsilon\Psi$ is sufficiently strong concentrated around integer multiple of the wave numbers $\pm k_0$. Then Theorem 1.1 can be proved by defining $\phi(k_0) = \begin{pmatrix} \mathrm{i}k_0 \\ -\mathrm{i}\omega(k_0) \end{pmatrix}$ in the approximation (1.3) and then using the estimate $\|\varepsilon f(\varepsilon\cdot)\|_{L^2} = \varepsilon^{\frac{1}{2}}\|f\|_{L^2}$.

In Section 2 we derive the NLS equation and estimate the formal approximate solutions and the residual terms that remain after inserting the approximation into (1.6). In Section 3 we perform the error estimate to prove Theorem 1.1.

**Notation** We denote the Fourier transform of a function $u \in L^2(\mathbb{R}, \mathbb{K})$, with $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ by

$$\widehat{u}(k) = \frac{1}{2\pi} \int_{\mathbb{R}} u(x) \mathrm{e}^{-\mathrm{i}kx} \mathrm{d}x.$$

Let $H^s(\mathbb{R},\mathbb{K})$ be the space of functions mapping from $\mathbb{R}$ into $\mathbb{K}$ for which

$$\|u\|_{H^s(\mathbb{R},\mathbb{K})} = \Big(\int_{\mathbb{R}} |\widehat{u}(k)|^2(1+|k|^2)^s \mathrm{d}k\Big)^{\frac{1}{2}} < \infty.$$

We usually write $L^2$ and $H^s$ instead of $L^2(\mathbb{R},\mathbb{R})$ and $H^s(\mathbb{R},\mathbb{R})$. We define the space $L^p(m)(\mathbb{R},\mathbb{K})$ of $u$ such that $\sigma^m u \in L^p(\mathbb{R},\mathbb{K})$, where $\sigma(x) = (1+x^2)^{\frac{1}{2}}$. We also write $A \lesssim B$ if $A \le CB$ for a constant $C > 0$, and $A = \mathcal{O}(B)$ if $|A| \lesssim B$.

## 2 The Derivation of the NLS Approximation

In this section, the NLS equation will be obtained formally as an approximation equation for system (1.9). We compute the Fourier transform of $U_j$ for (1.9) as

$$\partial_t \widehat{U}_j = \mathrm{i} j\omega(k)\widehat{U}_j + \int \sum_{m,n\in\{\pm 1\}} \eta^j_{mn}(k, k-\ell, \ell)\widehat{U}_m(k-\ell)\widehat{U}_n(\ell)\mathrm{d}\ell, \tag{2.1}$$

where the kernel function $\eta^j_{mn}$ of quadratic terms (1.10) satisfies

$$\eta^j_{mn}(k, k-\ell, \ell) = -\frac{n}{2}(k-\ell)\omega(\ell) - \frac{jk}{4\omega(k)}[mn\omega(k-\ell)\omega(\ell) + (k-\ell)\ell]. \tag{2.2}$$

Define the residual as follows

$$\mathrm{Res}(U) = -\partial_t U + \Lambda U + Q(U,U), \tag{2.3}$$

which is a measure of how much $U$ fails to be a solution of (1.9). To derive the NLS equation formally as an approximation equation for system (2.1), we make the residual smaller by approximating $U$ not just with the NLS terms, but rather by a more complicated approximation

$$\varepsilon\widetilde{\Psi}_j = \sum_{0\le j_1,|j_2|\le 5} \varepsilon^{\beta_j(j_2,j_1)}\widetilde{\psi}^{j_1}_{j_2,j}, \quad j\in\{\pm 1\} \tag{2.4}$$

with

$$\beta_1(j_2,j_1) = 1 + ||j_2|-1| + j_1, \quad \beta_{-1}(j_2,j_1) = \begin{cases} \beta_1(j_2,j_1) & \text{for } j_2 \ne 1, \\ \beta_1(1,j_1)+2 & \text{for } j_2 = 1. \end{cases}$$

Assume that the term $\widetilde{\psi}^{j_1}_{j_2 j}$ has the form

$$\widetilde{\psi}^{j_1}_{j_2 j} = A^{j_1}_{j_2 j}(\varepsilon(x+c_g t), \varepsilon^2 t)E^{j_2}, \tag{2.5}$$

where $E^{j_2} = \mathrm{e}^{\mathrm{i}j_2(k_0 x + \omega_0 t)}$, then we find that the amplitudes $A^0_{\pm 11}$ of the one order terms $\widetilde{\psi}^0_{\pm 11}$ satisfy the NLS equation, while the higher terms satisfy some algebraic relations or inhomogenous linear partial differential equations.

We now insert (2.4) with (2.5) into (1.9). For the dispersion relation $\omega = \omega(k)$ which occurs in terms of the form $\omega\widetilde{\psi}^{j_1}_{j_2 j}$, we take their Taylor expansions around $k = j_2 k_0$ in Fourier space. Similarly, for the quadratic terms such as $\eta^j_{mn}\widetilde{\psi}^{j_1}_{j_2 j}\widetilde{\psi}^{\widetilde{j}_1}_{\widetilde{j}_2\widetilde{j}}$, we take their Taylor expansions around $k = (j_2 + \widetilde{j}_2)k_0$, $k-\ell = j_2 k_0$ and $\ell = \widetilde{j}_2 k_0$ in Fourier space, respectively. For more details, one can refer to [19].

Now we equate the coefficients of $\varepsilon^l E^{j_2}$ to zero for $j_2 = 0, 1, 2, \cdots$ inductively. Firstly, we find that the coefficients of $\varepsilon E^1$ and $\varepsilon^2 E^1$ vanish identically because of the relations $\omega_0 = \omega(k_0)$ and $c_g = \partial_k \omega(k_0)$. By letting the coefficient of $\varepsilon^3 E^1$ be zero, we obtain

$$\partial_T A_{11}^0 = \frac{\mathrm{i}}{2} \partial_k^2 \omega(k_0) \partial_X^2 A_{11}^0 + q, \tag{2.6}$$

where $q$ is only related to $A_{11}^0 A_{0j}^0$ and $A_{-11}^0 A_{2j}^0$.

By letting the coefficient of $\varepsilon^2 E^0$ and $\varepsilon^2 E^2$ be zero, we obtain

$$\begin{aligned} \lim_{k \to 0^-} \omega(k) A_{01}^0 &= \kappa_{01}(A_{11}^0 A_{-11}^0), \\ \lim_{k \to 0^+} \omega(k) A_{0-1}^0 &= \kappa_{02}(A_{11}^0 A_{-11}^0) \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} (2\omega_0 + \omega(2k_0)) A_{21}^0 &= \kappa_{21}(A_{11}^0)^2, \\ (2\omega_0 + \omega(2k_0)) A_{2-1}^0 &= \kappa_{22}(A_{11}^0 A_{-11}^0), \end{aligned} \tag{2.8}$$

where $\kappa_{0j},\ \kappa_{2j} \in \mathbb{C}$. Then $A_{0j}^0$ and $A_{2j}^0$ can be expressed by $|A_{11}^0|^2$ and $(A_{11}^0)^2$, respectively, due to the fact $\lim\limits_{k \to 0^\pm} \omega(k) \neq 0$ and $2\omega_0 \pm \omega(2k_0) \neq 0$. Inserting (2.7)–(2.8) into (2.6), we can obtain the NLS equation

$$\partial_T A_{11}^0 = \frac{\mathrm{i}}{2} \partial_k^2 \omega(k_0) \partial_X^2 A_{11}^0 - \mathrm{i}\gamma(k_0) A_{11}^0 |A_{11}^0|^2, \tag{2.9}$$

where $\gamma(k_0) \in \mathbb{R}$.

