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Distance Between Unitary Orbits of Self-Adjoint Elements in C^*-Algebras of Tracial Rank One

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Abstract The note studies certain distance between unitary orbits. A result about Riesz interpolation property is proved in the first place. Weyl (1912) shows that $\mathrm{dist}(U(x), U(y)) = \delta(x, y)$ for self-adjoint elements in matrixes. The author generalizes the result to C^*-algebras of tracial rank one. It is proved that $\mathrm{dist}(U(x), U(y)) = D_c(x, y)$ in unital AT-algebras and in unital simple C^*-algebras of tracial rank one, where x, y are self-adjoint elements and $D_C(x, y)$ is a notion generalized from $\delta(x, y)$.

Keywords Unitary orbits, Riesz interpolation property, Tracial rank one, $D_c(x, y)$
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1 Introduction

It is an interesting and important problem to determine when two normal elements are unitary equivalent in a C^*-algebra. Let $\mathrm{dist}(U(x), U(y))$ denote the distance between the unitary orbits of x and y. For matrix M_n, let $x, y \in M_n$ be two normal elements with eigenvalues $\{\alpha_1, \cdots, \alpha_n\}$ and $\{\beta_1, \cdots, \beta_n\}$, respectively. Suppose

$$\delta(x, y) = \min_{\pi} \max_{1 \le i \le n} |\alpha_i - \beta_{\pi(i)}|,$$

where π runs over all permutations of $\{1, \cdots, n\}$. The equality $\mathrm{dist}(U(x), U(y)) = \delta(x, y)$ is well known for Hermitian matrices by Weyl [12]. Hu and Lin [3] study the distance between unitary orbits in separable simple C^*-algebras of real rank zero and stable rank one. Let A be separable simple C^*-algebra of real rank zero and stable rank one with weakly unperforated $K_0(A)$. Let $x, y \in A$ be two normal elements. Hu and Lin prove that

$$\mathrm{dist}(U(x), U(y)) \le D_c(x, y) \tag{1.1}$$

when $[\lambda - x] = [\mu - y] = 0$ in $K_1(A)$ for all $\lambda \notin sp(x)$ and $\mu \notin sp(y)$,

$$\mathrm{dist}(U(x), U(y)) \le D_c^e(x, y) \le 2D_c(x, y) \tag{1.2}$$

when $[\lambda - x] = [\lambda - y] = 0$ in $K_1(A)$ for all $\lambda \notin sp(x) \cup sp(y)$,

$$\mathrm{dist}(U(x), U(y)) \le D_c^e(x, y) + 2\min\{\rho_x(x, y), \rho_y(x, y)\} \tag{1.3}$$

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for all normal elements x, y.

It is well known that a simple C^*-algebra A of tracial rank zero has real rank zero and stable rank one with weakly unperforated $K_0(A)$. However, a simple C^*-algebra A of tracial rank one hasn't real rank zero in general, where the property plays an important role to study the distance between unitary orbits. This takes a difficulty when A is tracial rank one. Then Lin and Hu proposed the problem of the distance between unitary orbits in C^*-algebras of tracial rank one.

Suppose $R \subset \{1, \cdots, m\} \times \{1, \cdots, n\}$ and $A \subset \{1, \cdots, m\}$. Define $R_A \subset \{1, \cdots, n\}$ to be the subset of those j's such that $(i, j) \in R$ for some $i \in A$. Let (G, G_+) be an ordered abelian group. Let $\{a_i\}_{i=1}^m, \{b_j\}_{j=1}^n \subset G_+$ with $\sum\limits_{i=1}^m a_i = \sum\limits_{j=1}^n b_j$, $R \subset \{1, \cdots, m\} \times \{1, \cdots, n\}$ satisfying: For any $A \subset \{1, \cdots, m\}$,

$$\sum_{i \in A} a_i \leq \sum_{j \in R_A} b_j.$$

Hu and Lin [3] generalize a result in [2] and obtain that when G is a countable weakly unperforated with the Riesz interpolation, there are $\{c_{ij}\} \subset G_+$ such that

$$\sum_{j=1}^n c_{ij} = a_i, \quad \sum_{i=1}^m c_{ij} = b_j \quad \text{for all } i, j \text{ and } c_{ij} = 0 \text{ unless } (i, j) \in R.$$

It is the starting point that Hu and Lin study the distance between unitary orbits.

It is obvious that this conclusion implies that the Riesz interpolation property holds. Furthermore, we realize that the conclusion holds when an ordered abelian group (G, G_+) just has the Riesz interpolation property. This result is presented as a starting point in Section 3. To solve the problem Lin and Hu put forward, by using the result of Weyl, we present the distance between unitary orbits of self-adjoint elements in unital AT-algebras and unital simple C^*-algebras of tracial rank one in Section 4 (one can see Theorems 4.2–4.3).

2 Preliminaries

In this section, we need to recall some notations, definitions and elementary conclusions. One can see [3, 6] for more details.

Definition 2.1 *Let A be a unital C^*-algebra. Denote by $U(A)$ the unitary group of A. Let $x \in A$ be a normal element. Define $U(x)$ to be the closure of $\{u^*xu : u \in U(A)\}$. Denote by $T(A)$ the tracial state space of A.*

Definition 2.2 (see [3]) *Let A be a C^*-algebra and $a, b \in A$ be two positive elements. We write $a \lesssim b$ if there is a sequence of elements $\{x_n\} \subset A$ such that*

$$x_n^* b x_n \to a \quad \text{as } n \to \infty. \tag{2.1}$$

If $a \lesssim b$ and $b \lesssim a$, then we write $[a] = [b]$ and say that a and b are equivalent in Cuntz semigroup. We write $[a] \leq [b]$ if $a \lesssim b$.

If $p, q \in A$ are two projections, then $p \lesssim q$ means that there is a partial isometry $v \in A$ such that $v^*v = p$ and $vv^* \leq q$.

Definition 2.3 *Let Ω be a compact metric space and let $O \subset \Omega$ be a non-empty open subset. Denote by f_O a positive function with $0 \le f_O \le 1$ whose support is exactly O, i.e., $f_O > 0$ for all $t \in O$ and $f_O(t) = 0$ for all $t \notin O$. Define $f_\emptyset = 0$. Let $O \subset \Omega$ be a non-empty subset. Denote by χ_O the characteristic function associating to O.*

Suppose A is a unital C^*-algebra and $x \in A$ is a normal element with $X = sp(x) \subset \Omega$. Denote by $\varphi_X : C(\Omega) \to A$ the unital homomorphism. Define $\varphi_X(f) = f(x)$ for all $f \in C(\Omega)$.

Define

$$O_d = \{t \in \Omega : \operatorname{dist}(t, O) < d\} \tag{2.2}$$

for any subset $O \subset \Omega$.

Denote by $\overline{O}$ the closure of O.

The following lemma is a notation in [3] without a proof. We give the proof.

Lemma 2.1 *Let Ω be a compact metric space and $O \subset \Omega$ be an open subset. Let $k : C(\Omega) \to A$ be a unital homomorphism. Then $[k(f_O)]$ does not depend on the choice of f_O. Therefore, if open sets $O_1 \subset O_2 \subset \Omega$, then $k(f_{O_1}) \lesssim k(f_{O_2})$.*

Proof Let f_O, g_O be two positive functions whose support is exactly O. If O is a clopen subset of Ω, the function h_O, defined by $h_O(t) = f_O(t)/g_O(t)$ if $t \in O$ and $h_O(t) = 0$ if $t \notin O$, belongs to $C(\Omega)$. Therefore, $[k(f_O)] = [k(g_O)]$.

Now we assume that O is not a clopen subset of Ω. Suppose

$$F_\varepsilon = g_O^{-1}[0, \varepsilon) \cap f_O^{-1}[0, \varepsilon). \tag{2.3}$$

Then F_ε is an open subset. Note that $\partial O \neq \emptyset$ (the boundary of O) since O is not a clopen subset of Ω. Then for any $\varepsilon > 0$, there exists $t_\varepsilon \in F_\varepsilon \cap O$. Fix ε and t_ε. Choose a number ε' with $0 < \varepsilon' < \min\{f_0(t_\varepsilon), g_0(t_\varepsilon)\}$, then $\varepsilon' < \varepsilon$. Define $F_{\varepsilon'} = g_O^{-1}[0, \varepsilon') \cap f_O^{-1}[0, \varepsilon')$. Let $H_{\varepsilon'} = F_{\varepsilon'}^c \cap \overline{F_\varepsilon}$, where $F_{\varepsilon'}^c$ is the complementary set of $F_{\varepsilon'}$. Then $t_\varepsilon \in H_{\varepsilon'} \neq \emptyset$ and $H_{\varepsilon'}$ is a close subset. Since $g_O(F_{\varepsilon'}^c) > 0$, we may define $h'_{\varepsilon'} = f_O/g_O$ in $H_{\varepsilon'}$. Then $h'_{\varepsilon'} \in C(H_{\varepsilon'})$. Let

$$\pi : C(\overline{F_\varepsilon}) \to C(H_{\varepsilon'}) \tag{2.4}$$

be the restriction map. It is well known that π is surjective. So, $h'_{\varepsilon'}$ can be extended on $\overline{F_\varepsilon}$. Let $L_\varepsilon = \{t \in \overline{F_\varepsilon} : h'_{\varepsilon'}(t) g_O(t) \ge \varepsilon\}$. Then L_ε is a close set. We may define $s_\varepsilon(t) = \frac{\varepsilon}{g_O(t)}$ when $t \in L_\varepsilon$ and $s_\varepsilon(t) = h'_{\varepsilon'}(t)$ when $t \in \overline{F_\varepsilon} \backslash L_\varepsilon$. By pasting lemma (one can see [4, Theorem 3.2]), $s_\varepsilon \in C(\overline{F_\varepsilon})$. Note that for all $t \in \overline{F_\varepsilon}$,

$$s_\varepsilon(t) g_O(t) \le \varepsilon. \tag{2.5}$$

Let $h''_{\varepsilon'} = f_O/g_O$ on $F_{\varepsilon'}^c$. Note that $F_{\varepsilon'}^c \cup \overline{F_\varepsilon} = \Omega$, $F_{\varepsilon'}^c \cap \overline{F_\varepsilon} = H_{\varepsilon'}$ and $s_\varepsilon|_{H_{\varepsilon'}} = h''_{\varepsilon'}|_{H_{\varepsilon'}}$. Therefore, it follows from pasting lemma that there exists $h_\varepsilon \in C(\Omega)$ such that $h_\varepsilon|_{\overline{F_\varepsilon}} = s_\varepsilon$ and $h_\varepsilon|_{F_{\varepsilon'}^c} = h''_{\varepsilon'}$. So, for any $t \in F_{\varepsilon'}$,

$$|f_O(t) - h_\varepsilon(t) g_O(t)| \le |f_O(t)| + |h_\varepsilon(t) g_O(t)| = |f_O(t)| + |s_\varepsilon(t) g_O(t)| < 2\varepsilon. \tag{2.6}$$

For any $t \in F_{\varepsilon'}^c$, $f_O(t) = h_\varepsilon(t) g_O(t)$. Therefore,

$$\|f_O - h_\varepsilon g_O\| = \sup\{|f_O(t) - h_\varepsilon(t) g_O(t)| : t \in \Omega\} \le 2\varepsilon. \tag{2.7}$$

So, we get $k(f_O) \lesssim k(g_O)$. In the same way, $k(g_O) \lesssim k(f_O)$. Therefore, $[k(f_O)] = [k(g_O)]$.

