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Maximal Operators of Multilinear Singular Integrals on Weighted Hardy Type Spaces*

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Abstract In this paper, the authors show that the maximal operators of the multilinear Calderón-Zygmund singular integrals are bounded from a product of weighted Hardy spaces into a weighted Lebesgue spaces, which essentially extend and improve the previous known results obtained by Grafakos and Kalton (2001) and Li, Xue and Yabuta (2011). The corresponding estimates on variable Hardy spaces are also established.

Keywords Weighted Hardy spaces, Variable Hardy spaces, Maximal operators, Multilinear Calderón-Zygmund operators, Multiple weights
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1 Introduction

The study of Hardy spaces was derived from the context of Fourier series and complex analysis in the early 1900s. It is well-known that many important operators in harmonic analysis are well-behaved both on L^p for $p > 1$ and on Hardy spaces. In fact, Hardy type spaces have been shown to be more natural and useful than the corresponding extension of Lebesgue spaces in many respects. A complete treatment of Hardy spaces in $\mathbb{R}^n$ was due to Stein, Weiss, Coifman and Fefferman in [2–3, 10]. Subsequently, the weighted Hardy space was introduced by García-Cuerra [11]. While, the intensive study of weighted Hardy spaces was the work of Strömberg and Torchinsky [22], in which they showed that linear Calderón-Zygmund operators whose kernels have enough regularity are bounded from $H^p(\omega)$ into $L^p(\omega)$ or $H^p(\omega)$ for $0 < p < \infty$, $\omega \in A_\infty$. For more works of weighted Hardy spaces, we refer readers to [7, 12, 16, 19], and the references there in.

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Recently, the investigation of multilinear operators on Hardy space has attracted many researchers' interests. It was Grafakos and Kalton [13] who first proved that the multilinear Calderón-Zygmund operators were bounded from products of Hardy spaces into Lebesgue spaces. The bounds into Hardy spaces were given by Grafakos, Nakamura, Nguyen and Sawano in [14] for the multilinear Calderón-Zygmund operators and in [15] for the multilinear multiplier operators. The corresponding boundedness of multilinear Calderón-Zygmund operators from products of weighted Hardy spaces into weighted Lebesgue spaces or Hardy spaces were also studied in [6, 17, 27]. Among them, we would like to mention the excellent work of Cruz-Uribe, Moen and Nguyen [6], in which they generalized the results in [13–14, 17, 22, 27] by establishing a finite atomic decomposition theorem for weighted Hardy spaces.

Inspired by the work in [6], we will focus on the maximal multilinear singular integral operators. We first recall the relevant notations and definitions.

Definition 1.1 (**Multilinear Calderón-Zygmund operator**) *Let T be a multilinear operator originally defined on the m-fold product of Schwartz spaces $\mathcal{S}(\mathbb{R}^n)$ and taking values in the space of tempered distributions $\mathcal{S}'(\mathbb{R}^n)$,*

$$T : \mathcal{S}(\mathbb{R}^n) \times \cdots \times \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n).$$

T is said to be an m-linear Calderón-Zygmund operator if it can be extended to be a bounded multilinear operator from $L^{q_1}(\mathbb{R}^n) \times \cdots \times L^{q_m}(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for some $1 < q_1, \cdots, q_m < \infty$ and $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$, and if there is a function K defined away from the diagonal $x = y_1 = \cdots = y_m$ in $(\mathbb{R}^n)^{m+1}$, satisfying

$$T(f_1, \cdots, f_m)(x) = \int_{(\mathbb{R}^n)^m} K(x, y_1, \cdots, y_m) f_1(y_1) \cdots f_m(y_m) \mathrm{d}y_1 \cdots \mathrm{d}y_m$$

for all $x \notin \bigcap_{i=1}^{m} \operatorname{supp}(f_i)$, and

$$|\partial_{y_0}^{\alpha_0} \cdots \partial_{y_m}^{\alpha_m} K(y_0, y_1, \cdots, y_m)| \le \frac{A_\alpha}{\Big(\sum\limits_{k,l=0}^{m} |y_k - y_l|\Big)^{mn+|\alpha|}} \tag{1.1}$$

for all multiple indices $\alpha = (\alpha_0, \cdots, \alpha_m)$ such that $|\alpha| = |\alpha_0| + \cdots + |\alpha_m| \le N$, where N is a sufficiently large integer and A_α is a positive constant.

Definition 1.2 (**Maximal multilinear Calderón-Zygmund operator**) *The maximal multilinear Calderón-Zygmund operator is defined by*

$$T^*(\vec{f})(x) = \sup_{\delta > 0} |T_\delta(f_1, \cdots, f_m)(x)|, \tag{1.2}$$

where T_δ are given by

$$T_\delta(f_1, \cdots, f_m)(x) = \int_{(\mathbb{R}^n)^m} K_\delta(x, y_1, \cdots, y_m) f_1(y_1) \cdots f_m(y_m) \mathrm{d}\vec{y}$$

with $K_\delta(x,y_1,\cdots,y_m) = \phi\big(\frac{\sqrt{(x-y_1)^2+\cdots+(x-y_m)^2}}{2\delta}\big)K(x,y_1,\cdots,y_m)$, *where* $\mathrm{d}\vec{y} = \mathrm{d}y_1\cdots\mathrm{d}y_m$ *and* $\phi(x)$ *is a smooth function on* $\mathbb{R}^n$, *which vanishes if* $|x|\le\frac14$ *and is equal to* 1 *if* $|x|>\frac12$.

In 2001, Grafakos and Kalton [13] proved the following result.

Theorem 1.1 (see [13]) *Let* $0<p_1,\cdots,p_m,p\le 1$, $\frac{1}{p_1}+\cdots+\frac{1}{p_m}=\frac1p$. *Suppose that* K *satisfies* (1.1) *with* $N=\lfloor n\big(\frac1p-1\big)\rfloor$. *Then*

$$\|T^*(\vec{f})\|_{L^p(\mathbb{R}^n)} \lesssim \prod_{i=1}^m \|f_i\|_{H^{p_i}(\mathbb{R}^n)}.$$

This result was subsequently extended to the weighted cases by Li, Xue and Yabuta in [18], which can be listed as follows.