Repeating the above steps, we find that the higher-order terms satisfy some algebraic relations or inhomogeneous linear partial differential equations. For example, the term $A_{11}^1$ satisfies a linear inhomogeneous Schrödinger equation, in which all the inhomogeneous terms are known according to the prior steps.

To obtain the approximation property of the NLS equation (2.9), it is helpful to modify $\varepsilon\widetilde{\Psi}$ by a new approximation $\varepsilon\Psi$ by some cut-off function such that the modified approximation $\varepsilon\Psi$ in Fourier space has compact support in small neighborhoods of $j_2 k_0$ with $|j_2| \leq 5$. More precisely we define $\psi_{j_2 j}^{j_1}$ such that

$$\begin{cases} \widehat{\psi}_{j_2 j}^{j_1}(k) = \widehat{\widetilde{\psi}}{}_{j_2 j}^{j_1}(k) & \text{for } \{k \in \mathbb{R} \mid |k - j_2 k_0| \leq \delta\}, \\ \widehat{\psi}_{j_2 j}^{j_1}(k) = 0, & \text{otherwise}, \end{cases} \tag{2.10}$$

where $\delta > 0$ is a constant independent on $0 < \varepsilon \ll 1$. Then our modified approximation $\varepsilon\Psi$ is as follows

$$\varepsilon\Psi_j = \sum_{0 \leq j_1, |j_2| \leq 5} \varepsilon^{\beta_j(j_2, j_1)} \psi_{j_2 j}^{j_1}, \quad j \in \{\pm 1\},\ |\beta_j(j_2, j_1)| \leq 5. \tag{2.11}$$

Note that the approximation is only changed slightly by the above modification due to the concentration around the wave numbers $j_2 k_0$, but this will lead to a simpler control of the error and make the approximation an analytic function.

**Lemma 2.1** *Let $s_A \geq 6$ and $A \in C([0,T_0], H^{s_A}(\mathbb{R},\mathbb{C}))$ be a solution of the NLS equation* (2.9) *with*

$$\sup_{T\in[0,T_0]} \|A\|_{H^{s_A}} \leq C_A.$$

*Then for all $s \geq 0$, there exist $C_{\rm Res}, C_\Psi, \varepsilon_0 > 0$ depending on $C_A$ such that the following holds for all $\varepsilon \in (0,\varepsilon_0)$ : The modified approximation $\varepsilon\Psi$ exists for all $t \in \big[0, \frac{T_0}{\varepsilon^2}\big]$ and satisfies*

$$\begin{cases} \sup\limits_{t\in\left[0,\frac{T_0}{\varepsilon^2}\right]} \|{\rm Res}_U(\varepsilon\Psi)\|_{H^s} \leq C_{\rm Res}\varepsilon^{\frac{9}{2}}, & (2.12a)\\ \sup\limits_{t\in\left[0,\frac{T_0}{\varepsilon^2}\right]} \|\varepsilon\Psi - \varepsilon\Psi_{NLS}\|_{H^{s_A}} \leq C_\Psi\varepsilon^{\frac{3}{2}}, & (2.12b)\\ \sup\limits_{t\in\left[0,\frac{T_0}{\varepsilon^2}\right]} \|\widehat{\psi}^{j_1}_{j_2j}\|_{L^1(s+1)(\mathbb{R},\mathbb{C})} \leq C_\Psi. & (2.12c)\end{cases}$$

**Proof** We refer to [4, Lemma 2.6] (see also [2]) for the proof of Lemma 2.1. According to the form of original approximation solution, we have ${\rm Res}(\varepsilon\widetilde{\Psi}) = \mathcal{O}(\varepsilon^5)$ and $\varepsilon\widetilde{\Psi} - \varepsilon(\widetilde{\psi}^0_{11} + \widetilde{\psi}^0_{-11}) = \mathcal{O}(\varepsilon^2)$ on the time interval $[0, \frac{T_0}{\varepsilon^2}]$ if $A$ is a solution of the NLS equation (2.9) for $T \in [0, T_0]$. Since the modified approximation $\varepsilon\widehat{\Psi}$ has a compact support whose size depends on $k_0$, thus there exists a constant $C$ depending on $k_0$ such that $\|\Psi\|_{H^s} \leq C\|\Psi\|_{L^2}$ and $\|\widehat{\Psi}\|_{L^1(s)} \leq C\|\Psi\|_{L^1}$ for all $s > 0$.

Furthermore, by using the facts $\|f(\varepsilon\cdot)\|_{L^2} = \varepsilon^{-\frac{1}{2}}\|f\|_{L^2}$ and the estimate

$$\|(\chi_{[-\delta,\delta]-1})\varepsilon^{-1}\widehat{f}(\varepsilon^{-1}\cdot)\|_{L^2(m)} \leq C\varepsilon^{m+M-\frac{1}{2}}\|f\|_{H^{m+M}} \tag{2.13}$$

for all $m, M \geq 0$, where $\chi_{[-\delta,\delta]}$ is the characteristic function on $[-\delta,\delta]$, we can obtain (2.12a) and

$$\sup_{t\in\left[0,\frac{T_0}{\varepsilon^2}\right]} \|\varepsilon\Psi - \varepsilon(\psi^0_{11} + \psi^0_{-11})\|_{H^{s_A}} \leq C\varepsilon^{\frac{3}{2}}. \tag{2.14}$$

By combing (2.13)–(2.14), (1.3) and (1.8), we obtain (2.12b).

Finally, due to $\|\varepsilon^{-1}\widehat{f}(\varepsilon\cdot)\|_{L^1} = \|\widehat{f}\|_{L^1}$ and the construction of $\psi^{j_1}_{j_2j}$, we obtain (2.12c).

Note that the bound (2.12c) will be used to estimate

$$\|\psi^{j_1}_{j_2j}f\|_{H^s} \leq C\|\psi^{j_1}_{j_2j}\|_{C^s_b}\|f\|_{H^s} \leq C\|\widehat{\psi}^{j_1}_{j_2j}\|_{L^1(s)(\mathbb{R},\mathbb{C})}\|f\|_{H^s}$$

without loss of powers in $\varepsilon$, as it would be the case with $\|\psi^{j_1}_{j_2j}\|_{L^2(s)(\mathbb{R},\mathbb{C})}$. Moreover, by an analogous argument as in the proof of [4, Lemma 3.3], we have the following lemma.