Definition 2.4 (see [3]) *Let Ω be a compact metric space and let $O \subset \Omega$ be an open subset. If A is a unital C^*-algebra and $K_1, K_2 : C(\Omega) \to A$ are two unital homomorphisms, define*

$$D_c(K_1, K_2) = \sup_O \{\inf_d \{d > 0 : K_1(f_O) \lesssim K_2(f_{O_d})\} : O \subset \Omega \text{ open}\}. \tag{2.8}$$

Let $X = sp(x), Y = sp(y) \subset \Omega$, where $x, y \in A$ are two normal elements. Define $\varphi_X : C(\Omega) \to A,\ f \mapsto f(x)$ and $\varphi_Y : C(\Omega) \to A,\ f \mapsto f(y)$. Denote by $D_c(x, y)$ the $D_c(\varphi_X, \varphi_Y)$.

Definition 2.5 (see [3]) *Let $\varepsilon > 0$. Denote by f_ε the continuous function on $(-\infty, \infty)$ such that $0 \leq f_\varepsilon \leq 1$; $f(t) = 1$ if $t \in (-\infty, -\varepsilon] \cup [\varepsilon, \infty)$, $f(t) = 0$ if $t \in \left[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right]$ and $f(t)$ is linear in $\left(-\varepsilon, -\frac{\varepsilon}{2}\right)$ and $\left(\frac{\varepsilon}{2}, \varepsilon\right)$.*

Let $b \in A_+$. Define

$$d_\tau(b) = \lim_{\varepsilon \to 0} \tau(f_\varepsilon(b)) \tag{2.9}$$

for $\tau \in T(A)$.

A is said to has strict comparison for positive elements, if

$$d_\tau(a) < d_\tau(b) \quad \text{for all } \tau \in T(A) \tag{2.10}$$

implies that $a \lesssim b$.

Unital simple C^*-algebras of tracial rank no more than one have strict comparison for positive elements.

Definition 2.6 (see [3]) *Let A be a unital C^*-algebra with $T(A) \neq \emptyset$. Let $\Omega \subset \mathbb{C}$ be a compact set and $X = sp(x), Y = sp(y) \subset \Omega$, where $x, y \in A$ are two normal elements. Define*

$$r_O(\varphi_X, \varphi_Y) = \inf_r \{r > 0 : d_\tau(\varphi_X(f_O)) \leq d_\tau(\varphi_Y(f_{O_d})) \text{ for all } \tau \in T(A)\}. \tag{2.11}$$

Define

$$D_T(\varphi_X, \varphi_Y) = \sup_O \{r_O(\varphi_X, \varphi_Y) : O \subset \Omega \text{ open}\}. \tag{2.12}$$

Denote by $D_T(x, y)$ the $D_T(\varphi_X, \varphi_Y)$.

Lemma 2.2 *Let A be a unital C^*-algebra of stable rank one with $T(A) \neq \emptyset$. Then for any normal elements $x, y, z \in A$,*

$$D_c(x, z) \leq D_c(x, y) + D_c(y, z), \tag{2.13}$$
$$D_c(x, y) = D_c(y, x), \tag{2.14}$$
$$D_T(x, z) \leq D_T(x, y) + D_T(y, z), \tag{2.15}$$
$$D_T(x, y) = D_T(y, x), \tag{2.16}$$
$$D_T(x, y) \leq D_c(x, y). \tag{2.17}$$

Proof It follows from [3, Propositions 2.15 and 2.21].

Lemma 2.3 *Let A be a unital C^*-algebra with $T(A) \neq \emptyset$ and let $\Omega \subset \mathbb{C}$ be a compact subset. For any $\varepsilon > 0$, there exists $\delta > 0$ satisfying: For any normal elements $x, y \in A$ with $sp(x), sp(y) \subset \Omega$, if*

$$\|x - y\| < \delta, \tag{2.18}$$

then

$$D_c(x,y) < \varepsilon, \tag{2.19}$$

$$D_T(x,y) < \varepsilon. \tag{2.20}$$

Proof It follows from the proof of [3, Lemma 2.17] that (2.19) holds. It follows from Lemma 2.2 that (2.20) holds.

Definition 2.7 (see [6]) *Denote by M_n the matrix algebra $M_n(\mathbb{C})$.*

Denote by $I^{(1)}$ the class of all unital hereditary C^*-subalgebras of $C(X)\otimes F$, where X is one-dimensional finite CW complex and F is a finite dimensional C^*-algebra.

Let A be a C^*-algebra and B be a C^*-subalgebra of A. If $a \in A$, we say $a \in_\varepsilon B$ if there is an element $b \in B$ such that $\|a-b\| < \varepsilon$.

Denote by $TR(A) \le k$ the tracial rank no more than k (One can see the definition of tracial rank by [6, *Definition* 3.6.1]).

Remark 2.1 Note that those conclusions about spectrum in the following:

(1) Let A be a unital C^*-algebra. Suppose $x, y \in A$ are two normal elements. Let $\varepsilon > 0$. If $\|x-y\| < \varepsilon$, then for any $\lambda \in sp(x)$, $\mathrm{dist}(\lambda, sp(y)) < \varepsilon$.

(2) Let A be a unital C^*-algebra and I_A be the unit. Suppose B is a C^*-subalgebra of A and $x \in B$ is a normal element. If B has unit $I_B \neq I_A$, then $sp_A(x) = sp_B(x) \cup \{0\}$. Otherwise, $sp_B(x) = sp_A(x)$.

(3) Suppose $A = B \oplus C$, where A, B, C are unital C^*-algebras. Let $x \in B$ and $y \in C$ be normal. Then $sp_B(x) \cup sp_C(y) = sp_A(x+y)$.

Remark 2.2 Let A be a unital C^*-algebra with unit I_A. Let B be a unital C^*-subalgebra of A with unit I_B. Suppose $x \in B$ is a normal element and $X_A = sp_A(x)$. Set $f \in C(X_A)$. Note that the function calculation $f(x)$ in A and in B are different. Therefore, if necessary, we denote by $f^A(x)$ the function calculation $f(x)$ in A and denote by $f^B(x)$ the function calculation $f(x)$ in B. It is obvious that $P^A(x) = P^B(x) + P(0)(I_A - I_B)$ for any polynomial P. So $f^A(x) = f^B(x) + f(0)(I_A - I_B)$ for any $f \in C(X_A)$.

This following lemma is probably well-known.

Lemma 2.4 *Let A be a unital simple C^*-algebra and B be a unital hereditary C^*-subalgebra of A. If $\tau \in T(B)$, then there is only one tracial positive linear function τ' on A such that $\tau'|_B = \tau$.*

Proof Since A is simple, it follows from [6, Lemma 3.3.6] that there are $x_1, \cdots, x_n \in A$ such that $I_A = \sum\limits_{i=1}^{n} x_i I_B x_i^*$. Suppose $\{e_{ij} : i, j = 1, \cdots, n\}$ is a set of matrix units of M_n and

$$\begin{aligned} l : A &\to M_n \otimes A \\ a &\mapsto e_{11} \otimes a. \end{aligned} \tag{2.21}$$

Let τ_n be the normalized trace of M_n. Then $\tau_n \otimes \tau \in T(M_n \otimes B)$. Put $x = \sum\limits_{j=1}^{n} e_{1j} \otimes x_j$ and $P = I \otimes I_B \in M_n \otimes B$, where I is the unit of M_n. Then

$$e_{11} \otimes I_A = xPx^*. \tag{2.22}$$

Since $M_n \otimes B$ is a hereditary C^*-subalgebra of $M_n \otimes A$, then

$$Px^*l(a)xP \in M_n \otimes B \tag{2.23}$$

for any $a \in A$. Define $\tau'(l(a)) = \tau_n \otimes \tau(Px^*l(a)xP)$ for $a \in A$. Therefore τ' is a positive linear function on $l(A)$. Suppose $a, b \in A$. Then by (2.22)–(2.23),

$$\begin{aligned}\tau'(l(ab)) &= \tau_n \otimes \tau(Px^*l(a)l(b)xP) = \tau_n \otimes \tau(Px^*l(a)xPx^*l(b)xP) \\ &= \tau_n \otimes \tau(Px^*l(b)xPx^*l(a)xP) = \tau'(l(ba)).\end{aligned} \tag{2.24}$$

This implies that τ' is a trace on $l(A)$. For any $b \in B_+$,

$$\begin{aligned}\tau'(l(b)) &= \tau_n \otimes \tau(Px^*l(b)xP) = \tau_n \otimes \tau(Px^*l(b)^{\frac{1}{2}}Pl(b)^{\frac{1}{2}}xP) \\ &= \tau_n \otimes \tau(Pl(b)^{\frac{1}{2}}xPx^*l(b)^{\frac{1}{2}}P) = \tau_n \otimes \tau(Pl(b)P) = \tau_n \otimes \tau(l(b)).\end{aligned} \tag{2.25}$$

Therefore, $\tau'|_{l(B)} = \tau_n \otimes \tau$. In other words, $n\tau' \circ l|_B = \tau$.