Theorem 1.2 (see [18]) *Let* $1<q_1,\cdots,q_m,q<\infty$, $0<p_1,\cdots,p_m,p\le 1$ *with* $\frac{1}{q_1}+\cdots+\frac{1}{q_m}=\frac1q$ *and* $\frac{1}{p_1}+\cdots+\frac{1}{p_m}=\frac1p$. *Suppose that* K *satisfies* (1.1) *with* $N=\max\limits_{1\le i\le m}\big\{\lfloor n\big(\frac{(q_i)_\omega}{p_i}-1\big)\rfloor, \lfloor\big(\frac{q_i}{p_i}-1\big)mn\rfloor\big\}$. *Then*

(i) *for* $\omega\in A_{q_1}\cap\cdots\cap A_{q_m}$,

$$\|T^*(\vec{f})\|_{L^p(\omega)} \lesssim \prod_{i=1}^m \|f_i\|_{H^{p_i}(\omega)};$$

(ii) *for each* i, $\omega_i\in A_1$,

$$\|T^*(\vec{f})\|_{L^p(\nu_{\vec{\omega}})} \lesssim \prod_{i=1}^m \|f_i\|_{H^{p_i}(\omega_i)},$$

where $\nu_{\vec{\omega}}=\prod\limits_{i=1}^m \omega_i^{\frac{p}{p_i}}$, $r_\omega=\inf\{r\in(1,\infty): \omega\in A_r\}$.

Obviously, the ranges of weight functions in the above results are too narrow. It is quite natural to ask whether or not the restrictions imposed on the weights in Theorem 1.2 can be relaxed or removed.

The first main purpose of this paper is to address the question above. Our main result can be formulated as follows.

Theorem 1.3 (**Products of Hardy spaces into L^p space**) *Let* $m\in\mathbb{Z}^+$, $0<p_1,\cdots,p_m<\infty,\omega_k\in A_\infty$ $(1\le k\le m)$, T^* *be defined by* (1.2). *Suppose that* K *satisfies* (1.1) *for* N *such that*

$$N\ge \max\Big\{\Big\lfloor mn\Big(\frac{r_{\omega_k}}{p_k}-1\Big)\Big\rfloor_+, 1\le k\le m\Big\}+(m-1)n, \tag{1.3}$$

where $r_\omega := \inf\{r\in(1,\infty):\omega\in A_r\}$ *for* $\omega\in A_\infty$. *Then*

$$T^*: H^{p_1}(\omega_1)\times\cdots\times H^{p_m}(\omega_m)\to L^p(\nu_{\vec{\omega}}),$$

where $\nu_{\vec{\omega}}=\prod\limits_{k=1}^m \omega_k^{\frac{p}{p_k}}$ *and* $\frac1p=\frac{1}{p_1}+\cdots+\frac{1}{p_m}$.

Remark 1.1 Since $H^p(\omega) = L^p(\omega)$ when $p > 1$ and $\omega \in A_p$, we can replace $H^{p_k}(\omega_k)$ by $L^{p_k}(\omega_k)$ in Theorem 1.3. Comparing with the results of [13, 18], our theorem greatly relaxes the ranges of ω_i and p_i, p with $i = 1, \cdots, m$.

Another generalization of Hardy spaces in $\mathbb{R}^n$ is the variable Hardy spaces, which were independently introduced and studied in [9, 20] and was studied also in [21]. Recently, Tan [24] proved that bilinear Calderón-Zygmund operators are bounded from products of variable Hardy spaces into variable Lebesgue or Hardy spaces. While, the assumptions in [24] were immediately improved by Cruz-Uribe, Moen and Nguyen [6]. We refer readers to [5, 25–26] for other results about variable Hardy spaces.

Our next theorem will extend the above results to the maximal multilinear Calderón-Zygmund operators, which are analogue to Theorem 1.3. We first recall some basic facts of variable spaces.

Let $\mathcal{P}_0$ be the set of all measurable functions $p(\cdot) : \mathbb{R}^n \to (0, \infty)$, the variable Lebesgue space $L^{p(\cdot)}$ consists of all measurable functions f such that

$$\|f\|_{L^{p(\cdot)}} = \inf\Big\{\lambda > 0 : \int_{\mathbb{R}^n} \Big(\frac{|f(x)|}{\lambda}\Big)^{p(x)} \mathrm{d}x \le 1\Big\} < \infty.$$

These spaces were first introduced by Orlicz [23], for further details, see [4]. Given $p(\cdot) \in \mathcal{P}_0$, we define

$$p_- = \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} p(x), \quad p_+ = \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} p(x).$$

Denote $p(\cdot) \in \mathcal{B}$ if the Hardy-Littlewood maximal operator is bounded on $L^{p(\cdot)}$. A sufficient condition of $p(\cdot) \in \mathcal{B}$ is that $p(\cdot) \in \mathcal{P}_0, 1 < p_- \le p_+ < \infty$ and $p(\cdot)$ is log-Hölder continuous, i.e., there exist constants C_0, C_∞ and p_∞ such that

$$|p(x) - p(y)| \le \frac{C_0}{-\log(|x-y|)}, \quad |x-y| < \frac{1}{2}$$

and

$$|p(x) - p_\infty| \le \frac{C_\infty}{\log(\mathrm{e} + |x|)}.$$

Variable Hardy spaces $H^{p(\cdot)}$ (see its definition in Section 2) were independently introduced by Cruz-Uribe, Wang [9] and Nakai, Sawano [20]. Now, we state our results in the setting of variable exponents.

Theorem 1.4 *Let $m \in \mathbb{Z}^+$ and $p_1, \cdots, p_m \in \mathbb{R}^+$. Suppose that $q_1(\cdot), \cdots, q_m(\cdot)$ satisfy the* log*-Hölder continuous condition with $0 < p_k < (q_k)_- \le (q_k)_+ < \infty$. Let T^* be as in Theorem* 1.3 *and K satisfy* (1.1) *for N such that*

$$N \ge \max\Big\{\Big\lfloor mn\Big(\frac{1}{p_k} - 1\Big)\Big\rfloor_+, 1 \le k \le m\Big\} + (m-1)n.$$

Then

$$T^* : H^{q_1(\cdot)} \times \cdots \times H^{q_m(\cdot)} \to L^{q(\cdot)},$$

where $\frac{1}{q}(\cdot) = \frac{1}{q_1}(\cdot) + \cdots + \frac{1}{q_m}(\cdot)$.

The rest of this paper is organized as follows. In Section 2, we will recall some facts about weights and weighted inequalities as well as the weighted and variable exponent Hardy spaces. The proof of Theorem 1.3 will be given in Section 3. Finally, we will prove Theorem 1.4 in Section 4.

Through out the rest of the paper, we denote a positive constant by C, which may change at each occurrence. We denote $f \lesssim g$, $f \sim g$ if $f \le Cg$ and $f \lesssim g \lesssim f$, respectively. Given a cube Q and $\tau > 0$, denote $l(Q)$ the side-length of cube Q, let $Q^* = 2\sqrt{n}Q$, where τQ means the cube with the same center of Q such that $l(\tau Q) = \tau l(Q)$. Given a measurable set $\Gamma \subset \mathbb{R}^n$, we denote the Lebesgue measure of Γ by $|\Gamma|$. As usual, we will denote $t^{-n}\phi\big(\frac{x}{t}\big)$ by $\phi_t(x)$ and $\mathrm{d}\vec{y} = \mathrm{d}y_1 \cdots \mathrm{d}y_m$.

