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Liouville Type Theorems for Nonlinear p-Laplacian Equation on Complete Noncompact Riemannian Manifolds*

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Abstract In this paper, the authors study the gradient estimates for positive weak solutions to the following p-Laplacian equation

$$\Delta_p u + au^\sigma = 0$$

on complete noncompact Riemannian manifold, where a, σ are two nonzero real constants with $p \neq 2$. Using the gradient estimate, they can get the corresponding Liouville theorem. On the other hand, by virtue of the Poincaré inequality, they also obtain a Liouville theorem under some integral conditions with respect to positive weak solutions.

Keywords p-Laplacian, Liouville theorem, Positive weak solution
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1 Introduction

Let (M^n, g) be a Riemannian manifold with the dimension n. Recently, a lot of attention was given to the following nonlinear elliptic equation defined on M^n:

$$\Delta u + au^\sigma = 0, \tag{1.1}$$

where a, σ are two nonzero real constants. For example, when $a < 0$ and $\sigma < 0$, (1.1) defined on a bounded smooth domain in $\mathbb{R}^n$ is called thin film equation, which describes a steady state of the thin film (see [4]). Moreover, it is linked with the Yang-Mills' problem for $n = 4$ and $\sigma = \frac{n+2}{n-3}$ in physics (see [1, 3]). For some development in this direction, see [5, 8–10, 13] and references therein.

In this paper, we focus on positive weak solutions to the following p-Laplacian type equaiton:

$$\Delta_p u + au^\sigma = 0, \tag{1.2}$$

where a, σ are two nonzero real constants with $p \neq 2$, and the p-Laplacian with respect to u is defined by

$$\Delta_p u := \mathrm{div}(|\nabla u|^{p-2}\nabla u) \quad \text{for every } u \in W^{1,p},$$

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which is understood in distribution sense. When $\sigma = p-1$, (1.2) is corresponding to the eigenvalue problem

$$\Delta_p u = -\lambda |u|^{p-2} u, \tag{1.3}$$

where u is said to be the eigenfunction associated with λ (for the study to gradient estimates of (1.3), see [7, 11]).

Theorem 1.1 *Let M^n be an n-dimensional complete noncompact Riemannian manifold with the sectional curvature $K_M \geq -K^2$ for some positive constant K. If either* (1)*: $a > 0$ and $\sigma \leq \frac{n+2}{n}(p-1)$ or* (2)*: $a < 0$ and $\sigma \geq \frac{n+2}{n}(p-1)$, then any positive weak solution u to* (1.2) *with $p > 1$ satisfies*

$$\frac{|\nabla u|^2}{u^2} \leq n(n-1)\Big(\frac{K}{p-1}\Big)^2. \tag{1.4}$$

In particular, the following Liouville-type result follows immediately.

Corollary 1.1 *Let M^n be an n-dimensional complete noncompact Riemannian manifold with nonnegative sectional curvature. If either* (1)*: $a > 0$ and $\sigma \leq \frac{n+2}{n}(p-1)$ or* (2)*: $a < 0$ and $\sigma \geq \frac{n+2}{n}(p-1)$, then* (1.2) *with $p > 1$ does not admit any positive weak solution.*

Generalizing the results in [2] in which they derived a Liouville theorem for weakly p-harmonic functions with finite p-energy and the manifold satisfies a Poincaré inequality, Zhao [14] consider the similar problem for (1.2). In the next part, we generalize the results of Zhao in [14] as follows.

Theorem 1.2 *Let M^n be an n-dimensional complete noncompact Riemannian manifold with* $\mathrm{Ric} \geq -\tau\rho$ *and satisfies the Poincaré inequality*

$$\int_M \rho\psi^2 \leq \int_M |\nabla \psi|^2, \tag{1.5}$$

where $\rho = \rho(x)$ is a positive function defined on M^n and the constant τ satisfies

$$0 < \tau < \frac{\alpha + 2(p-2)}{(p-1)^2} \tag{1.6}$$

with $\alpha = \min\big\{2, \frac{(p-1)^2}{n-1} + 1\big\}$. If $a\sigma < 0$ and

$$p \in \Big(1, 2 - \frac{\sqrt{n-1}-1}{n-2}\Big] \sqcup \Big[2 + \frac{\sqrt{n-1}+1}{n-2}, +\infty\Big), \tag{1.7}$$

then any positive weak solution u to (1.2) *with*

$$\lim_{R\to\infty} \frac{1}{R^2} \int_{B_p(2R)\setminus B_p(R)} |\nabla u|^{2p-2} \to 0 \tag{1.8}$$

must be constant, where $B_p(R)$ denotes the geodesic ball of radius R centered at $p \in M^n$.

Remark 1.1 In [15], Zhao and Shen obtained Liouville type theorems for $a \geq 0$ and $\sigma \leq p-1$ with $1 < p \leq 2$. Clearly, our Liouville-type results in Corollary 1.1 generalize those of Zhao and Shen in [15].