Now suppose τ_1' and τ_2' are tracial positive linear functions as two extensions of τ. For any $a \in A_+$,

$$\begin{aligned}\tau_n \otimes \tau_1'(l(a)) &= \tau_n \otimes \tau_1'(l(a)^{\frac{1}{2}}xPx^*l(a)^{\frac{1}{2}}) = \tau_n \otimes \tau_1'(Px^*l(a)xP) \\ &= \tau_n \otimes \tau(Px^*l(a)xP) = \tau_n \otimes \tau_2'(Px^*l(a)xP) = \tau_n \otimes \tau_2'(l(a))\end{aligned} \tag{2.26}$$

This implies that $\tau_1' = \tau_2'$.

3 C^*-Algebras with the Riesz Interpolation Property

Definition 3.1 *Let I and J be two sets. Suppose $R \subset I \times J$, $A \subset I$ and $B \subset J$. Define $R_A \subset J$ to be the subset of those j's such that $(i, j) \in R$ for some $i \in A$. Define $R^B \subset I$ to be the subset of those i's such that $(i, j) \in R$ for some $j \in B$.*

Theorem 3.1 *Let (G, G_+) be an ordered abelian group with the Riesz interpolation property. If $\{a_i\}_{i=1}^m, \{b_j\}_{j=1}^n \subset G_+$ with $\sum\limits_{i=1}^m a_i = \sum\limits_{j=1}^n b_j$, $R \subset \{1, \cdots, m\} \times \{1, \cdots, n\}$ satisfying: For any $A \subset \{1, \cdots, m\}$,*

$$\sum_{i \in A} a_i \le \sum_{j \in R_A} b_j, \tag{3.1}$$

then there are $\{c_{ij}\} \subset G_+$ such that

$$\sum_{j=1}^n c_{ij} = a_i, \quad \sum_{i=1}^m c_{ij} = b_j \quad \text{for all } i, j \tag{3.2}$$

$$\text{and } c_{ij} = 0 \quad \text{unless } (i, j) \in R. \tag{3.3}$$

Proof It follows from [6, Lemma 3.3.14] that, if $a_1, a_2, b_1, b_2 \in G$ with $a_i \le b_j$ $(i, j = 1, 2)$, then there exists $c \in G$ such that $a_i \le c \le b_j$ $(i, j = 1, 2)$. Now, suppose a_1, a_2, $b_1, b_2, b_3 \in G$ with $a_i \le b_j$ $(i = 1, 2$ and $j = 1, 2, 3)$. Then for a_1, a_2 and b_1, b_2, there exists $c' \in G$ such that $a_i \le c' \le b_j$ for $i, j = 1, 2$. Therefore, for a_1, a_2 and c', b_3, we have $a_i \le c'$ and $a_i \le b_3$ for $i = 1, 2$. Then, we have $c \in G$ such that $a_i \le c \le c', b_3$ for $i = 1, 2$. Note that $c \le c' \le b_1, b_2$.

Therefore, we get $c \in G$ such that $a_i \le c \le b_j$ for $i = 1, 2$ and $j = 1, 2, 3$. In the same way, if $a_1, \cdots, a_m, b_1, \cdots, b_n \in G$ with $a_i \le b_j$ for all i, j, then there exists $c \in G$ such that $a_i \le c \le b_j$ for all i, j.

Now, suppose $\{a_i\}_{i=1}^m, \{b_j\}_{j=1}^n \subset G_+$ with $\sum\limits_{i=1}^m a_i = \sum\limits_{j=1}^n b_j$, $R \subset \{1, \cdots, m\} \times \{1, \cdots, n\}$ satisfying: For any $A \subset \{1, \cdots, m\}$,

$$\sum_{i \in A} a_i \le \sum_{j \in R_A} b_j. \tag{3.4}$$

When $m = 1$, it is trivial to check that there are $c_{1j} = b_j \subset G_+$ such that

$$\sum_{j=1}^n c_{1j} = a_1, \quad c_{1j} = b_j \quad \text{for all } j \tag{3.5}$$

and

$$c_{ij} = 0 \text{ unless } (i, j) \in R. \tag{3.6}$$

Then we assume that it holds for $m = k - 1$. Let $m = k$. Without loss of generality, suppose $(1, 1) \in R$, $R_{\{1\}} = \{1, \cdots, n'\}$ and $R^{\{1\}} = \{1, \cdots, m'\}$ where $1 \le m' \le m$, $1 \le n' \le n$. We first consider $m', n' \ge 2$. Let $\mathcal{A} = \{A \subset \{2, \cdots, m\} : A \cap \{2, \cdots, m'\} \ne \emptyset\}$ and $\mathcal{B} = \{B : B \subset \{m'+1, \cdots, m\}\}$. Note that $\emptyset \in \mathcal{B}$, $\mathcal{A}, \mathcal{B} \ne \emptyset$ and $\mathcal{A} \cup \mathcal{B} = \{C : C \subset \{2, \cdots, m\}\}$. So, we can suppose $\mathcal{A} = \{A_1, \cdots, A_{l_1}\}$, $\mathcal{B} = \{B_1, \cdots, B_{l_2}\}$. Define $\sum\limits_{i \in \emptyset} a_i = \sum\limits_{j \in \emptyset} b_j = 0$. Therefore, we have

$$a_1 + \sum_{i \in A_l} a_i \le \sum_{j \in R_{A_l \cup \{1\}}} b_j \quad \text{for all } l = 1, \cdots, l_1, \tag{3.7}$$

$$\sum_{i \in A_l} a_i \le \sum_{j \in R_{A_l}} b_j \quad \text{for all } l = 1, \cdots, l_1, \tag{3.8}$$

$$a_1 + \sum_{i \in B_l} a_i \le \sum_{j \in R_{B_l \cup \{1\}}} b_j \quad \text{for all } l = 1, \cdots, l_2, \tag{3.9}$$

$$\sum_{i \in B_l} a_i \le \sum_{j \in R_{B_l}} b_j, \quad \text{for all } l = 1, \cdots, l_2. \tag{3.10}$$

By (3.8), we have

$$\sum_{j \in R_{A_l}} b_j - \sum_{i \in A_l} a_i \ge 0 \quad \text{for all } l = 1, \cdots, l_1. \tag{3.11}$$

By (3.9), we have

$$a_1 + \sum_{i \in B_l} a_i - \sum_{j \in R_{B_l \cup \{1\}} \setminus \{1\}} b_j \le b_1 \quad \text{for all } l = 1, \cdots, l_2. \tag{3.12}$$

Note that for any $B \in \mathcal{B}$, $1 \notin R_B$. Then by (3.10),

$$\sum_{i \in B_l} a_i \le \sum_{j \in R_{B_l}} b_j \le \sum_{j \in R_{B_l \cup \{1\}} \setminus \{1\}} b_j. \tag{3.13}$$

So,

$$a_1+\sum_{i\in B_l}a_i-\sum_{j\in R_{B_l\cup\{1\}}\backslash\{1\}}b_j\le a_1\quad\text{for all } l=1,\cdots,l_2. \tag{3.14}$$

Let $B'=R_{A_l\cup B_{l'}\cup\{1\}}\backslash R_{A_l}$. We can find that $R_{A_l\cup B_{l'}\cup\{1\}}=R_{A_l}\cup R_{B_{l'}\cup\{1\}}$ and $R_{A_l\cap(B_{l'}\cup\{1\})}\subset R_{A_l}\cap R_{B_{l'}\cup\{1\}}$. So $B'=R_{B_{l'}\cup\{1\}}\backslash R_{A_l}$, $B'\cap R_{A_l\cap B_{l'}}=\emptyset$ and $B'\cup R_{A_l\cap B_{l'}}\subset R_{B_{l'}\cup\{1\}}\backslash\{1\}$. Therefore,

$$\begin{aligned}
&a_1+\sum_{i\in A_l}a_i+\sum_{i\in B_{l'}}a_i\\
&=a_1+\sum_{i\in A_l\cup B_{l'}}a_i+\sum_{i\in A_l\cap B_{l'}}a_i\\
&\le\sum_{j\in R_{A_l\cup B_{l'}\cup\{1\}}}b_j+\sum_{j\in R_{A_l\cap B_{l'}}}b_j\\
&=\sum_{j\in R_{A_l}}b_j+\sum_{j\in B'}b_j+\sum_{j\in R_{A_l\cap B_{l'}}}b_j\\
&\le\sum_{j\in R_{A_l}}b_j+\sum_{j\in R_{B_{l'}\cup\{1\}}\backslash\{1\}}b_j.
\end{aligned} \tag{3.15}$$

So,

$$a_1+\sum_{i\in B_{l'}}a_i-\sum_{j\in R_{B_{l'}\cup\{1\}}\backslash\{1\}}b_j\le\sum_{j\in R_{A_l}}b_j-\sum_{i\in A_l}a_i \tag{3.16}$$

for all $l=1,\cdots,l_1$ and $l'=1,\cdots,l_2$. Combine (3.11)–(3.12), (3.14) and (3.16), we have

$$0,a_1+\sum_{i\in B_{l'}}a_i-\sum_{j\in R_{B_{l'}\cup\{1\}}\backslash\{1\}}b_j\le a_1,b_1,\sum_{j\in R_{A_l}}b_j-\sum_{i\in A_l}a_i \tag{3.17}$$

for all $l=1,\cdots,l_1$ and $l'=1,\cdots,l_2$. Therefore, there exists $c_{11}\in G$ such that

$$0,a_1+\sum_{i\in B_{l'}}a_i-\sum_{j\in R_{B_{l'}\cup\{1\}}\backslash\{1\}}b_j\le c_{11}\le a_1,b_1,\sum_{j\in R_{A_l}}b_j-\sum_{i\in A_l}a_i \tag{3.18}$$

for all $l=1,\cdots,l_1$ and $l'=1,\cdots,l_2$.