2 Preliminaries

2.1 Weights and vector-valued inequality

In this subsection, we give some basic definitions of weights and weighted vector-valued inequality that we will use.

A weight ω is a nonnegative locally integrable function such that $0 < \omega(x) < \infty$ a.e. $x \in \mathbb{R}^n$. For $1 < p < \infty$, we say that a weight $\omega \in A_p$ if

$$[\omega]_{A_p} = \sup_Q \Big(\frac{1}{|Q|}\int_Q \omega(y)\mathrm{d}y\Big)\Big(\frac{1}{|Q|}\int_Q \omega(y)^{1-p'}\mathrm{d}y\Big)^{p-1} < \infty,$$

where $\frac{1}{p} + \frac{1}{p'} = 1$ and the supremum is taken over all cubes $Q \subset \mathbb{R}^n$. A weight $w \in A_1$ if there is a constant C such that for any cube Q,

$$\frac{1}{|Q|}\int_Q w(y)\mathrm{d}y \le C \inf_{y\in Q} w(y).$$

Then we define the A_∞ class of weights by $A_\infty = \bigcup\limits_{1\le p<\infty} A_p$.

The following weighted vector-valued inequality is frequently applied in our proof, which was proved by Anderson and John [1].

Lemma 2.1 (see [1]) *Given* $1 < p, q < \infty$ *and* $\omega \in A_p$, *it holds that*

$$\Big\|\Big(\sum_k (Mf_k)^q\Big)^{\frac{1}{q}}\Big\|_{L^p(\omega)} \lesssim \Big\|\Big(\sum_k |f_k|^q\Big)^{\frac{1}{q}}\Big\|_{L^p(\omega)},$$

where M is the Hardy-Littlewood maximal function.

We also need the following weighted inequalities.

Lemma 2.2 (see [6]) *Let $\omega \in A_\infty$, and fix $0 < p < \infty$ and $\max\{1, p\} < q < \infty$. Then given any collection of cubes $\{Q_k\}_{k=1}^\infty$ and nonnegative integrable functions g_k with $\operatorname{supp}(g_k) \subset Q_k$,*

$$\Big\| \sum_{k=1}^\infty g_k \Big\|_{L^p(\omega)} \le C(\omega, p, q, n) \Big\| \sum_{k=1}^\infty \Big(\frac{1}{\omega(Q_k)} \int_{Q_k} g_k(x)^q \omega(x) \mathrm{d}x \Big)^{\frac{1}{q}} \chi_{Q_k} \Big\|_{L^p(\omega)}.$$

Lemma 2.3 (see [6]) *Let T be the multilinear Calderón-Zygmund operator defined in Definition 1.1 and fix $\omega \in A_q$, $q > 1$. Then given any collection $f_1, \cdots, f_m$ of bounded functions of compact support,*

$$\|T(f_1, \cdots, f_m)\|_{L^q(\omega)} \le C \|f_1\|_{L^q(\omega)} \prod_{k=2}^m \|f_k\|_{L^\infty}.$$

2.2 Weighted and variable exponent Hardy spaces

In this subsection, we state some elementary results about weighted and variable Hardy spaces. For more details, see [7, 9, 20, 22].

Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz class of smooth functions. For a large integer N_0, denote

$$\mathcal{S}_{N_0} = \Big\{ \phi \in \mathcal{S}(\mathbb{R}^n) : \int_{\mathbb{R}^n} (1 + |x|)^{N_0} \Big(\sum_{|\beta| \le N_0} \Big| \frac{\partial^\beta}{\partial x^\beta} \phi(x) \Big|^2 \Big) \mathrm{d}x \le 1 \Big\}.$$

Given $\omega \in A_\infty$ and $0 < p < \infty$, the weighted Hardy spaces $H^p(\omega)$ is defined by

$$H^p(\omega) = \{ f \in \mathcal{S}'(\mathbb{R}^n) : \mathcal{M}_{N_0}(f) \in L^p(\omega) \}$$

with the quasi-norm

$$\|f\|_{H^p(\omega)} = \|\mathcal{M}_{N_0}(f)\|_{L^p(\omega)},$$

where $\mathcal{M}_{N_0}(f)$ is given by

$$\mathcal{M}_{N_0}(f)(x) = \sup_{\phi \in \mathcal{S}_{N_0}} \sup_{t > 0} |\phi_t * f(x)|.$$

Given an integer $N > 0$, we say that a function a is an (N, ∞)-atom if $\|a\|_{L^\infty} \le 1$, $\operatorname{supp}(a) \subset Q$ for some cube Q, and for $|\beta| \le N$,

$$\int_{\mathbb{R}^n} x^\beta a(x) \mathrm{d}x = 0.$$

For $\omega \in A_\infty$ and $0 < p < \infty$, denote $S_\omega := \lfloor n(\frac{r_\omega}{p} - 1) \rfloor_+$. If $N > S_\omega$, then any (N, ∞)-atom is in $H^p(\omega)$. Further more, in [22], it was proved that for $N > S_\omega$, there exists a sequence of nonnegative numbers $\{\lambda_k\}$ and a sequence of (N, ∞)-atoms $\{a_k\}$ with $\operatorname{supp}(a_k) \subset Q_k$ such that for every $f \in H^p(\omega)$,

$$f = \sum_k \lambda_k a_k, \quad \|f\|_{H^p(\omega)} \sim \Big\| \sum_k \lambda_k \chi_{Q_k} \Big\|_{L^p(\omega)}.$$

For $N > S_\omega$, we define

$$\mathcal{O}_N = \Big\{ f \in C_0^\infty : \int_{\mathbb{R}^n} x^\beta f(x) \mathrm{d}x = 0,\ 0 \le |\beta| \le N \Big\}.$$

Then $\mathcal{O}_N$ is dense in $H^p(\omega)$ (see [6–7]). In addition, we have the following finite atomic decomposition which was given in [7].