Remark 1.2 It needs to point out that Zhao [14] derives the Liouville type theorems for the case $a \le 0$ and $\sigma \le p-1$, while our Liouville type results in Theorem 1.2 hold for $a < 0$ and $\sigma > 0$, or $a > 0$ and $\sigma < 0$. On the other hand, the Liouville type results of Zhao [14] are true for $1 < p < \frac{n-1+\sqrt{(n-1)(n+3)}}{2}$, while our corresponding results hold for $p \ge 2 + \frac{\sqrt{n-1}+1}{n-2}$.

2 Proof of Theorem 1.1

We define

$$v = (p-1)\log u,$$

which is equivalent to $u = \mathrm{e}^{\frac{v}{p-1}}$. Then from (1.2), we obtain that v satisfies

$$\Delta_p v + |\nabla v|^p + a(p-1)^{(p-1)}\mathrm{e}^{(\frac{\sigma}{p-1}-1)v} = 0, \tag{2.1}$$

which is equivalent to

$$|\nabla v|^{p-2}\Delta v + \nabla v\nabla|\nabla v|^{p-2} + |\nabla v|^p + a(p-1)^{(p-1)}\mathrm{e}^{(\frac{\sigma}{p-1}-1)v} = 0. \tag{2.2}$$

Through direct computation, we get the following lemma.

Lemma 2.1 *We introduce the elliptic operator $\mathcal{L}$ defined by*

$$\mathcal{L} = \mathrm{div}(|\nabla v|^{p-2}A\nabla\cdot)$$

with $A = Id + (p-2)\frac{\nabla v\otimes\nabla v}{|\nabla v|^2}$. Then for $p > 1$ and

$$G = |\nabla v|^p, \tag{2.3}$$

we have

$$\begin{aligned}\mathcal{L}(G) \ge{} & \frac{p}{n}G^2 + ap(p-1)^{(p-1)}\Big(\frac{n+2}{n} - \frac{\sigma}{p-1}\Big)\mathrm{e}^{(\frac{\sigma}{p-1}-1)v}G \\ & - p|\nabla v|^{p-2}\nabla v\nabla G - (n-1)pK^2G^{\frac{2p-2}{p}} \\ & + \frac{1}{n}a^2p(p-1)^{2(p-1)}\mathrm{e}^{2(\frac{\sigma}{p-1}-1)v}.\end{aligned} \tag{2.4}$$

Proof Recall the following well-known Bochner formula (see [12, (2.3)]) with respect to the elliptic operator $\mathcal{L}$:

$$\begin{aligned}\frac{1}{p}\mathcal{L}(|\nabla v|^p) ={} & |\nabla v|^{2p-4}|\mathrm{Hess}\, v|_A^2 + |\nabla v|^{2p-4}\mathrm{Ric}(\nabla v, \nabla v) \\ & + |\nabla v|^{p-2}\nabla v\nabla\Delta_p v.\end{aligned} \tag{2.5}$$

Substituting (2.1) into (2.5), we get

$$\begin{aligned}\frac{1}{p}\mathcal{L}(|\nabla v|^p) ={} & |\nabla v|^{2p-4}|\mathrm{Hess}\, v|_A^2 + |\nabla v|^{2p-4}\mathrm{Ric}(\nabla v, \nabla v) \\ & - |\nabla v|^{p-2}\nabla v\nabla[|\nabla v|^p + a(p-1)^{(p-1)}\mathrm{e}^{(\frac{\sigma}{p-1}-1)v}] \\ ={} & |\nabla v|^{2p-4}|\mathrm{Hess}\, v|_A^2 + |\nabla v|^{2p-4}\mathrm{Ric}(\nabla v, \nabla v) - |\nabla v|^{p-2}\nabla v\nabla|\nabla v|^p \\ & - a(p-1)^{(p-1)}\Big(\frac{\sigma}{p-1} - 1\Big)\mathrm{e}^{(\frac{\sigma}{p-1}-1)v}|\nabla v|^p\end{aligned}$$

$$\begin{aligned}
&\geq \frac{1}{n}(\Delta_p v)^2 - (n-1)K^2|\nabla v|^{2p-2} - |\nabla v|^{p-2}\nabla v \nabla |\nabla v|^p \\
&\quad - a(p-1)^{(p-1)}\Big(\frac{\sigma}{p-1} - 1\Big)\mathrm{e}^{(\frac{\sigma}{p-1}-1)v}|\nabla v|^p \\
&= \frac{1}{n}|\nabla v|^{2p} + a(p-1)^{(p-1)}\Big(\frac{n+2}{n} - \frac{\sigma}{p-1}\Big)\mathrm{e}^{(\frac{\sigma}{p-1}-1)v}|\nabla v|^p \\
&\quad - |\nabla v|^{p-2}\nabla v \nabla |\nabla v|^p - (n-1)K^2|\nabla v|^{2p-2} \\
&\quad + \frac{1}{n}a^2(p-1)^{2(p-1)}\mathrm{e}^{2(\frac{\sigma}{p-1}-1)v}, \qquad (2.6)
\end{aligned}$$

where we used the following Cauchy inequality:

$$\begin{aligned}
|\nabla v|^{2p-4}|\mathrm{Hess}\, v|_A^2 &\geq \frac{1}{n}[|\nabla v|^{p-2}\mathrm{tr}_A(\mathrm{Hess}\, v)]^2 \\
&= \frac{1}{n}(\Delta_p v)^2 \qquad (2.7)
\end{aligned}$$

and (2.1), which completes the proof of Lemma 2.1.