Now, let $a_1'=a_1-c_{11}\ge 0$ and $b_1'=b_1-c_{11}\ge 0$. Let $a_i'=a_i$ and $b_j'=b_j$ for $i=2,\cdots,m$ and $j=2,\cdots,n$. Let $R'=R\backslash\{(1,1)\}$. Then, by (3.7), we have

$$a_1'+\sum_{i\in A_l}a_i'\le\sum_{j\in R_{A_l\cup\{1\}}}b_j'=\sum_{j\in R'_{A_l\cup\{1\}}}b_j'\quad\text{for all } l=1,\cdots,l_1. \tag{3.19}$$

By (3.18), we have

$$\sum_{i\in A_l}a_i'\le\sum_{j\in R_{A_l}}b_j'=\sum_{j\in R'_{A_l}}b_j'\quad\text{for all } l=1,\cdots,l_1 \tag{3.20}$$

and

$$a_1'+\sum_{i\in B_l}a_i'\le\sum_{j\in R_{B_l\cup\{1\}}\backslash\{1\}}b_j=\sum_{j\in R'_{B_l\cup\{1\}}}b_j'\quad\text{for all } l=1,\cdots,l_2. \tag{3.21}$$

By (3.10), we have

$$\sum_{i\in B_l} a_i' \leq \sum_{j\in R_{B_l}} b_j' \quad \text{for all } l=1,\cdots,l_2. \tag{3.22}$$

Therefore, when $m',n'\geq 2$, for $\{a_i'\}_{i=1}^m, \{b_j'\}_{j=1}^n \subset G_+$, we have $\sum\limits_{i=1}^m a_i' = \sum\limits_{j=1}^n b_j'$ and for any $A\subset\{1,\cdots,m\}$,

$$\sum_{i\in A} a_i' \leq \sum_{j\in R_A'} b_j'. \tag{3.23}$$

However, when $m'=1$ and $n'\geq 2$, let $c_{11}=b_1$, $a_1'=a_1-c_{11}=\sum\limits_{j=2}^n b_j - \sum\limits_{i=2}^m a_i \geq 0$ and $b_1'=0$. Let $a_i'=a_i$ and $b_j'=b_j$ for $i=2,\cdots,m$ and $j=2,\cdots,n$. Let $R'=R\backslash\{(1,1)\}$. It is trivial to check that $\sum\limits_{i=1}^m a_i' = \sum\limits_{j=1}^n b_j'$ and for any $A\subset\{1,\cdots,m\}$,

$$\sum_{i\in A} a_i' \leq \sum_{j\in R_A'} b_j'. \tag{3.24}$$

Therefore, we have $c_{11},\cdots,c_{1n'-1}\in G_+$, $a_i',b_j'\in G_+$ ($i=1,\cdots,m$ and $j=1,\cdots,n$) and $R'=R\backslash\{(1,1),\cdots,(1,n'-1)\}$ such that $b_j'=b_j-c_{1j}\geq 0$ for $j=1,\cdots,n'-1$, $a_1'=a_1-\sum\limits_{j=1}^{n'-1} c_{1j}\geq 0$, $a_i'=a_i$ for $i=2,\cdots,m$, $b_j'=b_j$ for $j=n'\cdots,n$, $\sum\limits_{i=1}^m a_i'=\sum\limits_{j=1}^n b_j'$ and for any $A\subset\{1,\cdots,m\}$,

$$\sum_{i\in A} a_i' \leq \sum_{j\in R_A'} b_j'. \tag{3.25}$$

Note that $a_1'\leq b_{n'}'$. Therefore, let $a_i''=a_i'$ for $i=2,\cdots,m$. Let $b_j''=b_j'$ for all $j\neq n'$. Let $c_{1n'}=a_1'$ and $b_{n'}''=b_{n'}'-c_{1n'}\geq 0$. Let

$$R''=R'\backslash\{(1,n')\}\subset\{2,\cdots,m\}\times\{1,\cdots,n\}. \tag{3.26}$$

We can check that $\sum\limits_{i=2}^m a_i''=\sum\limits_{j=1}^n b_j''$ and for any $A\subset\{2,\cdots,m\}$,

$$\sum_{i\in A} a_i'' \leq \sum_{j\in R_A''} b_j''. \tag{3.27}$$

Since we assume that theorem holds for $m-1$, there are $\{c_{ij}\}\subset G_+$ ($i=2,\cdots,m$ and $j=1,\cdots,n$) such that

$$\sum_{j=1}^n c_{ij}=a_i'', \quad \sum_{i=2}^m c_{ij}=b_j'' \quad \text{for all } i,j \tag{3.28}$$

and

$$c_{ij}=0 \quad \text{unless } (i,j)\in R''. \tag{3.29}$$

Let $c_{1n'+1} = \cdots = c_{1n} = 0$. Then for above all $c_{ij} \in G_+$, we have

$$\sum_{j=1}^{n} c_{ij} = a_i, \quad \sum_{i=1}^{m} c_{ij} = b_j \quad \text{for all } i, j \tag{3.30}$$

and

$$c_{ij} = 0 \quad \text{unless } (i, j) \in R. \tag{3.31}$$

In the end, by induction, for any m, theorem still holds.

Lemma 3.1 *Let A be a unital C^*-algebra of stable rank one. Suppose $(K_0(A), K_0(A)_+)$ has the Riesz interpolation property. Let $x = \sum\limits_{i=1}^{m} \alpha_i p_i$, $y = \sum\limits_{j=1}^{n} \beta_j q_j$, where $\{\alpha_i\}_{i=1}^m, \{\beta_j\}_{j=1}^n \subset \mathbb{C}$, $\{p_1, \cdots, p_m\}$ and $\{q_1, \cdots, q_n\}$ are two sets of mutually orthogonal non-zero projections in A such that $\sum\limits_{i=1}^{m} p_i = \sum\limits_{j=1}^{n} q_j = I_A$. Then $D_c(x, y) \le d$ if and only if, for any $\varepsilon > 0$, there are projections $p_{ij}, q_{ij} \in A$ $(i = 1, \cdots, m$ and $j = 1, \cdots, n)$ such that*

$$p_i = \sum_{j=1}^{n} p_{ij}, \quad q_i = \sum_{i=1}^{m} q_{ij}, \quad [p_{ij}] = [q_{ij}] \quad \textit{in } K_0(A) \textit{ for all } i, j \tag{3.32}$$

and

$$\max\{\mathrm{dist}(\alpha_i, \beta_j) : q_{ij} \ne 0\} < d + \varepsilon. \tag{3.33}$$

Proof Suppose $d = D_c(x, y)$, $\varepsilon > 0$,

$$R = \{(i, j) : \mathrm{dist}(\alpha_i, \beta_j) < d + \varepsilon\}. \tag{3.34}$$

For any $A \subset \{1, \cdots, m\}$, let $O_A = \{\alpha_i : i \in A\}$ and $O_{R_A} = \{\beta_j : j \in R_A\}$. Then

$$\sum_{i \in A} [p_i] = \Big[\sum_{i \in A} p_i\Big] = [f_{O_A}(x)] \le [f_{(O_A)_{d+\varepsilon}}(y)] = [f_{O_{R_A}}(y)] = \sum_{j \in R_A} [q_j]. \tag{3.35}$$

It follows from Theorem 3.1 that there are $\{r_{ij}\} \subset K_0(A)_+$ $(i = 1, \cdots, m$ and $j = 1, \cdots, n)$ such that

$$[p_i] = \sum_{j=1}^{n} r_{ij}, \quad [q_j] = \sum_{i=1}^{m} r_{ij} \quad \text{for all } i, j, \tag{3.36}$$

where $r_{ij} = 0$ unless $(i, j) \in R$. Since A has stable rank one, by [6, 3.1.13], there are two sets $\{p_{ij}\}, \{q_{ij}\}$ of mutually orthogonal projections with $[p_{ij}] = [q_{ij}] = r_{ij}$ such that

$$p_i = \sum_{j=1}^{n} p_{ij}, \quad q_j = \sum_{i=1}^{m} q_{ij} \quad \text{for all } i, j. \tag{3.37}$$

Furthermore,

$$\max\{\mathrm{dist}(\alpha_i, \beta_j) : q_{ij} \ne 0\} < d + \varepsilon. \tag{3.38}$$

The converse is obvious.

Hu and Lin [3] prove the following theorem when the following A is separable simple with weakly unperforated $K_0(A)$. We generalize the result.

Theorem 3.2 *Let A be a unital C^*-algebra of stable rank one. Suppose $(K_0(A), K_0(A)_+)$ has the Riesz interpolation property. Let $x, y \in A$ be two normal elements with finite spectrum. Then,*

$$\mathrm{dist}(U(x), U(y)) \leq D_c(x, y). \tag{3.39}$$

Proof Suppose $\varepsilon > 0$, $d = D_c(x, y)$, $x = \sum\limits_{i=1}^{m} \alpha_i p_i$, $y = \sum\limits_{j=1}^{n} \beta_j q_j$, where $\{\alpha_i\}_{i=1}^m, \{\beta_j\}_{j=1}^n \subset \mathbb{C}$, $\{p_1, \cdots, p_m\}$ and $\{q_1, \cdots, q_n\}$ are two sets of mutually orthogonal non-zero projections in A such that $\sum\limits_{i=1}^{m} p_i = \sum\limits_{j=1}^{n} q_j = I$. It follows from Lemma 3.1 that there are $\{p_{ij}\}, \{q_{ij}\}$ ($i = 1, \cdots, m$ and $j = 1, \cdots, n$) such that

$$p_i = \sum_{j=1}^{n} p_{ij}, \quad q_j = \sum_{i=1}^{m} q_{ij} \quad \text{for all } i, j. \tag{3.40}$$

and

$$\max\{\mathrm{dist}(\alpha_i, \beta_j) : q_{ij} \neq 0\} < d + \varepsilon. \tag{3.41}$$

Therefore, there exists $u \in U(A)$ such that $u^* p_{ij} u = q_{ij}$ for all i, j. So,

$$\|u^* x u - y\| = \Big\| \sum_{i,j} (\alpha_i - \beta_j) q_{ij} \Big\| \leq \max\{|\alpha_i - \beta_j| : q_{ij} \neq 0\} < d + \varepsilon. \tag{3.42}$$

Therefore, $\mathrm{dist}(U(x), U(y)) \leq D_c(x, y)$.

Hu and Lin [3] prove the following theorem when AF-algebras are simple.

Theorem 3.3 *Let A be a unital AF-algebra (may not simple) and $x, y \in A$ be two normal elements. Then*

$$\mathrm{dist}(U(x), U(y)) \leq D_c(x, y). \tag{3.43}$$

Proof Since a unital AF-algebra has real rank zero and stable rank one, so $(K_0(A), K_0(A)_+)$ has the Riesz interpolation property. Let $\varepsilon > 0$. Suppose δ is a number in Lemma 2.3 with $0 < \delta < \varepsilon$. It follows from [5] (or one can refer to [1]) that there is δ' with $\delta' < \frac{\delta}{2}$ such that for any integer n and $a \in M_n$, if $\|aa^* - a^*a\| < \delta'$, then there is a normal $a' \in M_n$ such that $\|a - a'\| < \frac{\delta}{2}$.