**Lemma 2.2** *For all $s \geq 0$ there exists a constant $C_\psi > 0$ such that*

$$\|\partial_t\widehat{\psi}^0_{\pm11} + {\rm i}\omega\widehat{\psi}^0_{\pm11}\|_{L^1(s)} \leq C_\psi\varepsilon^2. \tag{2.15}$$

For later convenience, we give the following lemma.

**Lemma 2.3** (see [13]) **[Commutator Estimate]** *Let $m \geq 1$ be an integer, and then the commutator $[\nabla^m, f]g := \nabla^m(fg) - f\nabla^m g$ can be bounded by*

$$\|[\nabla^m, f]g\|_{L^p} \leq \|\nabla f\|_{L^{p_1}}\|\nabla^{m-1}g\|_{L^{p_2}} + \|\nabla^m f\|_{L^{p_3}}\|g\|_{L^{p_4}}, \tag{2.16}$$

*where* $p, p_2, p_3 \in (1, \infty)$ *and*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

## 3 The Error Estimates

In order to prove Theorem 1.1, we need only to prove the NLS approximation for system (1.9) because of (1.7)–(1.8) and Lemma 2.1. Define the error $R$ by $U = \varepsilon\Psi + \varepsilon^\beta R$ $\big(\beta \geq \frac{5}{2}\big)$ and plug it into (1.9), we then obtain the following equation for $R$,

$$\begin{aligned}
\partial_t R_j &= j\Omega R_j + 2\varepsilon Q_j(\Psi, R) + \varepsilon^\beta Q_j(R,R) + \varepsilon^{-\beta}\mathrm{Res}_{U_j}(\varepsilon\Psi) \\
&= j\Omega R_j + \frac{j\varepsilon}{2}\partial_x(\Psi_j + \Psi_{-j})\Omega(R_j - R_{-j}) + \frac{j\varepsilon}{2}\Omega(\Psi_j - \Psi_{-j})\partial_x(R_j + R_{-j}) \\
&\quad + \frac{j\varepsilon\partial_x}{2\Omega}[\Omega(\Psi_j - \Psi_{-j})\Omega(R_j - R_{-j}) + \partial_x(\Psi_j + \Psi_{-j})\partial_x(R_j + R_{-j})] \\
&\quad + \frac{j\varepsilon^\beta}{2}\partial_x(R_j + R_{-j})\Omega(R_j - R_{-j}) + \frac{j\varepsilon^\beta\partial_x}{4\Omega}[(\Omega(R_j - R_{-j}))^2 + \partial_x((R_j + R_{-j}))^2] \\
&\quad + \varepsilon^{-\beta}\mathrm{Res}_{U_j}(\varepsilon\Psi) \\
&=: j\Omega R_j + \varepsilon A_1 + \varepsilon^\beta A_2 + \varepsilon^{-\beta}\mathrm{Res}_{U_j}(\varepsilon\Psi),
\end{aligned} \tag{3.1}$$

where $2Q_j(\Psi, R) =: A_1$, $Q_j(R,R) =: A_2$. By careful computation, we have

$$\begin{aligned}
A_1 &= j[\partial_x(\Psi_j + \Psi_{-j}) + \Omega(\Psi_j - \Psi_{-j})]\partial_x R_j \\
&\quad + \frac{j}{2}[\partial_x(\Psi_j + \Psi_{-j}) + \Omega(\Psi_j - \Psi_{-j})](\Omega - \partial_x)(R_j - R_{-j}) \\
&\quad + \frac{j}{2}\Big(\frac{\partial_x}{\Omega} - 1\Big)[\partial_x(\Psi_j + \Psi_{-j})\partial_x(R_j + R_{-j}) + \Omega(\Psi_j - \Psi_{-j})\Omega(R_j - R_{-j})] \\
&= \sum_{i=1}^{3} A_{1i}, \\
A_2 &= j[\partial_x R_j + (\Omega - \partial_x)(R_j - R_{-j})]\partial_x R_j \\
&\quad + \frac{j}{4}[(\Omega - \partial_x)(R_j - R_{-j})]^2 \\
&\quad + \frac{j}{4}\Big(\frac{\partial_x}{\Omega} - 1\Big)[(\Omega(R_j - R_{-j}))^2 + (\partial_x(R_j + R_{-j}))^2] \\
&= \sum_{i=1}^{3} A_{2i}.
\end{aligned} \tag{3.2}$$

According to the dispersive relation (1.5), we have

$$\begin{aligned}
&\widehat{(\Omega - \partial_x)}(k) = \mathrm{i}(\omega(k) - k) = \mathrm{i}(\sqrt{1+k^2} - k) = \mathrm{i}\frac{1}{\sqrt{1+k^2} + k} = \mathrm{i}\mathcal{O}(k^{-1}), \quad k \to \infty, \\
&\widehat{\Big(\frac{\partial_x}{\Omega} - 1\Big)}(k) = \frac{k}{\sqrt{1+k^2}} - 1 = \frac{-1}{\sqrt{1+k^2}(\sqrt{1+k^2} + k)} = \mathcal{O}(k^{-2}), \quad k \to \infty.
\end{aligned} \tag{3.3}$$

By (3.2)–(3.3), we note that the lost derivatives from the terms $2Q_j(\Psi, R)$ and $Q_j(R,R)$ are concentrated on $A_{11}$ and $A_{21}$, respectively, and the other terms such as $A_{12}, A_{13}$ and $A_{21}, A_{22}$

do not lose derivatives. In order to control the error $R$, we define the following energy function

$$\mathcal{E}_s = \sum_{\ell=0}^{s} E_\ell, \tag{3.4}$$

$$E_\ell = \sum_{j\in\{\pm1\}} \Big[ \int (\partial_x^\ell R_j)^2 \mathrm{d}x + (2\ell-1)\varepsilon \sum_{m\in\{\pm1\}} \int (\partial_x + jm\Omega)\Psi_m (\partial_x^\ell R_j)^2 \mathrm{d}x \Big], \tag{3.5}$$

where $s = s_A \geq 6$. The terms of order $\mathcal{O}(\varepsilon)$ in $E_\ell$ are used to counteract the effects of the quasilinearity. The evolution of these terms will cancel the terms with the highest derivatives from the $H^s$ norm. Note that

$$E_\ell \lesssim \|\partial_x^\ell R\|_{L^2}^2 + \varepsilon \|\partial_x \Psi\|_{L^\infty} \|\partial_x^\ell R\|_{L^2}^2,$$

then we can obtain the energy $\sqrt{\mathcal{E}_s}$ is equal to the $\|R\|_{H^s}$ by applying Lemma 2.1 and Sobolev embedding $H^1 \hookrightarrow L^\infty$.