Let θ be a cut-off function satisfying $\theta(t) = 1$ for $0 \leq t \leq \frac{1}{2}$ and $\theta(t) = 0$ for $t \geq 1$ such that

$$\frac{(\theta')^2}{\theta} \leq 40, \quad \theta'' \geq -40.$$

Define the function $\varphi : M^n \to \mathbb{R}$ by

$$\varphi(x) = \theta\Big(\frac{r(x)}{R}\Big),$$

where $r(x) = d(x_0, x)$ denotes the distance function on M^n centered at the point x_0. Then we have (see [6–7])

$$\frac{|\nabla \varphi|^2}{\varphi} \leq \frac{40}{R^2} \qquad (2.8)$$

and

$$\begin{aligned}
A^{ij}\varphi_{ij} &= \Delta\varphi + (p-2)\frac{\varphi_{ij}v^i v^j}{|\nabla v|^2} \\
&\geq -80(n+p-2)\frac{1+2KR}{R^2} - 40\max\{p-1, 1\}\frac{1}{R^2}, \qquad (2.9)
\end{aligned}$$

where $A^{ij} = \delta^{ij} + (p-2)\frac{v^i v^j}{|\nabla v|^2}$.

Let $\widetilde{G} = \varphi G$. Next we will apply the maximum principle to $\widetilde{G}$ on $B_R(x_0)$. We assume that $\widetilde{G} = \varphi G$ achieves its maximal value at x_1. Then at the point x_1, we have

$$\nabla G = -G\frac{\nabla \varphi}{\varphi}, \quad \mathcal{L}(\widetilde{G}) \leq 0. \qquad (2.10)$$

It is easy to see that at the point x_1,

$$0 \geq \mathcal{L}(\widetilde{G}) = \varphi\mathcal{L}G + G\mathcal{L}\varphi + 2|\nabla v|^{p-2}\nabla\varphi A(\nabla G) \qquad (2.11)$$

and

$$\varphi\mathcal{L}(G) \geq \frac{p}{n}\varphi G^2 + ap(p-1)^{(p-1)}\Big(\frac{n+2}{n} - \frac{\sigma}{p-1}\Big)\mathrm{e}^{(\frac{\sigma}{p-1}-1)v}\varphi G$$

$$
\begin{aligned}
&- pG^{\frac{2p-1}{p}}|\nabla\varphi| - (n-1)pK^2\varphi G^{\frac{2p-2}{p}} \\
&+ \frac{1}{n}a^2p(p-1)^{2(p-1)}\mathrm{e}^{2(\frac{\sigma}{p-1}-1)v}\varphi
\end{aligned}
\tag{2.12}
$$

by using (2.4).

Next, we estimate the last two terms on the right-hand side of (2.11). By the definition of A, we obtain

$$
\begin{aligned}
2|\nabla v|^{p-2}\nabla\varphi A(\nabla G) &= 2\Big[\nabla\varphi\nabla G + (p-2)\frac{(\nabla\varphi\nabla v)(\nabla v\nabla G)}{|\nabla v|^2}\Big]|\nabla v|^{p-2} \\
&= -2G\Big[\frac{|\nabla\varphi|^2}{\varphi} + (p-2)\frac{(\nabla v\nabla\varphi)^2}{\varphi|\nabla v|^2}\Big]|\nabla v|^{p-2} \\
&\geq -2\max\{p-1,1\}\frac{|\nabla\varphi|^2}{\varphi}G^{\frac{2p-2}{p}}.
\end{aligned}
\tag{2.13}
$$