Let $x, y \in A$ be two normal elements, then there are finite dimensional C^*-algebra $B \subset A$ such that $\mathrm{dist}(x, B) < \frac{\delta'}{4}$ and $\mathrm{dist}(y, B) < \frac{\delta'}{4}$. Let $x_1, y_1 \in B$ such that $\|x - x_1\| < \frac{\delta'}{4}$, $\|y - y_1\| < \frac{\delta'}{4}$. Then $\|x_1 x_1^* - x_1^* x_1\| < \delta'$ and $\|y_1 y_1^* - y_1^* y_1\| < \delta'$. So, there are normal elements $x_1', y_1' \in B$ such that $\|x_1' - x_1\| < \frac{\delta}{2}$ and $\|y_1' - y_1\| < \frac{\delta}{2}$. It is obvious that x_1', y_1' has finite spectrum with $\|x - x_1'\| < \delta$ and $\|y - y_1'\| < \delta$. Then by Theorem 3.2, there exists $u \in U(A)$ such that $\|x_1' - u^* y_1' u\| \leq D_c(x_1', y_1') + \varepsilon$. So,

$$\|x - u^* y u\| \leq \|x_1' - u^* y_1' u\| + 2\varepsilon \leq D_c(x_1', y_1') + 3\varepsilon \leq D_c(x, y) + 5\varepsilon. \tag{3.44}$$

Therefore, $\mathrm{dist}(U(x), U(y)) \leq D_c(x, y)$.

4 Distance Between Unitary Orbits of Self-Adjoint Elements

Lemma 4.1 *Let X be a compact Hausdorff space with* $\dim X \le 1$*. Then for any $\varepsilon > 0$ and self-adjoint element $x \in C(X)\otimes M_n$, there are self-adjoint elements $f_i(t) \in C(X)$ $(i = 1, \cdots, n)$ with $f_1(t) > \cdots > f_n(t)$ for all $t \in X$, mutually orthogonal rank one projections $e_1, \cdots, e_n \in M_n$ and unitary $u \in C(X) \otimes M_n$ such that*

$$\Big\| \sum_{i=1}^{n} f_i \otimes e_i - u^* x u \Big\| < \varepsilon. \tag{4.1}$$

Proof It follows from [11, Lemma 1.1], there are $f_1', \cdots, f_n' \in C(X)$ such that $f_1'(t) \ge \cdots \ge f_n'(t)$ and $sp(x(t)) = \{f_1'(t), \cdots, f_n'(t)\}$ for all $t \in X$. Let $\{e_{ij} : i, j = 1, \cdots, n\}$ be a set of matrix units of M_n. Then by [11, Corollary 1.3], $f_1'\otimes e_{11}+\cdots+f_n'\otimes e_{nn}$ and x are approximately unitarily equivalent. In other words, for $\varepsilon > 0$, there exists unitary $u \in C(X) \otimes M_n$ such that $\| \sum\limits_{i=1}^{n} f_i' \otimes e_{ii} - u^* x u\| < \frac{\varepsilon}{2}$. Let $I \in C(X)$ be the unit and $f_i = f_i' + \frac{(n-i)\varepsilon}{2n} I$, $i = 1, \cdots, n$. Then $f_1(t) > \cdots > f_n(t)$ for all $t \in X$ and

$$\Big\| \sum_{i=1}^{n} f_i \otimes e_{ii} - u^* x u \Big\| \le \Big\| \sum_{i=1}^{n} f_i \otimes e_{ii} - \sum_{i=1}^{n} f_i' \otimes e_{ii} \Big\| + \Big\| \sum_{i=1}^{n} f_i' \otimes e_{ii} - u^* x u \Big\| < \varepsilon. \tag{4.2}$$

Lemma 4.2 *Let X be a compact Hausdorff space with* $\dim X \le 1$*. Then for any self-adjoint elements $x, y \in C(X) \otimes M_n$,* $\mathrm{dist}(U(x), U(y)) \le D_T(x, y)$*.*

Proof Let $\varepsilon > 0$ and $d > D_T(x, y)$. Suppose $\delta < \varepsilon$. It follows from Lemma 4.1 that there are self-adjoint elements $f_i, g_i \in C(X)$ $(i = 1, \cdots, n)$ with $f_1 > \cdots > f_n$ and $g_1 > \cdots > g_n$, a set of mutually orthogonal rank one projections $\{e_1, \cdots, e_n\} \subset M_n$ and unitaries $u_1, u_2 \in C(X)\otimes M_n$ such that

$$\Big\| u_1^* x u_1 - \sum_{i=1}^{n} f_i \otimes e_i \Big\| < \delta \quad \text{and} \quad \Big\| u_2^* y u_2 - \sum_{i=1}^{n} g_i \otimes e_i \Big\| < \delta. \tag{4.3}$$

Let $x' = \sum\limits_{i=1}^{n} f_i \otimes e_i$ and $y' = \sum\limits_{i=1}^{n} g_i \otimes e_i$. It follows from Lemma 2.3, for sufficiently small δ, we may assume

$$|D_T(x', y') - D_T(x, y)| < 2\varepsilon. \tag{4.4}$$

Set

$$\tau_t : C(X) \to \mathbb{C} \tag{4.5}$$
$$f \to f(t). \tag{4.6}$$

Suppose τ is the tracial state of M_n. Then $\tau_t \otimes \tau$ is a tracial state of $C(X) \otimes M_n$. Let $d' > D_T(x', y')$. We claim that for any $t \in X$,

$$|f_i(t) - g_i(t)| < d' \tag{4.7}$$

for all i. Otherwise, there exists $t_0 \in X$ and positive integer $i_0 \le n$ such that

$$|f_{i_0}(t_0) - g_{i_0}(t_0)| \ge d'. \tag{4.8}$$

We may fix t_0 and assume i_0 is the minimum integer for t_0 such that (4.8) holds. Furthermore we may assume $f_{i_0}(t_0) \geq g_{i_0}(t_0)$. Then,

$$f_i(t_0) - g_j(t_0) \geq d' \tag{4.9}$$

for all $i \leq i_0$ and $j \geq i_0$.

Let $d'' = \frac{d'+D_T(x',y')}{2}$ and $O_1 = \left\{\lambda \in \mathbb{R} : |\lambda - f_1(t_0)| < \frac{d'-d''}{2}\right\}$. Note that $\tau_t \otimes \tau(a) = \tau(a(t))$. Note that $f(a)(t) = f(a(t))$ for any normal element $a \in C(X) \otimes M_n$, continuous $f \in C(sp(a))$ and $t \in X$. Since

$$d_{\tau_{t_0}\otimes\tau}(f_{O_1}(x')) \leq d_{\tau_{t_0}\otimes\tau}(f_{(O_1)_{d''}}(y')) \tag{4.10}$$

and

$$d_{\tau_{t_0}\otimes\tau}(f_{O_1}(x')) = \tau(\chi_{O_1}(x'(t_0))) \geq \frac{1}{n}, \tag{4.11}$$

then $(O_1)_{d''} \cap \{g_1(t_0), \cdots, g_n(t_0)\} \neq \emptyset$. Suppose $g_j(t_0) \in (O_1)_{d''} \cap \{g_1(t_0), \cdots, g_n(t_0)\}$ for some j, then

$$f_1(t_0) - g_1(t_0) \leq f_1(t_0) - g_j(t_0) < d'' + \frac{d'-d''}{2} < d'. \tag{4.12}$$

By (4.9), $i_0 > 1$.

Let $\varepsilon' > 0$ such that $\varepsilon' < d' - (f_{i_0-1}(t_0) - g_{i_0-1}(t_0))$. Moreover, we may assume $\varepsilon' < \{f_{i_0}(t_0) - f_{i_0+1}(t_0)\}$ if $i_0 < n$. Put $O = \{f_1(t_0), \cdots, f_{i_0}(t_0)\}_{\varepsilon'}$ and $O' = \{g_1(t_0), \cdots, g_{i_0-1}(t_0)\}_{\varepsilon'}$. Since $\varepsilon' < \{f_{i_0}(t_0) - f_{i_0+1}(t_0)\}$ if $i_0 < n$, then

$$O \cap \{f_1(t_0), \cdots, f_n(t_0)\} = \{f_1(t_0), \cdots, f_{i_0}(t_0)\} \tag{4.13}$$

when $i_0 < n$. It is obvious that (4.13) also holds when $i_0 = n$. Note that $O' \cap \{g_1(t_0), \cdots, g_n(t_0)\} = \{g_1(t_0), \cdots, g_{i_0-1}(t_0)\}$. Otherwise,

$$f_{i_0}(t_0) - g_{i_0}(t_0) \leq f_{i_0-1}(t_0) - g_{i_0-1}(t_0) + \varepsilon' < d', \tag{4.14}$$

a contradiction with (4.9).

Moreover, by (4.9),

$$(O)_{d''} \cap \{g_1(t_0), \cdots, g_n(t_0)\} \subset \{g_1(t_0), \cdots, g_{i_0-1}(t_0)\} \subset O'. \tag{4.15}$$

Therefore, as (4.11),

$$d_{\tau_{t_0}\otimes\tau}(f_O(x')) = \frac{i_0}{n} > \frac{i_0-1}{n} = d_{\tau_{t_0}\otimes\tau}(f_{O'}(y')) \geq d_{\tau_{t_0}\otimes\tau}(f_{(O)_{d''}}(y')). \tag{4.16}$$

It is contrary with $d'' > D_T(x', y')$. This proves the claim.

The claim shows that $\mathrm{dist}(U(x'), U(y')) \leq D_T(x', y')$. Therefore,

$$\mathrm{dist}(U(x), U(y)) \leq \mathrm{dist}(U(x'), U(y')) + 2\varepsilon \leq D_T(x', y') + 2\varepsilon \leq D_T(x, y) + 4\varepsilon. \tag{4.17}$$

This implies that $\mathrm{dist}(U(x), U(y)) \leq D_T(x, y)$.