Lemma 2.4 (see [7]) *Given $0 < p < \infty$ and $\omega \in A_\infty$, fix $N > S_\omega$. Then if $f \in \mathcal{O}_N$, there exists a finite sequence $\{a_i\}_{i=1}^M$ of (N, ∞)-atoms with supports Q_i, and a nonnegative sequence $\{\lambda_i\}_{i=1}^M$ such that $f = \sum\limits_i \lambda_i a_i$ and*

$$\Big\| \sum_{i=1}^M \lambda_i \chi_{Q_i} \Big\|_{L^p(\omega)} \lesssim \|f\|_{H^p(\omega)}.$$

Next we introduce some basic facts about variable Hardy space. Given $p(\cdot) \in \mathcal{P}_0$, the variable Hardy space $H^{p(\cdot)}$ is defined to be the set of all distributions

$$H^{p(\cdot)} = \{ f \in \mathcal{S}'(\mathbb{R}^n) : \mathcal{M}_{N_0} f \in L^{p(\cdot)} \}.$$

It was shown in [7] that every $f \in H^{p(\cdot)}$ can also be decomposed as the sum of atoms: Assume $p(\cdot) \in \mathcal{P}_0$, then for $N > \lfloor n(p_-^{-1} - 1) \rfloor_+$, there is a sequence of (N, ∞)-atoms a_k and $\lambda_k > 0$ such that for any $f \in H^{p(\cdot)}$,

$$f = \sum_k \lambda_k a_k, \quad \|f\|_{H^{p(\cdot)}} \sim \Big\| \sum_k \lambda_k \chi_{Q_k} \Big\|_{L^{p(\cdot)}}.$$

Given $N > S_{p(\cdot)} := \lfloor n(p_-^{-1} - 1) \rfloor_+$, then $\mathcal{O}_N$ is dense in $H^{p(\cdot)}$. Similarly to Lemma 2.4, the following finite atomic decomposition was also established in [7]

Lemma 2.5 (see [7]) *Given $p(\cdot) \in \mathcal{P}_0$, suppose that $0 < p_- \le p_+ < \infty$ and $p(\cdot)$ satisfies the* log-*Hölder continuous condition. Fix $0 < p_0 < p_-$. Then for any N such that*

$$N > \lfloor n(p_0^{-1} - 1) \rfloor_+,$$

given any $f \in \mathcal{O}_N$, there exists a finite sequence $\{a_i\}_{i=1}^M$ of (N, ∞)-atoms with supports Q_i, and a nonnegative sequence $\{\lambda_i\}_{i=1}^M$ such that

$$f = \sum_i \lambda_i a_i$$

and

$$\Big\| \sum_{i=1}^M \lambda_i \chi_{Q_i} \Big\|_{L^{p(\cdot)}} \lesssim \|f\|_{H^{p(\cdot)}}.$$

3 Proof of Theorem 1.3

This section is devoted to proving Theorem 1.3. First, we establish the following two lemmas.

Lemma 3.1 *Let T^* and K be stated as in Theorem 1.3 and a_i be an (N,∞)-atom supported in Q_i $(1\le i\le m)$. Then we have*

$$T^*(a_1,\cdots,a_m)\chi_{\big(\bigcap\limits_{i=1}^m Q_i^*\big)^c}\lesssim\prod_{i=1}^m(M(\chi_{Q_i}))^{\frac{n+N+1}{mn}},\tag{3.1}$$

where M is the Hardy-Littlewood maximal operator.

Proof Given any non-empty subset $\Gamma\subset\{1,\cdots,m\}$, to prove (3.1), it suffices to prove for all $y\notin\bigcup\limits_{k\in\Gamma}Q_k^*$,

$$T^*(a_1,\cdots,a_m)(y)\lesssim\frac{\min\{l(Q_i):i\in\Gamma\}^{n+N+1}}{\big(\sum\limits_{i\in\Gamma}|y-c_i|\big)^{n+N+1}},\tag{3.2}$$

where c_i is the center of Q_i.

Indeed, fix $y\in\big(\bigcap\limits_{i=1}^m Q_i^*\big)^c$, one can find a non-empty subset $\Gamma\subset\{1,\cdots,m\}$ such that $y\notin Q_i^*$ for all $i\in\Gamma$ and $y\in Q_k^*$ for all $k\notin\Gamma$. Then

$$\begin{aligned}&\prod_{i=1}^m\Big(\frac{l(Q_i)^n}{(l(Q_i)+|y-c_i|)^n}\Big)^{\frac{n+N+1}{mn}}\\&=\prod_{i\in\Gamma}\Big(\frac{l(Q_i)^n}{(l(Q_i)+|y-c_i|)^n}\Big)^{\frac{n+N+1}{mn}}\prod_{i\notin\Gamma}\Big(\frac{l(Q_i)^n}{(l(Q_i)+|y-c_i|)^n}\Big)^{\frac{n+N+1}{mn}}\\&\gtrsim\prod_{i\in\Gamma}\Big(\frac{l(Q_i)^n}{(l(Q_i)+|y-c_i|)^n}\Big)^{\frac{n+N+1}{mn}}.\end{aligned}$$

By (3.2), we have

$$\begin{aligned}T^*(a_1,\cdots,a_m)&\lesssim\frac{\min\{l(Q_i):i\in\Gamma\}^{n+N+1}}{\big(\sum\limits_{i\in\Gamma}|y-c_i|\big)^{n+N+1}}\\&\lesssim\prod_{i\in\Gamma}\Big(\frac{l(Q_i)^n}{(l(Q_i)+|y-c_i|)^n}\Big)^{\frac{n+N+1}{mn}}\\&\lesssim\prod_{i=1}^m\Big(\frac{l(Q_i)^n}{(l(Q_i)+|y-c_i|)^n}\Big)^{\frac{n+N+1}{mn}}\lesssim\prod_{i=1}^m(M(\chi_{Q_i}))^{\frac{n+N+1}{mn}}.\end{aligned}$$

Now, we turn to prove (3.2). For brevity, we may assume that $\Gamma=\{1,\cdots,r\}$ for some $1\le r\le m$. For $y\notin\bigcup\limits_{i\in\Gamma}Q_i^*$, since a_1 has zero vanishing moments up to order N, we deduce that

$$T^*(a_1,\cdots,a_m)(y)$$

$$= \sup_{\delta>0} \Big| \int_{\mathbb{R}^{mn}} [K_\delta(y, y_1, \cdots, y_m) - P_N(y, y_1, \cdots, y_m)] a_1(y_1) \cdots a_m(y_m) \mathrm{d}\vec{y} \Big|$$
$$=: \sup_{\delta>0} \Big| \int_{\mathbb{R}^{mn}} K_\delta^1(y, y_1, \cdots, y_m) a_1(y_1) \cdots a_m(y_m) \mathrm{d}\vec{y} \Big|,$$

where

$$P_N(y, y_1, \cdots, y_m) := \sum_{|\alpha| \le N} \frac{1}{\alpha!} \partial_2^\alpha K_\delta(y, c_1, y_2, \cdots, y_m)(y_1 - c_1)^\alpha.$$