On the other hand, due to

$$
\nabla\varphi\nabla|\nabla v|^{p-2} = \Big(\frac{2}{p}-1\Big)G^{\frac{p-2}{p}}\frac{|\nabla\varphi|^2}{\varphi}
$$

and

$$
|\nabla v|^{p-2}\nabla v\nabla\frac{\nabla\varphi\nabla v}{|\nabla v|^2} = |\nabla v|^{p-2}\Big(\frac{\varphi_{ij}v_iv_j}{|\nabla v|^2} - \frac{1}{p}\frac{|\nabla\varphi|^2}{\varphi} + \frac{2}{p}\frac{(\nabla\varphi\nabla v)^2}{|\nabla v|^2\varphi}\Big),
$$

we have

$$
\begin{aligned}
\mathcal{L}\varphi &= \operatorname{div}(|\nabla v|^{p-2}A(\nabla\varphi)) \\
&= |\nabla v|^{p-2}\Delta\varphi + (p-2)\frac{\nabla\varphi\nabla v}{|\nabla v|^2}\Delta_p v + \nabla\varphi\nabla|\nabla v|^{p-2} \\
&\quad + (p-2)|\nabla v|^{p-2}\nabla v\nabla\frac{\nabla\varphi\nabla v}{|\nabla v|^2} \\
&= |\nabla v|^{p-2}\Delta\varphi - (p-2)\frac{\nabla\varphi\nabla v}{|\nabla v|^2}(|\nabla v|^p + a(p-1)^{(p-1)}\mathrm{e}^{(\frac{\sigma}{p-1}-1)v}) \\
&\quad + (p-2)\frac{\varphi_{ij}v_iv_j}{|\nabla v|^2}|\nabla v|^{p-2} + 2\Big(\frac{2}{p}-1\Big)\frac{|\nabla\varphi|^2}{\varphi}G^{\frac{p-2}{p}} \\
&\quad + \frac{2(p-2)}{p}\frac{(\nabla\varphi\nabla v)^2}{|\nabla v|^2\varphi}|\nabla v|^{p-2} \\
&= \varphi_{ij}A^{ij}|\nabla v|^{p-2} - (p-2)\frac{\nabla\varphi\nabla v}{|\nabla v|^2}(|\nabla v|^p + a(p-1)^{(p-1)}\mathrm{e}^{(\frac{\sigma}{p-1}-1)v}) \\
&\quad + 2\Big(\frac{2}{p}-1\Big)\frac{|\nabla\varphi|^2}{\varphi}G^{\frac{p-2}{p}} + \frac{2(p-2)}{p}\frac{(\nabla\varphi\nabla v)^2}{|\nabla v|^2\varphi}|\nabla v|^{p-2},
\end{aligned}
\tag{2.14}
$$

which shows that

$$
\begin{aligned}
G\mathcal{L}\varphi &= \varphi_{ij}A^{ij}G^{\frac{2p-2}{p}} - (p-2)G^{\frac{2p-2}{p}}\nabla\varphi\nabla v \\
&\quad - a(p-2)(p-1)^{p-1}G^{\frac{p-2}{p}}\mathrm{e}^{(\frac{\sigma}{p-1}-1)v}\nabla\varphi\nabla v \\
&\quad + 2\Big(\frac{2}{p}-1\Big)\frac{|\nabla\varphi|^2}{\varphi}G^{\frac{2p-2}{p}} + \frac{2(p-2)}{p}\frac{(\nabla\varphi\nabla v)^2}{|\nabla v|^2\varphi}G^{\frac{2p-2}{p}} \\
&\geq \varphi_{ij}A^{ij}G^{\frac{2p-2}{p}} - |p-2|G^{\frac{2p-1}{p}}|\nabla\varphi|
\end{aligned}
$$

$$
\begin{aligned}
&- |a(p-2)|(p-1)^{p-1}G^{\frac{p-1}{p}}\mathrm{e}^{(\frac{\sigma}{p-1}-1)v}|\nabla\varphi| \\
&- \frac{2}{p}\max\{p-2,0\}\frac{|\nabla\varphi|^2}{\varphi}G^{\frac{2p-2}{p}}.
\end{aligned} \tag{2.15}
$$

Putting (2.12)–(2.13) and (2.15) into (2.11) yields

$$
\begin{aligned}
0 \geq\ & \frac{p}{n}\varphi G^2 - 2\max\{p-1,1\}G^{\frac{2p-1}{p}}|\nabla\varphi| \\
&+ \Big[\varphi_{ij}A^{ij} - (n-1)pK^2\varphi - \Big(2\max\{p-1,1\} \\
&+ \frac{2}{p}\max\{p-2,0\}\Big)\frac{|\nabla\varphi|^2}{\varphi}\Big]G^{\frac{2p-2}{p}} \\
&+ ap(p-1)^{(p-1)}\Big(\frac{n+2}{n} - \frac{\sigma}{p-1}\Big)\mathrm{e}^{(\frac{\sigma}{p-1}-1)v}\varphi G \\
&+ \frac{1}{n}a^2p(p-1)^{2(p-1)}\mathrm{e}^{2(\frac{\sigma}{p-1}-1)v}\varphi - |a(p-2)|(p-1)^{p-1}\mathrm{e}^{(\frac{\sigma}{p-1}-1)v}|\nabla\varphi|G^{\frac{p-1}{p}} \\
\geq\ & \frac{p}{n}\varphi G^2 - 2\max\{p-1,1\}G^{\frac{2p-1}{p}}|\nabla\varphi| \\
&+ \Big[\varphi_{ij}A^{ij} - (n-1)pK^2\varphi - \Big(2\max\{p-1,1\} \\
&+ \frac{2}{p}\max\{p-2,0\}\Big)\frac{|\nabla\varphi|^2}{\varphi}\Big]G^{\frac{2p-2}{p}} + \frac{1}{n}a^2p(p-1)^{2(p-1)}\mathrm{e}^{2(\frac{\sigma}{p-1}-1)v}\varphi \\
&- |a(p-2)|(p-1)^{p-1}\mathrm{e}^{(\frac{\sigma}{p-1}-1)v}|\nabla\varphi|G^{\frac{p-1}{p}}
\end{aligned} \tag{2.16}
$$