Lemma 4.3 *Let X be a compact Hausdorff space with $\dim X \le 1$. Then for any self-adjoint elements $x, y \in C(X)\otimes M_n$, $D_T(x,y) \le \operatorname{dist}(U(x), U(y))$.*

Proof We first prove that

$$T(C(X)\otimes M_n) = T(C(X))\otimes T(M_n). \tag{4.18}$$

Let $\tau \in T(C(X)\otimes M_n)$ and $\tau_1 = \tau|_{C(X)}$ and $\tau_2 = \tau|_{M_n}$. Then $\tau_1 \in T(C(X))$ and $\tau_2 \in T(M_n)$. Let $\{e_{ij}\}$ be a matrix units of M_n. Let I_1 be the unit of $C(X)$ and I_2 be the unit of M_n. Then for any $f \in C(X)$,

$$\begin{aligned}\tau(f\otimes e_{ii}) &= \tau((f\otimes e_{ii})(I_1\otimes e_{ii}))\\ &= \tau((f\otimes e_{ii})(I_1\otimes e_{ij})(I_1\otimes e_{ji}))\\ &= \tau((I_1\otimes e_{ji})(f\otimes e_{ii})(I_1\otimes e_{ij}))\\ &= \tau(f\otimes e_{jj})\end{aligned} \tag{4.19}$$

for any i, j. Therefore,

$$n\tau(f\otimes e_{ii}) = \sum_{i=1}^{n}\tau(f\otimes e_{ii}) = \tau(f\otimes I_2) = \tau_1(f) \tag{4.20}$$

for all i. In other words, $\tau(f\otimes e_{ii}) = \tau_1(f)\tau_2(e_{ii}) = (\tau_1\otimes\tau_2)(f\otimes e_{ii})$. Moreover, if $i \ne j$,

$$\tau(f\otimes e_{ij}) = \tau((f\otimes e_{ij})(I_1\otimes e_{jj})) = \tau((f\otimes e_{jj})(I_1\otimes e_{ij})) = 0 = (\tau_1\otimes\tau_2)(f\otimes e_{ij}). \tag{4.21}$$

This implies that $\tau = \tau_1\otimes\tau_2$. So $T(C(X)\otimes M_n)\subset T(C(X))\otimes T(M_n)$.

For any $\tau_1 \in T(C(X))$ and $\tau_2 \in T(M_n)$, we may check that $\tau_1\otimes\tau_2$ is a positive linear function on $C(X)\otimes M_n$. Since $\|\tau_1\otimes\tau_2\| = (\tau_1\otimes\tau_2)(I_1\otimes I_2) = 1$, $\tau_1\otimes\tau_2$ is state. Moreover, we may find that $\tau_1\otimes\tau_2$ is trace. Therefore, (4.18) holds.

As [11, Lemma 1.1], there are $\lambda_1, \cdots, \lambda_n, \eta_1, \cdots, \eta_n \in C(X)$ such that $sp(x(t)) = \{\lambda_1(t), \cdots, \lambda_n(t)\}$ and $sp(y(t)) = \{\eta_1(t), \cdots, \eta_n(t)\}$, respectively, for all $t \in X$. Define

$$\delta(x(t), y(t)) = \min_{\pi}\max_{1\le i\le n}|\lambda_i(t) - \eta_{\pi(i)}(t)|, \tag{4.22}$$

where π runs over all permutations of $\{1, \cdots, n\}$. The equality

$$\operatorname{dist}(U(x(t)), U(y(t))) = \delta(x(t), y(t)) \tag{4.23}$$

is well known for Hermitian matrices by Weyl [12]. Let $d(t) = \delta(x(t), y(t))$ and $d' = \|x - y\|$. Then $\sup_{t\in X}\{d(t)\} \le d'$. Let $\varepsilon > 0$. Let $\tau_t \in T(C(X))$ as in Lemma 4.2. Suppose τ_2 is the tracial state on M_n. For any open set O, choose $f_{(O)_\varepsilon}$ such that $f_{(O)_\varepsilon}(t) = 1$ when $t \in O$, choose $f_{(O)_{d'+3\varepsilon}}$ such that $f_{(O)_{d'+3\varepsilon}}(t) = 1$ when $t \in (O)_{d'+2\varepsilon}$. So by (4.23),

$$\begin{aligned}(\tau_t\otimes\tau_2)(f_{(O)_\varepsilon}(x)) &\le d_{\tau_t\otimes\tau_2}(f_{(O)_{2\varepsilon}}(x))\\ &= \tau_2(\chi_{(O)_{2\varepsilon}}(x(t)))\\ &\le \tau_2(\chi_{(O)_{d'+2\varepsilon}}(y(t)))\\ &= d_{\tau_t\otimes\tau_2}(f_{(O)_{d'+2\varepsilon}}(y))\end{aligned}$$

$$\leq (\tau_t \otimes \tau_2)(f_{(O)_{d'+3\varepsilon}}(y)). \tag{4.24}$$

As [11, Lemma 1.1], there are $\lambda'_1, \cdots, \lambda'_n, \eta'_1, \cdots, \eta'_n \in C(X)$ such that

$$\begin{aligned} sp((f_{(O)_\varepsilon}(x))(t)) &= \{\lambda'_1(t), \cdots, \lambda'_n(t)\}, \\ sp((f_{(O)_{d'+3\varepsilon}}(y))(t)) &= \{\eta'_1(t), \cdots, \eta'_n(t)\}, \end{aligned} \tag{4.25}$$

respectively, for all $t \in X$. Suppose $x' = \sum\limits_{i=1}^{n} \lambda'_i e_{ii}$ and $y' = \sum\limits_{j=1}^{n} \eta'_j e_{jj}$. It follows from [11, Corollary 1.3], $\mathrm{dist}(U(x'), U(f_{(O)_\varepsilon}(x))) = \mathrm{dist}(U(y'), U(f_{(O)_{d'+3\varepsilon}}(y))) = 0$. Then $(\tau_t \otimes \tau_2)(f_{(O)_\varepsilon}(x)) = (\tau_t \otimes \tau_2)(x')$ and $(\tau_t \otimes \tau_2)(f_{(O)_{d'+3\varepsilon}}(y)) = (\tau_t \otimes \tau_2)(y')$ for all $t \in X$. Therefore, by (4.24),

$$\frac{1}{n}\sum_{i=1}^{n} \lambda'_i(t) = (\tau_t \otimes \tau_2)(x') \leq (\tau_t \otimes \tau_2)(y') = \frac{1}{n}\sum_{j=1}^{n} \eta'_j(t) \tag{4.26}$$

for all $t \in X$. Now, choose any $\tau' \in T(C(X))$. Then τ' is a positive linear function on $C(X)$. It follows from the Riesz representation theorem (see [8, Theorem 2.14]), there is a positive measure μ such that $\tau'(f) = \int_X f \mathrm{d}\mu$ for all $f \in C(X)$. It follows from (4.26) that

$$(\tau' \otimes \tau_2)(x') = \frac{1}{n}\sum_{i=1}^{n} \tau'(\lambda'_i) = \int_X \frac{1}{n}\sum_{i=1}^{n} \lambda'_i \mathrm{d}\mu \leq \int_X \frac{1}{n}\sum_{j=1}^{n} \eta'_i \mathrm{d}\mu = (\tau' \otimes \tau_2)(y'). \tag{4.27}$$

So,

$$\begin{aligned} d_{\tau'\otimes\tau_2}(f_O(x)) &\leq (\tau' \otimes \tau_2)(f_{(O)_\varepsilon}(x)) \\ &= (\tau' \otimes \tau_2)(x') \\ &\leq (\tau' \otimes \tau_2)(y') \\ &= (\tau' \otimes \tau_2)(f_{(O)_{d'+3\varepsilon}}(y)) \\ &\leq d_{\tau'\otimes\tau_2}(f_{(O)_{d'+4\varepsilon}}(y)). \end{aligned} \tag{4.28}$$

Therefore, by (4.18), for all $\tau \in T(C(X) \otimes M_n)$, $d_\tau(f_O(x)) \leq d_\tau(f_{(O)_{d'+4\varepsilon}}(y))$. This implies that $D_T(x, y) \leq \mathrm{dist}(U(x), U(y))$.

Theorem 4.1 *Let $C \in I^{(1)}$. Then for any self-adjoint elements $x, y \in C$,*

$$\mathrm{dist}(U(x), U(y)) = D_T(x, y). \tag{4.29}$$

Proof Suppose $C = \oplus_{i=1}^{n} P_i(C(X_i) \otimes M_{k_i})P_i$ where $X_1, \cdots, X_n$ are one-dimensional connected finite CW complexes and $P_i \in C(X_i) \otimes M_{k_i}$ are projections, $i = 1, \cdots, n$. By [6, Theorem 2.6.15], we may furthermore assume $C = \oplus_{i=1}^{n} C(X_i) \otimes M_{k_i}$. Suppose $x = \sum\limits_{i=1}^{n} x_i$ and $y = \sum\limits_{i=1}^{n} y_i$, where $x_i, y_i \in C(X_i) \otimes M_{k_i}$ are self-adjoint elements for all i. It follows from Lemmas 4.2–4.3,

$$\mathrm{dist}(U(x), U(y)) = \max_i\{\mathrm{dist}(U(x_i), U(y_i)\} = \max_i\{D_T(x_i, y_i)\} = D_T(x, y). \tag{4.30}$$

Lemma 4.4 *Let $\Omega \subset \mathbb{C}$ be a compact subset. For any $\varepsilon > 0$, there exist finitely many open subsets $O_1, \cdots, O_n \subset \Omega$ satisfying the following:*

Let $d > 0$ and $f_{(O_i)_{2\varepsilon}}(t)$ $(i = 1, \cdots, n)$ be some continuous functions on Ω such that $f_{(O_i)_{2\varepsilon}}(t) = 1$ if $t \in (O_i)_\varepsilon$ and $f_{(O_i)_{2\varepsilon}}(t) = 0$ if $t \notin (O_i)_{2\varepsilon}$. Let A be a unital C^*-algebra *of stable rank one. Suppose $x, y \in A$ are two normal elements with $sp(x), sp(y) \subset \Omega$. If there exist $u_i \in U(A)$, $0 < \varepsilon_i < 1$ and $0 < \varepsilon_i' < 1$ $(i = 1, \cdots, n)$ such that for all i,*

$$u_i^* f_{\varepsilon_i'}(f_{\varepsilon_i}(f_{(O_i)_{2\varepsilon}}(x)))u_i \in \mathrm{Her}(f_{(O_i)_{d+2\varepsilon}}(y)), \tag{4.31}$$

then

$$D_c(x, y) \leq d + 4\varepsilon. \tag{4.32}$$

Proof Since Ω is compact, there are open subsets $O_1, \cdots, O_n \subset \Omega$ such that for any open set $G \subset \Omega$, there is an integer i,

$$G \subset O_i \subset G_\varepsilon. \tag{4.33}$$

Suppose $d > 0$. We assume $u_i \in U(A)$, $0 < \varepsilon_i < 1$ and $0 < \varepsilon_i' < 1$ $(i = 1, \cdots, n)$ satisfying for all i,

$$u_i^* f_{\varepsilon_i'}(f_{\varepsilon_i}(f_{(O_i)_{2\varepsilon}}(x)))u_i \in \mathrm{Her}(f_{(O_i)_{d+2\varepsilon}}(y)). \tag{4.34}$$