Set $y_0 = y$. Observe that $\sqrt{\sum\limits_{i=1}^m (y_0 - y_i)^2} \sim \delta$ when ϕ is not always a constant and $\sum\limits_{i=1}^m |y_0 - y_i| \sim \sum\limits_{k,l=0}^m |y_k - y_l|$, one can verify that the kernels K_δ satisfy (1.1) uniformly in $\delta > 0$. Hence, by (1.1) and the fact that $|y - y_i| \sim |y - c_i|$ for all $i \in \Gamma$ and $y_i \in Q_i$, we get

$$|K_\delta^1(y, y_1, \cdots, y_m)| \lesssim |y_1 - c_1|^{N+1} \Big(\sum_{i\in\Gamma} |y - c_i| + \sum_{k=2}^m |y - y_k| \Big)^{-mn-N-1}.$$

This together with using polar coordinates formula leads to

$$\begin{aligned} T^*(a_1, \cdots, a_m)(y) &\lesssim \sup_{\delta>0} \int_{\mathbb{R}^{mn}} |K_\delta^1(y, y_1, \cdots, y_m)||a_1(y_1) \cdots a_m(y_m)| \mathrm{d}\vec{y} \\ &\lesssim \int_{\mathbb{R}^{mn}} \frac{|y_1 - c_1|^{N+1} |a_1(y_1) \cdots a_m(y_m)|}{\big(\sum\limits_{i\in\Gamma} |y - c_i| + \sum\limits_{k=2}^m |y - y_k| \big)^{mn+N+1}} \mathrm{d}\vec{y} \\ &\lesssim \int_{\mathbb{R}^{(m-1)n}} \frac{l(Q_1)^{n+N+1}}{\big(\sum\limits_{i\in\Gamma} |y - c_i| + \sum\limits_{k=2}^m |y_k| \big)^{mn+N+1}} \mathrm{d}y_2 \cdots \mathrm{d}y_m \\ &\lesssim \frac{l(Q_1)^{n+N+1}}{\big(\sum\limits_{i\in\Gamma} |y - c_i| \big)^{n+N+1}}, \end{aligned} \tag{3.3}$$

which completes the proof of Lemma 3.1.

Lemma 3.2 *Let T^* and K be stated as in Theorem 1.3, $\omega \in A_q$ $(1 \le q < \infty)$ and a_i be an (N, ∞)-atom supported in Q_i. Assume $l(Q_1) = \min\{l(Q_i) : 1 \le i \le m\}$. Then*

$$\|T^*(a_1, \cdots, a_m)\chi_{Q_1^*}\|_{L^q(\omega)} \lesssim \omega(Q_1)^{\frac{1}{q}} \prod_{i=1}^m \inf_{z\in Q_1} (M(\chi_{Q_i}))(z)^{\frac{n+N+1}{mn}}. \tag{3.4}$$

Proof We will demonstrate (3.4) by considering the following two cases.

Case 1 $Q_1^* \cap Q_k^* \ne \emptyset$ for all $2 \le k \le m$. In this case, observe that $Q_1^* \subset 3Q_k^*$ for all k, we have

$$M(\chi_{Q_i})(z) \ge \frac{|Q_1 \cap Q_k|}{|Q_1|} \ge \frac{|\frac{1}{3}Q_1 \cap Q_k|}{|Q_1|} \gtrsim 1, \quad \forall\, z \in Q_1.$$

The same reasoning as in [18] may yield that, for all $\eta > 0$, there exists a constant $C > 0$ such that

$$T^*(\vec{f})(x) \le C((M(|T(\vec{f})|^\eta)(x))^{\frac{1}{\eta}} + \mathcal{M}(\vec{f})(x)),$$

where $\mathcal{M}(\vec{f})(x) = \sup\limits_{Q \ni x} \prod\limits_{i=1}^{m} |Q|^{-1} \int_Q |f_i(y_i)| \mathrm{d}y_i$.

Now, we choose η such that $\omega \in A_{\frac{q}{\eta}}$, which implies that

$$\begin{aligned}
&\|T^*(a_1, \cdots, a_m)\chi_{Q_1^*}\|_{L^q(\omega)} \\
&\le \|T^*(a_1, \cdots, a_m)\|_{L^q(\omega)} \\
&\lesssim \|(M(|T(a_1, \cdots, a_m)(x)|^\eta))^{\frac{1}{\eta}}\|_{L^q(\omega)} + \|\mathcal{M}(a_1, \cdots, a_m)\|_{L^q(\omega)} \\
&\lesssim \|T(a_1, \cdots, a_m)\|_{L^q(\omega)} + \|a_1\|_{L^q(\omega)}\|a_2\|_{L^\infty} \cdots \|a_m\|_{L^\infty} \\
&\lesssim \|a_1\|_{L^q(\omega)}\|a_2\|_{L^\infty} \cdots \|a_m\|_{L^\infty} \\
&\lesssim \omega(Q_1)^{\frac{1}{q}} \prod_{i=1}^{m} \inf_{z \in Q_1} M(\chi_{Q_i})(z),
\end{aligned} \tag{3.5}$$

where $q > 1$ and the last second inequality follows from Lemma 2.3. When $q = 1$, choosing $q_0 > 1$, using (3.5) and $\omega \in A_{q_0}$, we have

$$\begin{aligned}
\|T^*(a_1, \cdots, a_m)\chi_{Q_1^*}\|_{L^1(\omega)} &\le \|T^*(a_1, \cdots, a_m)\chi_{Q_1^*}\|_{L^{q_0}(\omega)}\omega(Q_1^*)^{\frac{1}{q_0'}} \\
&\lesssim \omega(Q_1)^{\frac{1}{q_0}}\omega(Q_1)^{\frac{1}{q_0'}} \prod_{i=1}^{m} \inf_{z \in Q_1} M(\chi_{Q_i})(z) \\
&= \omega(Q_1) \prod_{i=1}^{m} \inf_{z \in Q_1} M(\chi_{Q_i})(z).
\end{aligned}$$

Case 2 $Q_1^* \cap Q_k^* = \emptyset$ for some k. For $y_1 \in Q_1$, since K_δ^1 satisfies (1.1) uniformly in $\delta > 0$, we have that for some $\xi_1 \in Q_1$ and $y_i \in Q_i$ $(1 \le i \le m)$,

$$|K_\delta^1(y, y_1, \cdots, y_m)| \lesssim l(Q_1)^{N+1}\Big(|y - \xi_1| + \sum_{j=2}^{m} |y - y_j|\Big)^{-mn-N-1}.$$

For $y_1 \in Q_1^*$, $y_i \in Q_i$ $(i \in \Gamma)$, one can check that

$$l(Q_1)^{N+1}\Big(|y - \xi_1| + \sum_{j=2}^{m} |y - y_j|\Big)^{-mn-N-1} \lesssim l(Q_1)^{N+1}\Big(\sum_{i \in \Gamma} |c_1 - c_i| + \sum_{j=2}^{m} |y - y_j|\Big)^{-mn-N-1}.$$