because of

$$a\Big(\frac{n+2}{n} - \frac{\sigma}{p-1}\Big) \geq 0.$$

Inserting

$$
\begin{aligned}
&\frac{p}{n}a^2(p-1)^{2(p-1)}\mathrm{e}^{2(\frac{\sigma}{p-1}-1)v}\varphi - |a(p-2)|(p-1)^{p-1}\mathrm{e}^{(\frac{\sigma}{p-1}-1)v}|\nabla\varphi|G^{\frac{p-1}{p}} \\
\geq\ & -\frac{n(p-2)^2}{4p}\frac{|\nabla\varphi|^2}{\varphi}G^{\frac{2p-2}{p}}
\end{aligned} \tag{2.17}
$$

into (2.16) gives

$$0 \geq \frac{p}{n}\varphi G^2 - 2\max\{p-1,1\}|\nabla\varphi|G^{\frac{2p-1}{p}} + D_1G^{\frac{2p-2}{p}}, \tag{2.18}$$

which is equivalent to

$$0 \geq \frac{p}{n}\varphi^{1-\frac{2}{p}}\widetilde{G}^{\frac{2}{p}} - 2\max\{p-1,1\}\frac{|\nabla\varphi|}{\varphi^{\frac{1}{2}}}\varphi^{1-\frac{p+2}{2p}}\widetilde{G}^{\frac{1}{p}} + D_1, \tag{2.19}$$

where

$$
\begin{aligned}
D_1 :=\ & \varphi_{ij}A^{ij} - (n-1)pK^2\varphi - \Big(2\max\{p-1,1\} + \frac{2}{p}\max\{p-2,0\}\Big)\frac{|\nabla\varphi|^2}{\varphi} \\
&- \frac{n(p-2)^2}{4p}\frac{|\nabla\varphi|^2}{\varphi}.
\end{aligned}
$$

Since x_1 is the maximum point of the function $\widetilde{G}$ and $\varphi = 1$ on $\mathbb{B}_R(x_0)$,

$$\varphi(x_1)|\nabla v|^p(x_1) \geq \sup_{\mathbb{B}_R(x_0)}|\nabla v|^p(x).$$

On the other hand, using the fact that

$$\varphi(x_1)|\nabla v|^p(x_1) \le \varphi(x_1) \sup_{\mathbb{B}_{2R}(x_0)} |\nabla v|^p(x),$$

it is easy to see that

$$\sigma(R) \le \varphi(x_1) \le 1,$$

where $\sigma(R)$ is defined by

$$\sigma(R) := \frac{\sup_{\mathbb{B}_R(x_0)} |\nabla v|^p(x)}{\sup_{\mathbb{B}_{2R}(x_0)} |\nabla v|^p(x)}.$$

Applying

$$\sigma(R) \to 1$$

and

$$D_1 \to -(n-1)pK^2$$

as $R \to \infty$, and using the fact if

$$ax^2 + bx + c \le 0$$

with $a > 0$, then

$$x \le \frac{-b + \sqrt{b^2 - 4ac}}{2a},$$

we obtain from (2.19) that

$$\begin{aligned} |\nabla v|(x) = \varphi(x)|\nabla v|(x) &\le \widetilde{G}^{\frac{1}{p}}(x_0) \\ &\le \sqrt{n(n-1)}K, \end{aligned} \tag{2.20}$$

which gives the estimate (1.4) and the proof of Theorem 1.1 is finished.

3 Proof of Theorem 1.2

Firstly, we prove the following lemma.