In order to prove rigorously the NLS approximation for system (1.9), we have to prove that $R$ is of order $\mathcal{O}(1)$ for all $t \in [0, \frac{T_0}{\varepsilon^2}]$. In detail, we want to show $\|R\|_{H^s} \leq C$ for a constant $C$ independent of $\varepsilon$ in the time interval $[0, \frac{T_0}{\varepsilon^2}]$, i.e., we want to show that

$$\partial_t \mathcal{E}_s \lesssim \varepsilon^2 (1 + \mathcal{E}_s + \varepsilon^{\frac{3}{2}} \mathcal{E}_s^{\frac{3}{2}}).$$

Then we will conclude that

$$\sup_{t\in\left[0,\frac{T_0}{\varepsilon^2}\right]} \mathcal{E}_s(t) \leq C.$$

Furthermore, we can obtain that $\sup_{t\in\left[0,\frac{T_0}{\varepsilon^2}\right]} R(t) \leq C.$

Now we consider the evolution of $E_\ell$,

$$\begin{aligned}
\partial_t E_\ell = &\sum_{j\in\{\pm1\}} \Big\{ 2\int \partial_x^\ell R_j \partial_x^\ell \partial_t R_j \mathrm{d}x + 2(2\ell-1)\varepsilon \sum_{m\in\{\pm1\}} \Big[ \int (\partial_x + jm\Omega)\Psi_m \partial_x^\ell R_j \partial_x^\ell \partial_t R_j \mathrm{d}x \\
&+ \frac{1}{2} \int (\partial_x + jm\Omega)\partial_t \Psi_m (\partial_x^\ell R_j)^2 \mathrm{d}x \Big] \Big\} \\
= &\sum_{j\in\{\pm1\}} \Big\{ 2j \int \partial_x^\ell R_j \partial_x^\ell \Omega R_j \mathrm{d}x + 2\varepsilon \int \partial_x^\ell R_j \partial_x^\ell A_1 \mathrm{d}x \\
&+ 2(2\ell-1)\varepsilon \sum_{m\in\{\pm1\}} \int (j\partial_x + m\Omega)\Psi_m \partial_x^\ell R_j \partial_x^\ell \Omega R_j \mathrm{d}x \\
&+ (2\ell-1)\varepsilon \sum_{m\in\{\pm1\}} \int (\partial_x + jm\Omega)\partial_t \Psi_m (\partial_x^\ell R_j)^2 \mathrm{d}x \\
&+ 2\varepsilon^\beta \int \partial_x^\ell R_j \partial_x^\ell A_2 \mathrm{d}x + 2(2\ell-1)\varepsilon^2 \sum_{m\in\{\pm1\}} \int (\partial_x + jm\Omega)\Psi_m \partial_x^\ell R_j \partial_x^\ell A_1 \mathrm{d}x \\
&+ 2(2\ell-1)\varepsilon^{\beta+1} \sum_{m\in\{\pm1\}} \int (\partial_x + jm\Omega)\Psi_m \partial_x^\ell R_j \partial_x^\ell A_2 \mathrm{d}x \\
&+ 2\varepsilon^{-\beta} \int \partial_x^\ell R_j \partial_x^\ell \mathrm{Res}_{U_j}(\varepsilon\Psi) \mathrm{d}x
\end{aligned}$$

$$+ 2(2\ell-1)\varepsilon^{1-\beta}\sum_{m\in\{\pm1\}}\int(\partial_x+jm\Omega)\Psi_m\partial_x^\ell R_j\partial_x^\ell \mathrm{Res}_{U_j}(\varepsilon\Psi)\mathrm{d}x\Big\}$$

$$=: \sum_{j\in\{\pm1\}}\sum_{i=1}^{9} I_i. \tag{3.6}$$

Due to the skew symmetry of $\Omega$, the first term $I_1$ is equal to zero.

Now we first estimate the terms $I_5 - I_9$ which is at least of order $\mathcal{O}(\varepsilon^2)$.

For $I_5$, recalling (3.2), we have

$$I_5 = 2\varepsilon^\beta\int\partial_x^\ell R_j\partial_x^\ell\Big(\sum_{i=1}^{3}A_{2i}\Big)\mathrm{d}x =: \sum_{i=1}^{3} I_{5i}.$$

For $I_{51}$, we have

$$\begin{aligned} I_{51} &= 2j\varepsilon^\beta\int\partial_x^\ell[(\partial_x R_j + (\Omega-\partial_x)(R_j-R_{-j})\partial_x R_j)\partial_x R_j]\partial_x^\ell R_j\mathrm{d}x \\ &= 2j\varepsilon^\beta\int\partial_x^\ell(\partial_x R_j)^2\partial_x^\ell R_j\mathrm{d}x + 2j\varepsilon^\beta\int\partial_x^\ell((\Omega-\partial_x)(R_j-R_{-j})\partial_x R_j)\partial_x^\ell R_j\mathrm{d}x \\ &=: I_{511}+I_{512}. \end{aligned}$$

For $I_{511}$, we have

$$\begin{aligned} I_{511} &= 4j\varepsilon^\beta\int\partial_x R_j\partial_x^{\ell+1}R_j\partial_x^\ell R_j\mathrm{d}x + 4\ell j\varepsilon^\beta\int\partial_x^2 R_j(\partial_x^\ell R_j)^2\mathrm{d}x \\ &\quad + 2j\varepsilon^\beta\int\sum_{i=2}^{\ell-2}C_{\ell-2}^2\partial_x^{i+1}R_j\partial_x^{\ell-i+1}R_j\partial_x^\ell R_j\mathrm{d}x \\ &= 2(2\ell-1)j\varepsilon^\beta\int\partial_x^2 R_j(\partial_x^\ell R_j)^2\mathrm{d}x + 2j\varepsilon^\beta\int\sum_{i=2}^{\ell-2}C_{\ell-2}^2\partial_x^{i+1}R_j\partial_x^{\ell-i+1}R_j\partial_x^\ell R_j\mathrm{d}x \\ &= \varepsilon^\beta\mathcal{O}(\mathcal{E}_s^{\frac32}), \end{aligned} \tag{3.7}$$

where we have used integration by parts, Sobolev embedding inequality and Young's inequality. For $I_{512}$, we have

$$\begin{aligned} I_{512} &= 2j\varepsilon^\beta\int((\Omega-\partial_x)(R_j-R_{-j}))\partial_x^{\ell+1}R_j\partial_x^\ell R_j\mathrm{d}x \\ &\quad + 2j\varepsilon^\beta\int[\partial_x^\ell,(\Omega-\partial_x)(R_j-R_{-j})]\partial_x R_j\partial_x^\ell R_j\mathrm{d}x \\ &= -j\varepsilon^\beta\int\partial_x((\Omega-\partial_x)(R_j-R_{-j}))(\partial_x^\ell R_j)^2\mathrm{d}x \\ &\quad + 2j\varepsilon^\beta\int[\partial_x^\ell,(\Omega-\partial_x)(R_j-R_{-j})]\partial_x R_j\partial_x^\ell R_j\mathrm{d}x \\ &= \varepsilon^\beta\mathcal{O}(\mathcal{E}_s^{\frac32}), \end{aligned} \tag{3.8}$$

where we have used (3.3) and the commutator estimates in Lemma 2.3. By combining (3.7) with (3.8), we have $I_{51} = \varepsilon^\beta\mathcal{O}(\mathcal{E}_s^{\frac32})$. By estimation directly, the terms $I_{52}$ and $I_{53}$ can be bounded by $\varepsilon^\beta\mathcal{O}(\mathcal{E}_s^{\frac32})$ by using (3.3) and integration by parts. Then we obtain

$$I_5 = \varepsilon^\beta\mathcal{O}(\mathcal{E}_s^{\frac32}). \tag{3.9}$$

For $I_6$, recalling (3.3), we have

$$I_6 = 2(2\ell-1)\varepsilon^2 \sum_{m\in\{\pm1\}} \varepsilon \int (\partial_x + jm\Omega)\Psi_m \partial_x^\ell R_j \partial_x^\ell \sum_{i=1}^{3} A_{1i} \mathrm{d}x =: \sum_{i=1}^{3} I_{6i}.$$