Keeping in mind the definition of T^*, similarly to (3.3), we have

$$\begin{aligned}
T^*(a_1, \cdots, a_m)(y) &= \sup_{\delta > 0}\Big|\int_{\mathbb{R}^{mn}} K_\delta^1(y, y_1, \cdots, y_m)a_1(y_1)\cdots a_m(y_m)\mathrm{d}\vec{y}\Big| \\
&\lesssim \int_{\mathbb{R}^{mn}} \frac{l(Q_1)^{N+1}|a_1(y_1)|\cdots|a_m(y_m)|}{\big(\sum\limits_{i \in \Gamma} |c_1 - c_i| + \sum\limits_{j=2}^{m} |y - y_j|\big)^{mn+N+1}}\mathrm{d}\vec{y} \\
&\lesssim \frac{l(Q_1)^{n+N+1}}{\big(\sum\limits_{i \in \Gamma} |c_1 - c_i|\big)^{n+N+1}} \\
&\lesssim \frac{l(Q_1)^{n+N+1}}{\big(\sum\limits_{i \in \Gamma} (|c_1 - c_i| + l(Q_1) + l(Q_i))\big)^{n+N+1}}.
\end{aligned} \tag{3.6}$$

For any $z \in Q_1$, it is not hard to prove that

$$\frac{l(Q_1)^{n+N+1}}{\big(\sum\limits_{k\in\Gamma}[l(Q_1)+|c_1-c_k|+l(Q_k)]\big)^{n+N+1}} \lesssim \prod_{k=1}^{m} \inf_{z\in Q_1} M(\chi_{Q_k})(z)^{\frac{n+N+1}{mn}}.$$

This together with (3.6) yields that

$$\|T^*(a_1,\cdots,a_m)\|_{L^\infty} \lesssim \prod_{k=1}^{m} \inf_{z\in Q_1} M(\chi_{Q_k})(z)^{\frac{n+N+1}{mn}}.$$

Then, for $q \geq 1$, by the double condition of ω, we have

$$\begin{aligned}\|T^*(a_1,\cdots,a_m)\chi_{Q_1^*}\|_{L^q(\omega)} &\leq \omega(Q_1^*)^{\frac{1}{q}}\|T^*(a_1,\cdots,a_m)\|_{L^\infty}\\ &\lesssim \omega(Q_1)^{\frac{1}{q}} \prod_{i=1}^{m} \inf_{z\in Q_1} (M(\chi_{Q_i}))(z)^{\frac{n+N+1}{mn}}.\end{aligned}$$

This finishes the proof of Lemma 3.2.

Now we are in the position to prove Theorem 1.3.

Proof of Theorem 1.3 By Lemma 2.4, it will suffice to fix a finite sum of (N,∞)-atoms,

$$f_i = \sum_{j_i=1}^{N_0} \lambda_{i,j_i} a_{i,j_i}$$

with a_{i,j_i} are (N,∞)-atoms, $\operatorname{supp}(a_i,j_i) \subset Q_{i,j_i}$, and prove that

$$\|T^*(f_1,\cdots,f_m)\|_{L^p(\nu_{\vec{\omega}})} \lesssim \prod_{i=1}^{m} \Big\|\sum_{j_i} \lambda_{i,j_i}\chi_{Q_{i,j_i}}\Big\|_{L^{p_i}(\omega_i)}.$$

Denote the smallest cube among $Q^*_{1,j_1},\cdots,Q^*_{m,j_m}$ by $S_{j_1,\cdots,j_m}$. Since

$$T^*(f_1,\cdots,f_m)(x) \leq \sum_{j_1,\cdots,j_m} \lambda_{1,j_1},\cdots,\lambda_{m,j_m} T^*(a_{1,j_1},\cdots,a_{m,j_m})(x),$$

we will split $T^*(f_1,\cdots,f_m)$ into two terms $I_1(x), I_2(x)$, where

$$I_1(x) := \sum_{j_1,\cdots,j_m} \lambda_{1,j_1},\cdots,\lambda_{m,j_m} T^*(a_{1,j_1},\cdots,a_{m,j_m})(x)\chi_{S_{j_1,\cdots,j_m}}(x)$$

and

$$I_2(x) := \sum_{j_1,\cdots,j_m} \lambda_{1,j_1},\cdots,\lambda_{m,j_m} T^*(a_{1,j_1},\cdots,a_{m,j_m})(x)\chi_{(S_{j_1,\cdots,j_m})^c}(x).$$

We first estimate $I_1(x)$. Note that $\nu_{\vec{\omega}} \in A_\infty$, one can select $q > \max\{1,p\}$ such that $\nu_{\vec{\omega}} \in A_q$. Invoking Lemma 3.2, we have

$$\Big(\frac{1}{\nu_{\vec{\omega}}(S_{j_1,\cdots,j_m})}\int_{S_{j_1,\cdots,j_m}} T^*(a_{1,j_1},\cdots,a_{m,j_m})(x)^q \nu_{\vec{\omega}}(x)\mathrm{d}x\Big)^{\frac{1}{q}}$$

$$\lesssim \prod_{i=1}^{m} \inf_{z\in S_{j_1,\cdots,j_m}} M(\chi_{Q_i,j_i})(z)^{\frac{n+N+1}{mn}}.$$

Thus Lemma 2.2 yields that,

$$\begin{aligned}\|I_1\|_{L^p(\nu_{\vec{\omega}})} &\lesssim \Big\| \sum_{j_1,\cdots,j_m} \prod_{i=1}^{m} \lambda_{i,j_i} \Big(\frac{1}{\nu_{\vec{\omega}}(S_{j_1,\cdots,j_m})} \int_{S_{j_1,\cdots,j_m}} T^*(a_{1,j_1},\cdots,a_{m,j_m})(x)^q \\ &\qquad \times \nu_{\vec{\omega}}(x)\mathrm{d}x \Big)^{\frac{1}{q}} \chi_{S_{j_1,\cdots,j_m}} \Big\|_{L^p(\nu_{\vec{\omega}})} \\ &\lesssim \Big\| \sum_{j_1,\cdots,j_m} \prod_{i=1}^{m} \lambda_{i,j_i} \Big(\prod_{i=1}^{m} \inf_{z\in S_{j_1,\cdots,j_m}} M(\chi_{Q_i,j_i})(z)^{\frac{n+N+1}{mn}} \Big) \chi_{S_{j_1,\cdots,j_m}} \Big\|_{L^p(\nu_{\vec{\omega}})} \\ &\le \Big\| \prod_{i=1}^{m} \Big(\sum_{j_i} \lambda_{i,j_i} M(\chi_{Q_i,j_i})^{\frac{n+N+1}{mn}} \Big) \Big\|_{L^p(\nu_{\vec{\omega}})}.\end{aligned}$$