Then for any open subset $G \subset \Omega$, suppose $G \subset O_i \subset G_\varepsilon$ for some i. Let $f_{G_{d+4\varepsilon}}(t)$ be a continuous function on Ω such that $f_{G_{d+4\varepsilon}}(t) = 1$ if $t \in G_{d+3\varepsilon}$ and $f_{G_{d+4\varepsilon}}(t) = 0$ if $t \notin G_{d+4\varepsilon}$. Since the support of f_G is contained in $(O_i)_\varepsilon$ and $(O_i)_\varepsilon$ is contained in the support of $f_{\varepsilon_i'}(f_{\varepsilon_i}(f_{(O_i)_{2\varepsilon}}(t)))$, it follows from Lemma 2.1 that

$$f_G(x) \lesssim f_{\varepsilon_i'}(f_{\varepsilon_i}(f_{(O_i)_{2\varepsilon}}(x))). \tag{4.35}$$

Since $f_{(O_i)_{d+2\varepsilon}}(y) \leq f_{G_{d+4\varepsilon}}(y)$, $\mathrm{Her}(f_{(O_i)_{d+2\varepsilon}}(y)) \subset \mathrm{Her}(f_{G_{d+4\varepsilon}}(y))$. By [9, Proposition 2.4], for any $\varepsilon' > 0$, there is a unitary $u_i' \in A$ such that

$$\begin{aligned} {u'}_i^* f_{\varepsilon'}(f_G(x))u_i' &\in \mathrm{Her}(u_i^* f_{\varepsilon_i'}(f_{\varepsilon_i}(f_{(O_i)_{2\varepsilon}}(x)))u_i) \\ &\subset \mathrm{Her}(f_{(O_i)_{d+2\varepsilon}}(y)) \subset \mathrm{Her}(f_{G_{d+4\varepsilon}}(y)). \end{aligned} \tag{4.36}$$

Therefore, by [9, Proposition 2.4] again, $f_G(x) \lesssim f_{G_{d+4\varepsilon}}(y)$. This implies that $D_c(x, y) \leq d + 4\varepsilon$.

Lemma 4.5 *Let A be a unital simple C^*-algebra with $TR(A) \leq 1$. Let $x, y \in A$ be two self-adjoint elements. Let $e \in A$ be a projection and $\varepsilon > 0$. Suppose $d > D_c(x, y)$. Then there exists a projection $P \in A$, a C^*-subalgebra $C \in I^{(1)}$ with $I_C = P$, four self-adjoint elements $x_1, y_1 \in C$ and $x_2, y_2 \in (1 - P)A(1 - P)$ such that*

(1) *$\|x - (x_1 + x_2)\| < \varepsilon$ and $\|y - (y_1 + y_2)\| < \varepsilon$,*

(2) *$D_c(x_1, y_1) \leq d + \varepsilon$ in C,*

(3) *$D_c(x_2, y_2) \leq d + \varepsilon$ in $(1 - P)A(1 - P)$,*

(4) *$1 - P \lesssim e$.*

Proof Let $0 < \varepsilon < 1$ and $0 < \varepsilon_1 \leq \frac{\varepsilon}{5}$. Suppose Ω is the closure of $(sp(x))_{2\varepsilon} \cup (sp(y))_{2\varepsilon}$. Let $O_1, \cdots, O_n \subset \Omega$ be those open sets in Lemma 4.4 corresponding to Ω and ε_1(replace of ε in Lemma 4.4). Denote by g_i the function $f_{\varepsilon_1}(f_{(O_i)_{2\varepsilon_1}})(t)$ $(i = 1, \cdots, n)$, where $f_{(O_i)_{2\varepsilon_1}}(t)$ are those functions in Lemma 4.4 $(i = 1, \cdots, n)$. Denote by g_{n+i} the function $f_{(O_i)_{d+2\varepsilon_1}}(t)$ $(i = 1, \cdots, n)$. Since $g_i(x) \lesssim g_{n+i}(y)$, there are $r_i \in A$ such that $\|r_i^* g_{n+i}(y) r_i - g_i(x)\| < \varepsilon_1$, $i = 1, \cdots, n$.

Let $M = \max\{1, \|x\|, \|y\|, \|r_i\| : i = 1, \cdots, n\}$. $\varepsilon_2 = \frac{\varepsilon_1}{(M+1)^2}$. Let $\delta > 0$ with $\delta < \varepsilon_2$ satisfying [6, Lemma 2.5.11]. In other words, for number M and functions $g_1, \cdots, g_{2n}$ as above, let B be a unital C^*-algebra, then for any projection $P \in B$, if $a \in B$ is a normal element with $\|a\| \le M$ and $\|Pa - aP\| < 4\delta$, then $\|Pg_i(a) - g_i(a)P\| < \varepsilon_2$ for $i = 1, \cdots, 2n$; if $a, b \in B$ are normal elements with $\|a\|, \|b\| \le M$ and $\|a-b\| < 4\delta$, then $\|g_i(a) - g_i(b)\| < \varepsilon_2$ for $i = 1, \cdots, 2n$.

Since $TR(A) \le 1$, let e be a projection and

$$\mathcal{F} = \{x, y, g_i(x), g_{n+i}(y), r_i : i = 1, \cdots, n\}, \tag{4.37}$$

then there exists a projection $P \in A$, a C^*-subalgebra $C \in I^{(1)}$ with $I_C = P$ such that

$$\|Pa - aP\| < \delta \quad \text{for all } a \in \mathcal{F}, \tag{4.38}$$

$$PaP \in_\delta C \quad \text{for all } a \in \mathcal{F}, \tag{4.39}$$

$$1 - P \lesssim e. \tag{4.40}$$

Since x, y are self-adjoint elements, there are self-adjoint elements $x_1, y_1 \in C$, $x_2, y_2 \in A'$, where $A' = (1-P)A(1-P)$, such that

$$\|x - (x_1 + x_2)\| < 4\delta \quad \text{and} \quad \|y - (y_1 + y_2)\| < 4\delta. \tag{4.41}$$

Note that

$$sp_C(x_1) \subset sp_{C+A'}(x_1 + x_2) = sp_A(x_1 + x_2) \subset \Omega. \tag{4.42}$$

In the same way, $sp_C(y_1), sp_{A'}(x_2), sp_{A'}(y_2) \subset \Omega$. So

$$\|g_i^A(x) - (g_i^C(x_1) + g_i^{A'}(x_2))\| < \varepsilon_2, \tag{4.43}$$

$$\|g_{n+i}^A(y) - (g_{n+i}^C(y_1) + g_{n+i}^{A'}(y_2))\| < \varepsilon_2 \tag{4.44}$$

for all i. Let $s_i \in C$ such that $\|s_i - Pr_iP\| < \delta$. Then $\|s_i\| \le M + 1$, $i = 1, \cdots, n$. Therefore,

$$\begin{aligned}
&\|s_i^* g_{n+i}^C(y_1) s_i - g_i^C(x_1)\| \\
&\le \|Pr_i^* P g_{n+i}^C(y_1) Pr_i P - P g_i^C(x_1) P\| + 2\varepsilon_1 \\
&= \|Pr_i^* P(g_{n+i}^C(y_1) + g_{n+i}^{A'}(y_2)) Pr_i P - P(g_i^C(x_1) + g_i^{A'}(x_2))P\| + 2\varepsilon_1 \\
&\le \|Pr_i^* g_{n+i}^A(y) r_i P - P g_i^A(x) P\| + 4\varepsilon_1 \\
&\le \|r_i^* g_{n+i}^A(y) r_i - g_i^A(x)\| + 4\varepsilon_1 \\
&\le 5\varepsilon_1.
\end{aligned} \tag{4.45}$$

It follows from [9, Proposition 2.2] that

$$f_{5\varepsilon_1}^C(g_i^C(x_1)) \lesssim s_i^* g_{n+i}^C(y_1) s_i \tag{4.46}$$

in C. Since $s_i^* g_{n+i}^C(y_1) s_i \lesssim g_{n+i}^C(y_1)$, $f_{5\varepsilon_1}^C(g_i^C(x_1)) \lesssim g_{n+i}^C(y_1)$ in C. Note that $5\varepsilon_1 < 1$, then there exists $0 < \varepsilon' < 1$ such that the support of $f_{\varepsilon'}^C f_{(O_i)_{2\varepsilon}}^C$ is contained in the support of $f_{5\varepsilon_1}^C f_{\varepsilon_1}^C f_{(O_i)_{2\varepsilon}}^C$. So $f_{\varepsilon'}^C(f_{(O_i)_{2\varepsilon}}^C(x_1)) \lesssim f_{5\varepsilon_1}^C(g_i^C(x_1)) \lesssim g_{n+i}^C(y_1)$ in C, $i = 1, \cdots, n$. By [9, Proposition 2.4], there are unitaries $u_1, \cdots, u_n \in C$ and ε'' with $0 < \varepsilon'' < 1$ such that

$$u_i^* f_{\varepsilon''}^C(f_{\varepsilon'}^C(f_{(O_i)_{2\varepsilon}}^C(x_1))) u_i \in \mathrm{Her}(f_{n+i}^C(y_1)) \tag{4.47}$$

in C, $i = 1, \cdots, n$. By applying Lemma 4.4, we have $D_c(x_1, y_1) < d + \varepsilon$ in C. It follows from [7, Theorem 6.9] that A has stable rank one. Therefore, we may apply the same argument to show that $D_c(x_2, y_2) < d + \varepsilon$ in $(1-P)A(1-P)$.