Lemma 3.1 *We introduce the elliptic operator $\mathcal{L}$ associated with u, which is defined by*

$$\mathcal{L} = \operatorname{div}(|\nabla u|^{p-2} A \nabla \cdot)$$

with $A = Id + (p-2)\frac{\nabla u \otimes \nabla u}{|\nabla u|^2}$, where u is a positive weak solution to (1.2). Then for $p > 1$ and

$$\omega = |\nabla u|^p, \tag{3.1}$$

we have

$$\begin{aligned} \mathcal{L}(\omega) \ge\ & \frac{\alpha + 2(p-2)}{p} \omega^{-\frac{2}{p}} |\nabla \omega|^2 + \frac{pa^2}{n-1} u^{2\sigma} \\ & + \frac{2a(p-1)}{n-1} u^{\sigma} \omega^{-\frac{2}{p}} \nabla u \nabla \omega + p\Big(\frac{p-2}{p}\Big)^2 \omega^{-\frac{4}{p}} (\nabla u \nabla \omega)^2 \\ & + p|\nabla u|^{2p-4} \operatorname{Ric}(\nabla u, \nabla u) - pa\sigma u^{\sigma-1} \omega. \end{aligned} \tag{3.2}$$

Proof As in (2.5), we have

$$\begin{aligned}\frac{1}{p}\mathcal{L}(|\nabla u|^p) &= |\nabla u|^{2p-4}|\mathrm{Hess}\, u|_A^2 + |\nabla u|^{2p-4}\mathrm{Ric}(\nabla u, \nabla u) + |\nabla u|^{p-2}\nabla u \nabla \Delta_p u \\ &= |\nabla u|^{2p-4}|\mathrm{Hess}\, u|_A^2 + |\nabla u|^{2p-4}\mathrm{Ric}(\nabla u, \nabla u) - a\sigma u^{\sigma-1}|\nabla u|^p. \end{aligned} \tag{3.3}$$

For any point $p \in M^n$ where $|\nabla u| \neq 0$, we choose an orthonormal frame $\{e_i\}_{i=1}^n$ such that $\nabla u = |\nabla u| e_1$, and hence $u_1 = |\nabla u|$, $u_2 = \cdots = u_n = 0$. Then, at the point p, we have $\omega = u_1^p$ and

$$\omega_k = p u_1^{p-1} u_{1k} \quad \text{for all } k \geq 1. \tag{3.4}$$

Since (1.2) is equivalent to

$$|\nabla u|^{p-2}\Delta u + \nabla u \nabla |\nabla u|^{p-2} + a u^\sigma = 0, \tag{3.5}$$

and hence at the point p it holds that

$$\Delta u = -(p-2)u_{11} - a u^\sigma u_1^{2-p}. \tag{3.6}$$

Thus,

$$\begin{aligned}\sum_{k=2}^n u_{kk} &= \Delta u - u_{11} \\ &= -(p-1)u_{11} - a u^\sigma u_1^{2-p} \end{aligned} \tag{3.7}$$

and

$$\begin{aligned}\sum_{i,j} u_{ij}^2 &\geq u_{11}^2 + 2\sum_{k=2} u_{1k}^2 + \sum_{k=2}^n u_{kk}^2 \\ &\geq u_{11}^2 + 2\sum_{k=2} u_{1k}^2 + \frac{1}{n-1}[(p-1)u_{11} + a u^\sigma u_1^{2-p}]^2 \\ &\geq u_{11}^2 + 2\sum_{k=2} u_{1k}^2 + \frac{(p-1)^2}{n-1}u_{11}^2 + \frac{a^2}{n-1}u^{2\sigma}u_1^{4-2p} \\ &\quad + \frac{2a(p-1)}{n-1}u^\sigma u_1^{2-p}u_{11} \\ &\geq \alpha \sum_{k=1} u_{1k}^2 + \frac{a^2}{n-1}u^{2\sigma}u_1^{4-2p} + \frac{2a(p-1)}{n-1}u^\sigma u_1^{2-p}u_{11}, \end{aligned} \tag{3.8}$$

where $\alpha = \min\left\{2, \frac{(p-1)^2}{n-1} + 1\right\}$. Using $p\omega u_{11} = \nabla u \nabla \omega$ and

$$\sum_{k=1} u_{1k}^2 = \frac{1}{p^2}u_1^{2-2p}|\nabla \omega|^2,$$

then we have

$$\sum_{i,j} u_{ij}^2 \geq \frac{\alpha}{p^2}\omega^{\frac{2-2p}{p}}|\nabla \omega|^2 + \frac{a^2}{n-1}u^{2\sigma}\omega^{\frac{4-2p}{p}} + \frac{2a(p-1)}{(n-1)p}u^\sigma \omega^{\frac{2-2p}{p}}\nabla u \nabla \omega. \tag{3.9}$$

Therefore,

$$\begin{aligned}|\mathrm{Hess}\, u|_A^2 &= [\delta_{ik} + (p-2)|\nabla u|^{-2}u_iu_k][\delta_{jl} + (p-2)|\nabla u|^{-2}u_ju_l]u_{ij}u_{kl} \\ &= u_{ij}^2 + 2(p-2)|\nabla u|^{-2}u_{ij}u_{il}u_ju_l + (p-2)^2|\nabla u|^{-4}u_{ij}u_{kl}u_iu_ju_ku_l \\ &= u_{ij}^2 + 2(p-2)\sum_{k=1} u_{1k}^2 + (p-2)^2u_{11}^2 \\ &\geq \frac{\alpha + 2(p-2)}{p^2}\omega^{\frac{2-2p}{p}}|\nabla\omega|^2 + \frac{a^2}{n-1}u^{2\sigma}\omega^{\frac{4-2p}{p}} \\ &\quad + \frac{2a(p-1)}{(n-1)p}u^\sigma\omega^{\frac{2-2p}{p}}\nabla u\nabla\omega + \Big(\frac{p-2}{p}\Big)^2\omega^{-2}(\nabla u\nabla\omega)^2, \end{aligned} \tag{3.10}$$