Observe that

$$\begin{aligned}&\Big\| \prod_{i=1}^{m} \Big(\sum_{j_i} \lambda_{i,j_i} M(\chi_{Q_i,j_i})^{\frac{n+N+1}{mn}} \Big) \Big\|_{L^p(\nu_{\vec{\omega}})} \\ &\le \prod_{i=1}^{m} \Big\| \sum_{j_i} \lambda_{i,j_i} M(\chi_{Q_i,j_i})^{\frac{n+N+1}{mn}} \Big\|_{L^{p_i}(\omega_i)} \\ &= \prod_{i=1}^{m} \Big\| \Big(\sum_{j_i} M(\lambda_{i,j_i}^{\frac{mn}{n+N+1}} \chi_{Q_{i,j_i}})^{\frac{n+N+1}{mn}} \Big)^{\frac{mn}{n+N+1}} \Big\|_{L^{p_i \frac{n+N+1}{mn}}(\omega_i)}^{\frac{n+N+1}{mn}} \\ &\lesssim \prod_{i=1}^{m} \Big\| \Big(\sum_{j_i} (\lambda_{i,j_i}^{\frac{mn}{n+N+1}} \chi_{Q_{i,j_i}})^{\frac{n+N+1}{mn}} \Big)^{\frac{mn}{n+N+1}} \Big\|_{L^{p_i \frac{n+N+1}{mn}}(\omega_i)}^{\frac{n+N+1}{mn}} \\ &= \prod_{i=1}^{m} \Big\| \sum_{j_i} \lambda_{i,j_i} \chi_{Q_i,j_i} \Big\|_{L^{p_i}(\omega_i)},\end{aligned} \tag{3.7}$$

where the last second inequality follows from the assumption $N \ge mn\big(\frac{r_{\omega_i}}{p_i} - 1\big) - 1 + n(m-1)$ and Lemma 2.1. Then

$$\|I_1\|_{L^p(\nu_{\vec{\omega}})} \le \prod_{i=1}^{m} \Big\| \sum_{j_i} \lambda_{i,j_i} \chi_{Q_i,j_i} \Big\|_{L^{p_i}(\omega_i)}.$$

Next, we estimate $\|I_2\|_{L^p(\nu_{\vec{\omega}})}$. By Lemma 3.1, we have

$$T^*(a_{1,j_1},\cdots,a_{m,j_m})(x)\chi_{(S_{j_1,\cdots,j_m})^c}(x) \lesssim \prod_{i=1}^{m} M(\chi_{Q_i,j_i})(x)^{\frac{n+N+1}{mn}}.$$

Therefore

$$I_2(x) \lesssim \prod_{i=1}^{m} \Big[\sum_{j_i} \lambda_{i,j_i} M(\chi_{Q_i,j_i})^{\frac{n+N+1}{mn}}(x) \Big].$$

Then by exploiting (3.7), we obtain

$$\|I_2\|_{L^p(\nu_{\vec{\omega}})} \lesssim \prod_{i=1}^{m} \Big\| \sum_{j_i} \lambda_{i,j_i} \chi_{Q_i,j_i} \Big\|_{L^{p_i}(\omega_i)}.$$

Summing up the estimates of $\|I_1\|_{L^p(\nu_{\vec{\omega}})}$ and $\|I_2\|_{L^p(\nu_{\vec{\omega}})}$ completes the proof of Theorem 1.3.

4 Proof of Theorem 1.4

This section is concerned with the proof of Theorem 1.4, which will be done by using the following extrapolation theorem.

Theorem 4.1 *Let $\mathcal{F} = \{(F, f_1, \cdots, f_m)\}$ be a family of $(m+1)$-tuples of nonnegative, measurable functions defined on $\mathbb{R}^n$. Assume that there are indices $0 < p_1, \cdots, p_m, p < \infty$ with $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ such that for every $\omega_k \in A_1$ $(1 \le k \le m)$ and $\nu_{\vec{\omega}} = \prod_{i=1}^{m} \omega_i^{\frac{p}{p_i}}$,*

$$\|F\|_{L^p(\nu_{\vec{\omega}})} \lesssim \|f_1\|_{H^{p_1}(\omega_1)} \cdots \|f_m\|_{H^{p_m}(\omega_m)},$$

provided that the left-hand side is finite, and where the implicit constant depends only on $p, n, [\omega_k]_{A_1}$. Suppose further that $q(\cdot), q_1(\cdot), \cdots, q_m(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$ with $\frac{1}{q(\cdot)} = \frac{1}{q_1(\cdot)} + \cdots + \frac{1}{q_m(\cdot)}$, $p_k < (q_k)_-$ $(1 \le k \le m)$ and each $q_k(\cdot)$ satisfies the log-*Hölder continuous condition. Then*

$$\|F\|_{L^{q(\cdot)}} \lesssim \|f_1\|_{H^{p_1(\cdot)}} \cdots \|f_m\|_{H^{p_m(\cdot)}}$$

for all $(F, f_1, \cdots, f_m) \in \mathcal{F}$ such that the left-hand side is finite and where the implicit constant depends only on $n, q_k(\cdot)$.

Proof We follow the idea in [8] and extend the method in [8] to multilinear case. Denote $\overline{q}(\cdot) = \frac{q(\cdot)}{p}$, $\overline{p}_k(\cdot) = \frac{q_k(\cdot)}{p_k}$, where $k = 1, 2, \cdots, m$. Since $p_k < (q_k)_-$, we have $(\overline{p}_k)_-, \overline{q}_- > 1$.

Hence, given a nonnegative function h, we can define Rubio de Francia iteration algorithms by:

$$\mathcal{R}_k h(x) = \sum_{l=0}^{\infty} \frac{M^l h(x)}{2^l \|M\|^l_{L^{\overline{p}'_k(\cdot)} \to L^{\overline{p}'_k(\cdot)}}},$$

where M is the Hardy-Littlewood maximal function. By our hypothesis and $(\overline{p}_k)_- > 1$, $\mathcal{R}_k$ enjoys the following properties:

(1) $\|\mathcal{R}_k h\|_{L^{\overline{p}'_k(\cdot)}} \le 2\|h\|_{L^{\overline{p}'_k(\cdot)}}$;

(2) $\mathcal{R}_k(h) \in A_1$.