For $I_{61}$, we have

$$\begin{aligned} I_{61} &= 2(2\ell-1)\varepsilon^2 \sum_{m,n\in\{\pm1\}} \varepsilon \int (\partial_x + jm\Omega)\Psi_m \partial_x^\ell R_j \partial_x^\ell [(j\partial_x + n\Omega)\Psi_n \partial_x R_j] \mathrm{d}x \\ &= (2\ell-1)^2\varepsilon^2 \sum_{m,n\in\{\pm1\}} \varepsilon \int (\partial_x + jm\Omega)\Psi_m \partial_x (j\partial_x + n\Omega)\Psi_n (\partial_x^\ell R_j)^2 \mathrm{d}x \\ &\quad + 2(2\ell-1)\varepsilon^2 \sum_{m,n\in\{\pm1\}} \varepsilon \int (\partial_x + jm\Omega)\Psi_m \partial_x^\ell R_j \sum_{i=2}^{\ell} C_\ell^i \partial_x^i (j\partial_x + n\Omega)\Psi_n \partial_x^{\ell-i+1} R_j \mathrm{d}x \\ &= \varepsilon^2 \mathcal{O}(\mathcal{E}_s), \end{aligned}$$

where we have used integration by parts, Sobolev embedding inequality, Young's inequality and Lemma 2.1. By (3.2)–(3.3), we obtain $I_{62} + I_{63} = \varepsilon^2\mathcal{O}(\mathcal{E}_s)$ by applying Sobolev embedding inequality and Young's inequality once more. Then we have

$$I_6 = \varepsilon^2 \mathcal{O}(\mathcal{E}_s). \tag{3.10}$$

For $I_7$, recalling (3.2), we have

$$\begin{aligned} I_7 &= 2(2\ell-1)\varepsilon^{\beta+1} \sum_{m\in\{\pm1\}} \int (\partial_x + jm\Omega)\Psi_m \partial_x^\ell R_j \partial_x^\ell A_2 \mathrm{d}x \\ &= \varepsilon^\beta \mathcal{O}(\mathcal{E}_s + \varepsilon^{\frac{3}{2}} \mathcal{E}_s^{\frac{3}{2}}), \end{aligned} \tag{3.11}$$

where we have used integration by parts, Sobolev embedding inequality, Young's inequality and Lemma 2.1 once more.

For $I_8$, by applying the estimation of the residual term in Lemma 2.1, we can obtain

$$\begin{aligned} I_8 &= 2\varepsilon^{-\beta} \int \partial_x^\ell R_j \partial_x^\ell \mathrm{Res}_{U_j}(\varepsilon\Psi) \mathrm{d}x \\ &= \varepsilon^2 \mathcal{O}(1 + \mathcal{E}_s). \end{aligned} \tag{3.12}$$

For $I_9$, similar to $I_8$, by applying the estimates of the residual term and approximated solution in Lemma 2.1, we have

$$\begin{aligned} I_9 &= 2(2\ell-1)\varepsilon^{1-\beta} \sum_{m\in\{\pm1\}} \int (\partial_x + jm\Omega)\Psi_m \partial_x^\ell R_j \partial_x^\ell \mathrm{Res}_{U_j}(\varepsilon\Psi) \mathrm{d}x \\ &= \varepsilon^3 \mathcal{O}(1 + \mathcal{E}_s). \end{aligned} \tag{3.13}$$

Thus by above equalities (3.9)–(3.13), we have

$$\sum_{i=5}^{9} I_i = \varepsilon^2 \mathcal{O}(1 + \mathcal{E}_s + \varepsilon^{\frac{3}{2}} \mathcal{E}_s^{\frac{3}{2}}). \tag{3.14}$$

Next, we estimate the leaving terms $I_2 - I_4$, which are of order $\mathcal{O}(\varepsilon)$.

For $I_2$, according to the relation (3.2), we have

$$I_2 = 2\varepsilon \int \partial_x^\ell R_j \partial_x^\ell \Big( \sum_{i=1}^{3} A_{1i} \Big) \mathrm{d}x =: \sum_{i=1}^{3} I_{2i}.$$

For $I_{21}$, by integration by parts, we have

$$\begin{aligned}
I_{21} &= 2\varepsilon \int \partial_x^\ell [j(\partial_x(\Psi_j + \Psi_{-j}) + \Omega(\Psi_j - \Psi_{-j}))\partial_x R_j] \partial_x^\ell R_j \mathrm{d}x \\
&= 2\varepsilon \sum_{m\in\{\pm 1\}} \int \partial_x^\ell [(j\partial_x + m\Omega)\Psi_m \partial_x R_j] \partial_x^\ell R_j \mathrm{d}x \\
&= -\varepsilon \sum_{m\in\{\pm 1\}} \int \partial_x [(j\partial_x + m\Omega)\Psi_m] (\partial_x^\ell R_j)^2 \mathrm{d}x \\
&\quad + 2\varepsilon \sum_{m\in\{\pm 1\}} \int \sum_{i=1}^{\ell} C_\ell^i \partial_x^i ((j\partial_x + m\Omega)\Psi_m) \partial_x^{\ell-i+1} R_j \partial_x^\ell R_j \mathrm{d}x \\
&= (2\ell - 1)\varepsilon \sum_{m\in\{\pm 1\}} \int \partial_x ((j\partial_x + m\Omega)\Psi_m) (\partial_x^\ell R_j)^2 \mathrm{d}x \\
&\quad + 2\varepsilon \sum_{m\in\{\pm 1\}} \int \sum_{i=2}^{\ell} C_\ell^i \partial_x^i ((j\partial_x + m\Omega)\Psi_m) \partial_x^{\ell-i+1} R_j \partial_x^\ell R_j \mathrm{d}x \\
&=: I_{211} + I_{212}.
\end{aligned}$$

For $I_3$, by integration by parts, we have

$$\begin{aligned}
I_3 &= 2(2\ell - 1)\varepsilon \sum_{m\in\{\pm 1\}} \int (j\partial_x + m\Omega)\Psi_m \partial_x^\ell R_j \partial_x^\ell \Omega R_j \mathrm{d}x \\
&= -(2\ell - 1)\varepsilon \sum_{m\in\{\pm 1\}} \int \partial_x [(j\partial_x + m\Omega)\Psi_m] (\partial_x^\ell R_j)^2 \mathrm{d}x \\
&\quad + 2(2\ell - 1)\varepsilon \sum_{m\in\{\pm 1\}} \int (j\partial_x + m\Omega)\Psi_m \partial_x^\ell R_j \partial_x^\ell (\Omega - \partial_x) R_j \mathrm{d}x \\
&= I_{31} + I_{32}.
\end{aligned}$$

Note that the terms $I_{31}$ and $I_{211}$ can be cancelled and this is the reason why we choose the modified energy $E_\ell$ in (3.4).