and hence from (3.3) we obtain

$$\begin{aligned}\frac{1}{p}\mathcal{L}(|\nabla u|^p) \geq{}& \frac{\alpha + 2(p-2)}{p^2}\omega^{-\frac{2}{p}}|\nabla\omega|^2 + \frac{a^2}{n-1}u^{2\sigma} \\ &+ \frac{2a(p-1)}{(n-1)p}u^\sigma\omega^{-\frac{2}{p}}\nabla u\nabla\omega + \Big(\frac{p-2}{p}\Big)^2\omega^{-\frac{4}{p}}(\nabla u\nabla\omega)^2 \\ &+ |\nabla u|^{2p-4}\mathrm{Ric}(\nabla u, \nabla u) - a\sigma u^{\sigma-1}\omega, \end{aligned} \tag{3.11}$$

finishing the proof of the estimate (3.2).

Inserting

$$\begin{aligned}&\frac{2a(p-1)}{n-1}u^\sigma\omega^{-\frac{2}{p}}\nabla u\nabla\omega + p\Big(\frac{p-2}{p}\Big)^2\omega^{-\frac{4}{p}}(\nabla u\nabla\omega)^2 \\ &= \Big(\frac{p-2}{\sqrt{p}}\omega^{-\frac{2}{p}}\nabla u\nabla\omega + \frac{a\sqrt{p}(p-1)}{(n-1)(p-2)}u^\sigma\Big)^2 - \frac{pa^2(p-1)^2}{(n-1)^2(p-2)^2}u^{2\sigma} \\ &\geq -\frac{pa^2(p-1)^2}{(n-1)^2(p-2)^2}u^{2\sigma}\end{aligned}$$

into (3.2) gives

$$\begin{aligned}\mathcal{L}(\omega) &\geq \frac{\alpha + 2(p-2)}{p}\omega^{-\frac{2}{p}}|\nabla\omega|^2 + \frac{pa^2}{n-1}\Big(1 - \frac{(p-1)^2}{(n-1)(p-2)^2}\Big)u^{2\sigma} \\ &\quad + p|\nabla u|^{2p-4}\mathrm{Ric}(\nabla u, \nabla u) - pa\sigma u^{\sigma-1}\omega \\ &\geq \frac{\alpha + 2(p-2)}{p}\omega^{-\frac{2}{p}}|\nabla\omega|^2 + p|\nabla u|^{2p-4}\mathrm{Ric}(\nabla u, \nabla u), \end{aligned} \tag{3.12}$$

provided

$$a\sigma < 0$$

and

$$p \in \Big(1, 2 - \frac{\sqrt{n-1}-1}{n-2}\Big] \sqcup \Big[2 + \frac{\sqrt{n-1}+1}{n-2}, +\infty\Big).$$

As in [14], we choose the cut-off function η such that $0 \leq \eta(x) \leq 1$ on $x \in M$ satisfying: $\eta(x) = 1$ if $x \in \overline{B_p(R)}$; $|\nabla\eta(x)| \leq \frac{10}{R}$ if $x \in B_p(2R)\backslash\overline{B_p(R)}$; $\eta(x) = 10$ if $x \in M\backslash B_p(2R)$. Then, we have

$$\int_M \eta^2\mathcal{L}(\omega) = -\int_M |\nabla u|^{p-2}A_{ij}\omega_i(\eta^2)_j$$

$$\le 2(1+|p-2|)\int_M \eta|\nabla\eta||\nabla u|^{p-2}|\nabla\omega|$$

$$\le \varepsilon_1\int_M \eta^2\omega^{-\frac{2}{p}}|\nabla\omega|^2+\frac{(1+|p-2|)^2}{\varepsilon_1}\int_M |\nabla\eta|^2\omega^{\frac{2p-2}{p}}, \tag{3.13}$$

which combining with (3.12) gives

$$\frac{(1+|p-2|)^2}{\varepsilon_1}\int_M |\nabla\eta|^2\omega^{\frac{2p-2}{p}}$$
$$\ge \Big(\frac{\alpha+2(p-2)}{p}-\varepsilon_1\Big)\int_M \eta^2\omega^{-\frac{2}{p}}|\nabla\omega|^2$$
$$+p\int_M \eta^2|\nabla u|^{2p-4}\mathrm{Ric}(\nabla u,\nabla u). \tag{3.14}$$