Now we fix $(F, f_1, f_2, \cdots, f_m) \in \mathcal{F}$ with $\|F\|_{L^{q(\cdot)}} < \infty$. Then by duality,

$$\|F\|^p_{L^{q(\cdot)}} = \|F^p\|_{L^{\overline{q}(\cdot)}} \sim \sup_{\|h\|_{L^{\overline{q}'(\cdot)}}=1} \int_{\mathbb{R}^n} F(x)^p h(x) \mathrm{d}x.$$

As a consequence, it suffices to prove that for any h with $\|h\|_{L^{\overline{q}'(\cdot)}} = 1$,

$$\int_{\mathbb{R}^n} F(x)^p h(x) \mathrm{d}x \lesssim \prod_{k=1}^{m} \|f_k\|^p_{H^{p_k(\cdot)}}.$$

For each k, since $h \le \mathcal{R}_k(h)$ and $\sum_{k=1}^{m} \frac{\overline{q}'(\cdot)}{\overline{p}'_k(\cdot)} \frac{p}{p_k} = 1$, we have

$$\int_{\mathbb{R}^n} F(x)^p h(x) \mathrm{d}x \lesssim \int_{\mathbb{R}^n} F(x)^p \mathcal{R}_1(h^{\frac{\overline{q}'(\cdot)}{\overline{p}'_1(\cdot)}})(x)^{\frac{p}{p_1}} \mathcal{R}_2(h^{\frac{\overline{q}'(\cdot)}{\overline{p}'_2(\cdot)}})(x)^{\frac{p}{p_2}} \mathrm{d}x \tag{4.1}$$

and the right-hand side is finite. Then by our hypothesis and (2), we deduce that

$$\begin{aligned}&\int_{\mathbb{R}^n} F(x)^p \prod_{k=1}^m \mathcal{R}_k(h^{\frac{\overline{q}'(\cdot)}{\overline{p}'_k(\cdot)}})(x)^{\frac{p}{p_k}}\mathrm{d}x\\ \lesssim &\prod_{k=1}^m \Big(\int_{\mathbb{R}^n} \mathcal{M}_{N_0} f_k(x)^{p_k} \mathcal{R}_k(h^{\frac{\overline{q}'(\cdot)}{\overline{p}'_k(\cdot)}})(x)\mathrm{d}x\Big)^{\frac{p}{p_k}}.\end{aligned} \tag{4.2}$$

By the Hölder inequality in the setting of variable Lebesgue spaces, $\|h^{\frac{\overline{q}'(\cdot)}{\overline{p}'_k(\cdot)}}\|_{L^{\overline{p}'_k(\cdot)}} \le 1$ and (1), we have

$$\begin{aligned}&\Big(\int_{\mathbb{R}^n} \mathcal{M}_{N_0} f_k(x)^{p_k} \mathcal{R}_k(h^{\frac{\overline{q}'(\cdot)}{\overline{p}'_k(\cdot)}})(x)\mathrm{d}x\Big)^{\frac{p}{p_k}}\\ \lesssim &\|\mathcal{M}_{N_0} f_k^{p_k}\|_{L^{\overline{p}_k(\cdot)}}^{\frac{p}{p_k}} \|\mathcal{R}_k(h^{\frac{\overline{q}'(\cdot)}{\overline{p}'_k(\cdot)}})\|_{L^{\overline{p}'_k(\cdot)}}^{\frac{p}{p_k}}\\ \lesssim &\|\mathcal{M}_{N_0} f_k\|_{L^{q_k(\cdot)}}^p \|h^{\frac{\overline{q}'(\cdot)}{\overline{p}'_k(\cdot)}}\|_{L^{\overline{p}'_k(\cdot)}}^{\frac{p}{p_k}} \le \|f_k\|_{H^{q_k(\cdot)}}^p.\end{aligned}$$

Then Theorem 4.1 follows from (4.1)–(4.2).

Proof of Theorem 1.4 Let $N_0 \in \mathbb{Z}^+$ such that

$$N_0 > \max\Big\{\Big\lfloor n\Big(\frac{1}{p_k} - 1\Big)\Big\rfloor_+,\ 1 \le k \le m\Big\}.$$

Denote the family $\{(F, f_1, \cdots, f_m)\}$ by $\mathcal{F}$. Then by the choice of N_0 and Lemma 2.5, it suffices to prove that for $f_k = \sum\limits_{j=1}^{K_0} \lambda_j a_j$, where a_j are (N_0, ∞)-atoms, we have

$$\|T^*(f_1, \cdots, f_m)\|_{L^{q(\cdot)}} \lesssim \prod_{k=1}^m \|f_k\|_{H^{q_k(\cdot)}}.$$

To see this, we first show that $\|F\|_{L^p(\nu_{\vec{\omega}})}$ and $\|F\|_{L^{q(\cdot)}}$ are finite, where

$$F = \min\{T^*(f_1, \cdots, f_m), R\}\chi_{B(0,R)}, \quad 0 < R < \infty.$$

Indeed, for weights $\omega_k \in A_1$ $(1 \le k \le m)$, since

$$N \ge \max\Big\{\Big\lfloor mn\Big(\frac{1}{p_k} - 1\Big)\Big\rfloor_+\Big\} + (m-1)n$$

and $f_k \in H^{p_k}(\omega_k)$, we can exploit Theorem 1.3 to get that

$$\|F\|_{L^p(\nu_{\vec{\omega}})} \le \|T^*(f_1, \cdots, f_m)\|_{L^p(\nu_{\vec{\omega}})} \lesssim \prod_{k=1}^m \|f_k\|_{H^{p_k}(\omega_k)} < \infty, \tag{4.3}$$

where $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$. Again by the choice of N_0 and Lemma 2.5, we have $f_k \in H^{q_k(\cdot)}$. Moreover,

$$\|F\|_{L^{q(\cdot)}} \le R\|\chi_{B(0,R)}\|_{L^{q(\cdot)}} < \infty. \tag{4.4}$$

In view of (4.3)–(4.4), it follows from Theorem 4.1 that

$$\|F\|_{L^{q(\cdot)}} \lesssim \prod_{k=1}^{m} \|f_k\|_{H^{q_k(\cdot)}}.$$

Finally, Fatou's lemma in the context of variable Lebesgue space (see [9]) yields that

$$\begin{aligned}\|T^*(f_1, \cdots, f_m)\|_{L^{q(\cdot)}} &= \Big\| \lim_{R\to\infty} \min\{T^*(f_1, \cdots, f_m), R\}\chi_{B(0,R)} \Big\|_{L^{q(\cdot)}} \\ &\le \liminf_{R\to\infty} \|F\|_{L^{q(\cdot)}} \lesssim \prod_{k=1}^{m} \|f_k\|_{H^{q_k(\cdot)}},\end{aligned}$$

which completes the proof of Theorem 1.4.

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