For $I_4$, we have

$$\begin{aligned}
I_4 &= (2\ell - 1)\varepsilon \int (\partial_x + jm\Omega)\partial_t \Psi_m (\partial_x^\ell R_j)^2 \mathrm{d}x \\
&= (2\ell - 1)\varepsilon \int (\partial_x + jm\Omega)(\partial_t + \Omega)\Psi_m (\partial_x^\ell R_j)^2 \mathrm{d}x - (2\ell - 1)\varepsilon \int (\partial_x + jm\Omega)\Omega\Psi_m (\partial_x^\ell R_j)^2 \mathrm{d}x \\
&= I_{41} + I_{42}.
\end{aligned}$$

An application of Lemma 2.2 with (2.15) and Lemma 2.1 leads to $I_{41} = \varepsilon^3 \mathcal{O}(\mathcal{E}_s)$.

So far, only the terms $I_{212}, I_{22}, I_{23}, I_{32}, I_{42}$ of order $\mathcal{O}(\varepsilon)$ need to be estimated. In order to prove that $R$ is of order $\mathcal{O}(1)$ for all $t \in \left[0, \frac{T_0}{\varepsilon^2}\right]$, we need to translate these terms of order $\mathcal{O}(\varepsilon)$ to be of order $\mathcal{O}(\varepsilon^2)$. For this sake we take the following coordinate transform

$$f = \mathrm{e}^{-\Omega t} R, \quad g = \mathrm{e}^{-\Omega t}\Psi, \tag{3.15}$$

and then by using (3.1) and (3.15) we have

$$\partial_t f_j = 2\varepsilon \mathrm{e}^{-\Omega t} Q_j(\mathrm{e}^{\Omega t} g, \mathrm{e}^{\Omega t} f) + \varepsilon^{\beta}\mathrm{e}^{-\Omega t} Q_j(\mathrm{e}^{\Omega t} f, \mathrm{e}^{\Omega t} f) + \varepsilon^{-\beta}\mathrm{e}^{-\Omega t}\mathrm{Res}_{U_j}(\varepsilon\Psi). \tag{3.16}$$

It is noted that the $\mathcal{O}(1)$ term no longer appears in this coordinate frame. We analyze the remaining terms of $\mathcal{O}(\varepsilon)$ in Fourier space

$$\begin{aligned}
&I_{212} + I_{22} + I_{23} + I_{32} + I_{42}\\
&= \varepsilon(-1)^{\ell+1} \sum_{m,n\in\{\pm1\}} \iint \Big\{ (j+n)\sum_{i=2}^{\ell} C_\ell^i (k-l)^i (j(k-l) + m\omega(k-l)) l^{\ell-i+1} k^\ell\\
&\quad + k^{2\ell}(j(k-l) + m\omega(k-l)) n(\omega(l) - l)\\
&\quad + jk^{2\ell}\Big(\frac{k}{\omega(k)} - 1\Big)((k-l)l + mn\omega(k-l)\omega(l))\\
&\quad + (2\ell-1)(j+n)(j(k-l) + m\omega(k-l)) l^\ell k^\ell(\omega(k) - k)\\
&\quad - \Big(\ell - \frac{1}{2}\Big)(j+n)((k-l) + jm\omega(k-l))\omega(k-l) k^\ell l^\ell \Big\} \overline{\widehat{R}_j(k)}\widehat{\Psi}_m(k-l)\widehat{R}_n(l)\mathrm{d}l\mathrm{d}k\\
&=: \varepsilon(-1)^{\ell+1} \sum_{m,n\in\{\pm1\}} \iint \alpha_{mn}^j(k, k-l, l)\overline{\widehat{R}_j(k)}\widehat{\Psi}_m(k-l)\widehat{R}_n(l)\mathrm{d}l\mathrm{d}k.
\end{aligned} \tag{3.17}$$

That is to say $\alpha_{mn}^j =: \sum_{i=1}^{5}\gamma_i$ is the kernel function of $I_{212} + I_{22} + I_{23} + I_{32} + I_{42} =: \varepsilon J$. By using (3.3), we find that the exponent sum of $k$ and $l$ from every term of $\widehat{\alpha}_{mn}^j(k, k-l, l)$ less than $2\ell$ except for $-\big(\ell - \frac{1}{2}\big)(k+n)((k-l) + jm\omega(k-l))\omega(k-l)k^\ell l^\ell$ for $m, n, j \in \{\pm1\}$. By using (3.15), we have

$$\begin{aligned}
\varepsilon\int_0^t J\mathrm{d}s &= \varepsilon(-1)^{\ell+1}\sum_{m,n\in\{\pm1\}}\int_0^t\iint \alpha_{mn}^j(k, k-l, l)\overline{\widehat{R}_j}(k)\widehat{\Psi}_m(k-l)\widehat{R}_n(l)\mathrm{d}l\mathrm{d}k\mathrm{d}s\\
&= \varepsilon(-1)^{\ell+1}\sum_{m,n\in\{\pm1\}}\int_0^t\iint \mathrm{e}^{\mathrm{i}\phi_{mn}^j(k,l)s}\alpha_{mn}^j(k, k-l, l)\overline{\widehat{f}_j(k)}\widehat{g}_m(k-l)\widehat{f}_n(l)\mathrm{d}l\mathrm{d}k\mathrm{d}s\\
&= \varepsilon(-1)^{\ell+1}\sum_{m,n\in\{\pm1\}}\iint \frac{\alpha_{mn}^j(k, k-l, l)}{\mathrm{i}\phi_{mn}^j(k,l)}\mathrm{e}^{\mathrm{i}\phi_{mn}^j(k,l)s}\overline{\widehat{f}_j(k)}\widehat{g}_m(k-l)\widehat{f}_n(l)\mathrm{d}l\mathrm{d}k\Big|_0^t\\
&\quad - \varepsilon(-1)^{\ell+1}\sum_{m,n\in\{\pm1\}}\int_0^t\iint \frac{\alpha_{mn}^j(k, k-l, l)}{\mathrm{i}\phi_{mn}^j(k,l)}\mathrm{e}^{\mathrm{i}\phi_{mn}^j(k,l)s}\partial_s(\overline{\widehat{f}_j(k)}\widehat{g}_m(k-l)\widehat{f}_n(l))\mathrm{d}l\mathrm{d}k\mathrm{d}s\\
&=: (-1)^{\ell+1}\sum_{m,n\in\{\pm1\}}(J_1 + J_2),
\end{aligned} \tag{3.18}$$

where

$$\phi_{mn}^j(k,l) = -j\omega(k) + m\omega(k-l) + n\omega(l), \quad m, n, j \in \{\pm1\}. \tag{3.19}$$