It is easy to see that

$$\int_M \eta^2\omega^{-\frac{2}{p}}|\nabla\omega|^2=\Big(\frac{p}{p-1}\Big)^2\int_M \eta^2|\nabla\omega^{\frac{p-1}{p}}|^2$$
$$=\Big(\frac{p}{p-1}\Big)^2\int_M |\nabla(\eta\omega^{\frac{p-1}{p}})-\omega^{\frac{p-1}{p}}\nabla\eta|^2$$
$$\ge\Big(\frac{p}{p-1}\Big)^2(1-\varepsilon_2)\int_M |\nabla(\eta\omega^{\frac{p-1}{p}})|^2$$
$$+\Big(\frac{p}{p-1}\Big)^2\Big(1-\frac{1}{\varepsilon_2}\Big)\int_M |\nabla\eta|^2\omega^{\frac{2p-2}{p}}. \tag{3.15}$$

Inserting it into (3.14) yields

$$\Big[\frac{(1+|p-2|)^2}{\varepsilon_1}-\Big(\frac{\alpha+2(p-2)}{p}-\varepsilon_1\Big)\Big(1-\frac{1}{\varepsilon_2}\Big)\Big(\frac{p}{p-1}\Big)^2\Big]\int_M |\nabla\eta|^2\omega^{\frac{2p-2}{p}}$$
$$\ge\Big(\frac{\alpha+2(p-2)}{p}-\varepsilon_1\Big)(1-\varepsilon_2)\Big(\frac{p}{p-1}\Big)^2\int_M |\nabla(\eta\omega^{\frac{p-1}{p}})|^2$$
$$+p\int_M \eta^2|\nabla u|^{2p-4}\mathrm{Ric}(\nabla u,\nabla u). \tag{3.16}$$

If $0<\varepsilon_2<1$ and

$$0<\varepsilon_1<\frac{\alpha+2(p-2)}{p}, \tag{3.17}$$

then using the Poincaré inequality (1.5) with $\psi=\eta\omega^{\frac{p-1}{p}}$ gives

$$\Big[\Big(\frac{\alpha+2(p-2)}{p}-\varepsilon_1\Big)(1-\varepsilon_2)\Big(\frac{p}{p-1}\Big)^2-\tau p\Big]\int_{B_p(2R)} \eta^2\rho\omega^{\frac{2p-2}{p}}$$

$$\leq \Big[\frac{(1+|p-2|)^2}{\varepsilon_1} - \Big(\frac{\alpha+2(p-2)}{p} - \varepsilon_1\Big)\Big(1-\frac{1}{\varepsilon_2}\Big)\Big(\frac{p}{p-1}\Big)^2\Big] \int\limits_{B_p(2R)\backslash B_p(R)} |\nabla\eta|^2 \omega^{\frac{2p-2}{p}}$$

$$\leq \Big[\frac{(1+|p-2|)^2}{\varepsilon_1} - \Big(\frac{\alpha+2(p-2)}{p} - \varepsilon_1\Big)\Big(1-\frac{1}{\varepsilon_2}\Big)\Big(\frac{p}{p-1}\Big)^2\Big] \frac{100}{R^2} \int\limits_{B_p(2R)\backslash B_p(R)} |\nabla\eta|^2 \omega^{\frac{2p-2}{p}}$$

from $\mathrm{Ric}(\nabla u, \nabla u) \geq -\tau\rho|\nabla u|^2$. Since

$$0 < \tau < \frac{\alpha+2(p-2)}{(p-1)^2},$$

there exist $\widetilde{\varepsilon}_1, \widetilde{\varepsilon}_2$ small enough such that

$$\Big(\frac{\alpha+2(p-2)}{p} - \widetilde{\varepsilon}_1\Big)(1-\widetilde{\varepsilon}_2)\Big(\frac{p}{p-1}\Big)^2 - \tau p > 0,$$

and hence

$$\begin{aligned}
&\Big[\Big(\frac{\alpha+2(p-2)}{p} - \widetilde{\varepsilon}_1\Big)(1-\widetilde{\varepsilon}_2)\Big(\frac{p}{p-1}\Big)^2 - \tau p\Big] \int\limits_{B_p(2R)} \eta^2 \rho \omega^{\frac{2p-2}{p}} \\
&\leq \Big[\frac{(1+|p-2|)^2}{\widetilde{\varepsilon}_1} - \Big(\frac{\alpha+2(p-2)}{p} - \widetilde{\varepsilon}_1\Big)\Big(1-\frac{1}{\widetilde{\varepsilon}_2}\Big)\Big(\frac{p}{p-1}\Big)^2\Big] \frac{100}{R^2} \int\limits_{B_p(2R)\backslash B_p(R)} |\nabla\eta|^2 \omega^{\frac{2p-2}{p}} \\
&\to 0
\end{aligned} \tag{3.18}$$

coming from the condition (1.8) by letting $R \to \infty$. Thus, from (3.18) we obtain that the function u is constant and the proof of Theorem 1.2 is completed.

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