Note that $\phi^j_{mn}(k,l) \neq 0$ for all $k, l \in \mathbb{R}$ according to the dispersive relation (1.5), thus the right-hand terms of (3.17) are well-defined. When $|k| \to \infty$, we have

$$\begin{aligned}\phi^j_{mj}(k,l) &= m\omega(k-l) - j(\omega(k) - \omega(l)) \\ &= m\omega(k-l) - j(k-l)\omega'(k - \theta(k,l)(k-l)) \\ &= m\omega(k-l) - j(k-l)(1 + \mathcal{O}(|k|^{-2})), \\ \phi^j_{m,-j}(k,l) &= -j\omega(k) + m\omega(k-l) - j\omega(l) \\ &= m\omega(k-l) - j(\omega(k) + \omega(k-(k-l))) \\ &= 2\omega(k)(1 + \mathcal{O}(|k|^{-1})).\end{aligned} \tag{3.20}$$

Note that the boundary term $J_1$ of the right-hand of (3.17) can be subtracted in the left-hand of our estimate, which does not change the energy because it can be estimated by $C\varepsilon\mathcal{E}_s$ due to (3.20). The second term $J_2$ is of order $\mathcal{O}(\varepsilon^2)$ when the time derivative is applied to either factor of $f$ or $g$.

For the last term $J_2$ of (3.17), we have

$$\begin{aligned}J_2 &= -\varepsilon \int_0^t \iint \frac{\widehat{\alpha}^j_{mn}(k,k-l,l)}{\mathrm{i}\phi^j_{mn}(k,l)} \mathrm{e}^{\mathrm{i}\phi^j_{mn}(k,l)s} \partial_s(\overline{\widehat{f}_j(k)}\widehat{g}_m(k-l)\widehat{f}_n(l))\mathrm{d}l\mathrm{d}k\mathrm{d}s \\ &= -\varepsilon \int_0^t \iint \frac{\widehat{\alpha}^j_{mn}(k,k-l,l)}{\mathrm{i}\phi^j_{mn}(k,l)} \mathrm{e}^{\mathrm{i}\phi^j_{mn}(k,l)s} \partial_s(\overline{\widehat{f}_j(k)})\widehat{g}_m(k-l)\widehat{f}_n(l)\mathrm{d}l\mathrm{d}k\mathrm{d}s \\ &\quad - \varepsilon \int_0^t \iint \frac{\widehat{\alpha}^j_{mn}(k,k-l,l)}{\mathrm{i}\phi^j_{mn}(k,l)} \mathrm{e}^{\mathrm{i}\phi^j_{mn}(k,l)s} \overline{\widehat{f}_j(k)}\partial_s(\widehat{g}_m(k-l))\widehat{f}_n(l)\mathrm{d}l\mathrm{d}k\mathrm{d}s \\ &\quad - \varepsilon \int_0^t \iint \frac{\widehat{\alpha}^j_{mn}(k,k-l,l)}{\mathrm{i}\phi^j_{mn}(k,l)} \mathrm{e}^{\mathrm{i}\phi^j_{mn}(k,l)s} \overline{\widehat{f}_j(k)}\widehat{g}_m(k-l)\partial_s(\widehat{f}_n(l))\mathrm{d}l\mathrm{d}k\mathrm{d}s \\ &=: J_{21} + J_{22} + J_{23}.\end{aligned} \tag{3.21}$$

For $J_{21}$, recalling (3.16), we have

$$\begin{aligned}J_{21} &= -\varepsilon \int_0^t \int\int \frac{\widehat{\alpha}^j_{mn}(k,k-l,l)}{\mathrm{i}\phi^j_{mn}(k,l)} \mathrm{e}^{\mathrm{i}(m\omega(k-l)+n\omega(l))s} \overline{(2\varepsilon\widehat{Q}_j(\mathrm{e}^{\mathrm{i}\omega s}\widehat{g}, \mathrm{e}^{\mathrm{i}\omega s}\widehat{f})(k)} \\ &\qquad \overline{+\varepsilon^{\beta}\widehat{Q}_j(\mathrm{e}^{\mathrm{i}\omega s}\widehat{f}, \mathrm{e}^{\mathrm{i}\omega s}\widehat{f})(k) + \varepsilon^{-\beta}\mathrm{Res}_{U_j}(\varepsilon \mathrm{e}^{\mathrm{i}\omega s}\widehat{g})})\widehat{g}_m(k-l)\widehat{f}_n(l)\mathrm{d}l\mathrm{d}k\mathrm{d}s \\ &= -2\varepsilon^2 \int_0^t \int\int \frac{\widehat{\alpha}^j_{mn}(k,k-l,l)}{\mathrm{i}\phi^j_{mn}(k,l)} \overline{\widehat{Q}_j(\widehat{\Psi}, \widehat{R})(k)}\widehat{\Psi}_m(k-l)\widehat{R}_n(l)\mathrm{d}l\mathrm{d}k\mathrm{d}s \\ &\quad - \varepsilon^{\beta+1} \int_0^t \int\int \frac{\widehat{\alpha}^j_{mn}(k,k-l,l)}{\mathrm{i}\phi^j_{mn}(k,l)} \overline{\widehat{Q}_j(\widehat{R}, \widehat{R})(k)}\widehat{\Psi}_m(k-l)\widehat{R}_n(l)\mathrm{d}l\mathrm{d}k\mathrm{d}s \\ &\quad - \varepsilon^{1-\beta} \int_0^t \int\int \frac{\widehat{\alpha}^j_{mn}(k,k-l,l)}{\mathrm{i}\phi^j_{mn}(k,l)} \overline{\mathrm{Res}_{U_j}(\varepsilon\widehat{\Psi})}\widehat{\Psi}_m(k-l)\widehat{R}_n(l)\mathrm{d}l\mathrm{d}k\mathrm{d}s \\ &=: J_{211} + J_{212} + J_{213}.\end{aligned} \tag{3.22}$$

The term $J_{213}$ including the residual can be bounded directly by $\varepsilon^2\mathcal{O}(1 + \mathcal{E}_s)$ by applying Lemma 2.1, Young's inequality and Cauchy-Schwarz. Recalling the kernel function $\alpha^j_{mn}$ from (3.17), we have

$$J_{211} + J_{212} = -2\varepsilon^2 \int_0^t \int\int \frac{\sum_{i=1}^{5}\gamma_i}{\mathrm{i}\phi^j_{mn}(k,l)} \overline{\widehat{Q}_j(\widehat{\Psi}, \widehat{R})(k)}\widehat{\Psi}_m(k-l)\widehat{R}_n(l)\mathrm{d}l\mathrm{d}k\mathrm{d}s$$

$$- \varepsilon^{\beta+1}\int_0^t\int\int \frac{\sum\limits_{i=1}^{5}\gamma_i}{\mathrm{i}\phi_{mn}^j(k,l)}\overline{\widehat{Q}_j(\widehat{R},\widehat{R})(k)}\widehat{\Psi}_m(k-l)\widehat{R}_n(l)\mathrm{d}l\mathrm{d}k\mathrm{d}s$$

$$=: \sum_{i=1}^{5}(J_{211i}+J_{212i}). \tag{3.23}$$

According to (3.3) and (3.17), we obtain $\gamma_i (i = 1, 2, 3, 4)$ is of order less than or equal to $2\ell - 1$ and the quadratic terms $Q_j(\Psi, R)$ or $Q_j(R, R)$ only lose one derivative by (3.2), so we have

$$\sum_{i=1}^{4}(J_{211i}+J_{212i}) = \varepsilon^2\mathcal{O}(\mathcal{E}_s + \varepsilon^{\frac{3}{2}}\mathcal{E}_s^{\frac{3}{2}}), \tag{3.24}$$

where we have used Lemma 2.1, Young's inequality and Cauchy-Schwarz once more.

For $J_{2115}$ with $j = n$, we have

$$\begin{aligned}
J_{2115} &= -2(2\ell-1)\varepsilon^2\int_0^t\iint \frac{((k-l)+jm\omega(k-l))\omega(k-l)k^\ell l^\ell}{\mathrm{i}\phi_{mj}^j(k,l)}\\
&\quad\times \overline{\widehat{Q}_j(\widehat{\Psi},\widehat{R})(k)}\widehat{\Psi}_m(k-l)\widehat{R}_j(l)\mathrm{d}l\mathrm{d}k\mathrm{d}s\\
&= -(2\ell-1)\varepsilon^2\int_0^t\iint \frac{((k-l)+jm\omega(k-l))\omega(k-l)k^\ell l^\ell}{\mathrm{i}\phi_{mj}^j(k,l)}\\
&\quad\times \overline{\sum_{i=1}^{3}\widehat{A}_{1i}(k)}\widehat{\Psi}_m(k-l)\widehat{R}_j(l)\mathrm{d}l\mathrm{d}k\mathrm{d}s\\
&=: \sum_{i=1}^{3}J_{2115}^i.
\end{aligned} \tag{3.25}$$

According to (3.20), we have

$$\left|\frac{k}{\mathrm{i}\phi_{mj}^j(k,l)}+\mathrm{i}k\right|\le C, \quad \left|\frac{k}{\mathrm{i}\phi_{m-j}^j(k,l)}+\mathrm{i}(k-l)\right|\le C. \tag{3.26}$$

Since $A_{12} + A_{13}$ from (3.2) does not lose derivatives, by applying (3.26), Lemma 2.1, Young's inequality and Cauchy-Schwarz, we have

$$J_{2115}^2 + J_{2115}^3 = \varepsilon^2\mathcal{O}(\mathcal{E}_s) \tag{3.27}$$

and

$$\begin{aligned}
J_{2115}^1 &= -(2\ell-1)\varepsilon^2\sum_{q\in\{\pm1\}}\int_0^t\iiint \mathrm{i}((k-l)+jm\omega(k-l))\omega(k-l)k^\ell l^\ell\\
&\quad\times\overline{(j(k-p)+q(k-p))p\widehat{\Psi}_q(k-p)\widehat{R}_j(p)}\widehat{\Psi}_m(k-l)\widehat{R}_j(l)\mathrm{d}p\mathrm{d}l\mathrm{d}k\mathrm{d}s\\
&\quad+\varepsilon^2\mathcal{O}(\mathcal{E}_s)\\
&= (-1)^{\ell+1}(2\ell-1)\varepsilon^2\sum_{q\in\{\pm1\}}\int_0^t\int \partial_x^\ell((j\partial_x+q\Omega)\Psi_q(\partial_x+jm\Omega)\Omega\Psi_m\partial_x R_j)\partial_x^\ell R_j\mathrm{d}x\mathrm{d}s\\
&\quad+\varepsilon^2\mathcal{O}(\mathcal{E}_s)
\end{aligned}$$

$$
\begin{aligned}
&= (-1)^{\ell+1}\frac{(2\ell-1)^2}{2}\varepsilon^2 \sum_{q\in\{\pm1\}}\int_0^t\int \partial_x((j\partial_x+q\Omega)\Psi_q(\partial_x+jm\Omega)\Omega\Psi_m)(\partial_x^\ell R_j)^2\mathrm{d}x\mathrm{d}s \\
&\quad + (-1)^{\ell+1}(2\ell-1)\varepsilon^2 \sum_{q\in\{\pm1\}}\int_0^t\int \sum_{i=2}^{\ell} C_i^\ell \partial_x^i (j\partial_x+q\Omega)\Psi_q(\partial_x+jm\Omega)\Omega\Psi_m \\
&\quad \times \partial_x^{\ell-i+1}R_j\partial_x^\ell R_j\mathrm{d}x\mathrm{d}s \\
&\quad + \varepsilon^2\mathcal{O}(\mathcal{E}_s) \\
&= \varepsilon^2\mathcal{O}(\mathcal{E}_s).
\end{aligned}
\tag{3.28}
$$

Combining (3.27) with (3.28), we obtain

$$
J_{2115} = \varepsilon^2\mathcal{O}(\mathcal{E}_s). \tag{3.29}
$$

Similarly, by using (3.2)–(3.3) and (3.17), we have

$$
J_{2125} = \varepsilon^2\mathcal{O}(\mathcal{E}_s + \varepsilon^{\frac{3}{2}}\mathcal{E}_s^{\frac{3}{2}}). \tag{3.30}
$$

By (3.22)–(3.24) and (3.29)–(3.30), we have

$$
J_{21} = \varepsilon^2\mathcal{O}(1+\mathcal{E}_s + \varepsilon^{\frac{3}{2}}\mathcal{E}_s^{\frac{3}{2}}).
$$

Similarly to $J_{21}$, $J_{22}$ and $J_{23}$ can be bounded by $\varepsilon^2\mathcal{O}(1+\mathcal{E}_s+\varepsilon^{\frac{3}{2}}\mathcal{E}_s^{\frac{3}{2}})$. Then we have

$$
J_2 = \varepsilon^2\mathcal{O}(\mathcal{E}_s + \varepsilon^{\frac{3}{2}}\mathcal{E}_s^{\frac{3}{2}}). \tag{3.31}
$$

Recalling the evolution of $E_\ell$ in (3.6), associate (3.14), (3.17)–(3.18) with (3.31), we have

$$
\partial_t(E_\ell + \varepsilon h_\ell) = \varepsilon^2\mathcal{O}(1+\mathcal{E}_s + \varepsilon^{\frac{3}{2}}\mathcal{E}_s^{\frac{3}{2}}), \tag{3.32}
$$

where $h_\ell$ is the modified term for the energy $E_\ell$ coming from $J_1$ in (3.18) and can be estimated by $\varepsilon\mathcal{O}(\mathcal{E}_s)$. Then by using Gronwall's inequality, we have

$$
\sup_{t\in\left[0,\frac{T_0}{\varepsilon^2}\right]} \|R(t)\|_{H^s} < C, \tag{3.33}
$$

independent of $\varepsilon$ as desired.

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