Lemma 4.6 *Let $A = C(X) \otimes M_n$, where X is a compact Hausdorff space with $\dim X \leq 1$. Let $x, y \in A$ be two self-adjoint elements. Then $D_c(x, y) = D_T(x, y)$.*

Proof By (2.17), it suffices to prove that $D_c(x, y) \leq D_T(x, y)$.

Let $a, b \in A_+$. We first prove that if $d_\tau(a) \leq d_\tau(b)$ for all $\tau \in T(A)$, then $a \lesssim b$. There are continuous functions $0 \leq \lambda_1 \leq \cdots \leq \lambda_n \in C(X)$ and $0 \leq \eta_1 \leq \cdots \leq \eta_n \in C(X)$ such that $sp(a(t)) = \{\lambda_1(t), \cdots, \lambda_n(t)\}$ and $sp(b(t)) = \{\eta_1(t), \cdots, \eta_n(t)\}$. Let $e_1, \cdots, e_n \in M_n$ be mutually disjoint projections. Then $\mathrm{dist}\big(U(a), U\big(\sum\limits_{i=1}^{n} \lambda_i \otimes e_i\big)\big) = \mathrm{dist}\big(U(b), U\big(\sum\limits_{i=1}^{n} \eta_i \otimes e_i\big)\big) = 0$. So we may assume that $a = \sum\limits_{i=1}^{n} \lambda_i \otimes e_i$ and $b = \sum\limits_{i=1}^{n} \eta_i \otimes e_i$. Suppose $d_\tau(a) \leq d_\tau(b)$ for all $\tau \in T(A)$. Let $t \in X$. If $a(t) \neq 0$, we may assume $\lambda_1(t) = \cdots = \lambda_{i_0}(t) = 0 < \lambda_{i_0+1}(t) \leq \lambda_n(t)$. Let $0 < \varepsilon < \lambda_{i_0+1}(t)$. Then

$$\frac{n - i_0}{n} = \tau'(f_\varepsilon(a(t))) = d_{\tau_t \otimes \tau'}(a) \leq d_{\tau_t \otimes \tau'}(b), \tag{4.48}$$

where τ' is the tracial state of M_n. This implies $0 < \eta_{i_0+1}(t) \leq \cdots \leq \eta_n(t)$. By the arbitrary of t, we obtain that the support of λ_i is containing in support of η_i. It follows from Lemma 2.1, $\lambda_i \lesssim \eta_i$ in $C(X)$. This implies that $a \lesssim b$.

Let $d = D_T(x, y)$. Then $d_\tau(f_O(x)) \leq d_\tau(f_{(O)_d}(y))$ for all $\tau \in T(A)$ implies $f_O(x) \lesssim f_{(O)_d}(y)$. Therefore, $D_c(x, y) = D_T(x, y)$.

Note that the following A may not be simple, it is different from Theorem 4.3.

Theorem 4.2 *Let A be a unital AT-algebra and $x, y \in A$ be two self-adjoint elements. Then*

$$\mathrm{dist}(U(x), U(y)) = D_c(x, y). \tag{4.49}$$

Proof We may assume $A = \overline{\cup A_n}$, where each $\{A_n\}$ is a finite direct sum of circle algebras with $I_{A_n} = I_A$. We may assume furthermore that $\{A_n\}$ is an increase sequence. Suppose $d > D_c(x, y)$ in A. The proof of Lemma 4.5 also shows that there exists A_n, two self-adjoint elements $x', y' \in A_n$ such that $\|x - x'\| < \varepsilon$, $\|y - y'\| < \varepsilon$ and $D_c(x', y') \leq d + \varepsilon$ in A_n. It follows from Lemma 2.2, $D_T(x', y') \leq D_c(x', y')$ in A_n. As Theorem 4.1, we also have $u \in U(A_n) \subset U(A)$ such that $\|u^* x' u - y'\| < d + 2\varepsilon$. Therefore, $\|u^* x u - y\| < d + 4\varepsilon$. This implies

$$\mathrm{dist}(U(x), U(y)) \leq D_c(x, y). \tag{4.50}$$

Let $\varepsilon > 0$, $\delta > 0$ with $\delta < \varepsilon$, $d_1 = \mathrm{dist}(U(x), U(y))$ and $d_2 = D_c(x, y)$ in A. There are two self-adjoint elements $x', y' \in A_n$ for some n such that $\|x' - x\| < \delta$ and $\|y' - y\| < \delta$. Suppose $u \in U(A)$ such that $d_1 \leq \|u^* x u - y\| < d_1 + \delta$. It follows from [6, Lemma 4.1.1], there is a unitary $u_1 \in A_{n_1}$ for some n_1 such that $\|u_1 - u\| < \delta$. Let $N_1 > n, n_1$ be an integer number. Then $x_1, y_1, u_1 \in A_{N_1}$ since $\{A_n\}$ is an increase sequence. So

$$\mathrm{dist}(U(x_1), U(y_1)) \leq \|u_1^* x_1 u_1 - y_1\| < d_1 + (3 + 2\|x\|)\delta < d_1 + (3 + 2\|x\|)\varepsilon \tag{4.51}$$

in A_{N_1}. It follows from Lemma 2.3, if δ is sufficiently small, then $D_c(x_1, x) < \varepsilon$ and $D_c(y_1, y) < \varepsilon$ in A. Suppose $d_3 = D_c(x_1, y_1)$ in A_{N_1}. Then by Lemma 4.6,

$$d_2 - 2\varepsilon \leq d_3 = D_T(x_1, y_1) = \mathrm{dist}(U(x_1), U(y_1)) \leq d_1 + (3 + 2\|x\|)\varepsilon. \tag{4.52}$$

By arbitrary of ε, $d_2 \leq d_1$.

We recall the notation of ε-path connected. One can see [10] for more details.

Definition 4.1 *Let $X \subset \mathbb{R}$ and $\varepsilon > 0$. We say X is ε-path connected if for any $a, b \in X$, there are finite points $c_1, \cdots, c_n \in X$ such that $|a - c_1| < \varepsilon$, $|c_n - b| < \varepsilon$ and $|c_i - c_{i+1}| < \varepsilon$ for $i = 1, \cdots, n-1$. We define a relation on X by setting $a \sim b$ if there is an ε-path connected subset of X containing both a and b. This relation is equivalence relation on X. This equivalence classes will be called ε-path connected component. If X is compact, then X determine finitely many mutually disjoint ε-path connected component.*

Lemma 4.7 *Let $R \subset \{1, \cdots, m\} \times \{1, \cdots, n\}$ be a subset such that for any non-empty subset $A \subset \{1, \cdots, m\}$ and $B \subset \{1, \cdots, n\}$, $R_A \neq \emptyset$ and $R^B \neq \emptyset$. Then there are $a_{ij} \in \mathbb{Z}_+$ ($i = 1, \cdots, m$ and $j = 1, \cdots, n$) such that $\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n} a_{ij} = mn$, $\sum\limits_{j=1}^{n} a_{ij} > 0$, $\sum\limits_{i=1}^{m} a_{ij} > 0$ for all i, j and $a_{ij} = 0$ unless $(i, j) \in R$.*

Proof Suppose $b_1 = \cdots = b_m = 1$. Choose $i_1 \in R^{\{1\}}$. Set $b_{i_1 1} = 1$ and $b_{i_1 j} = 0$ if $j \neq 1$. Choose $j_i \in R_{\{i\}}$ for each $i \neq i_1$. For each $i \neq i_1$, set $b_{ij_i} = 1$ and $b_{ij} = 0$ if $j \neq j_i$. It is obvious that $b_i = \sum\limits_{j=1}^{n} b_{ij}$ for $i = 1, \cdots, m$, $\sum\limits_{i=1}^{m} b_{i1} > 0$ and $b_{ij} = 0$ unless $(i, j) \in R$.

This proof implies that for each k, where $k \in \{1, \cdots, n\}$, there are $a_{ijk} \in \{0, 1\}$ ($i = 1, \cdots, m$ and $j = 1, \cdots, n$) such that $\sum\limits_{j=1}^{n} a_{ijk} = 1$ for $i = 1, \cdots, m$, $\sum\limits_{i=1}^{m} a_{ikk} > 0$ and $a_{ijk} = 0$ unless $(i, j) \in R$. Let $a_{ij} = \sum\limits_{k=1}^{n} a_{ijk}$. Then $a_{ij} = 0$ unless $(i, j) \in R$, $\sum\limits_{j=1}^{n} a_{ij} = n$ for each $i = 1, \cdots, m$ and $\sum\limits_{i=1}^{m} a_{ij} \geq \sum\limits_{i=1}^{m} a_{ijj} > 0$ for each $j = 1, \cdots, n$.

Lemma 4.8 *Let A be a unital C^*-algebra. Suppose A has stable rank one and $K_0(A)$ has the Riesz interpolation property. Let $x, y \in A$ be two self-adjoint elements. Let $\varepsilon > 0$. Suppose $X = \{\lambda_1, \cdots, \lambda_m\} \subset sp(x)$ is a ε-dense subset of $sp(x)$ and $Y = \{\eta_1, \cdots, \eta_n\} \subset sp(y)$ is a ε-dense subset of $sp(y)$. Suppose $X'_1, \cdots, X'_{m'}$ are mutually disjoint ε-path connected components of $sp(x)$ and $Y'_1, \cdots, Y'_{n'}$ are mutually disjoint ε-path connected components of $sp(y)$. Let $X_k = X'_k \cap X$ and $Y_l = Y'_l \cap Y$ for all k, l. Then*

(1) $(X_k)_\varepsilon \cap sp(x) = X'_k$, $(Y_l)_\varepsilon \cap sp(y) = Y'_l$ are clopen subsets of $sp(x)$ and $sp(y)$ respectively, $k = 1, \cdots, m'$, $l = 1, \cdots, n'$;

(2) moreover, $\sum\limits_{k\in O} \chi_{(X_k)_\varepsilon}(x) = \chi_{\cup_{k\in O}(X_k)_\varepsilon}(x)$ for any $O \subset \{1, \cdots, m'\}$ and $\sum\limits_{l\in O'} \chi_{(Y_l)_\varepsilon}(y) = \chi_{\cup_{l\in O'}(Y_l)_\varepsilon}(y)$ for any $O' \subset \{1, \cdots, n'\}$.

If $R \subset X \times Y$ such that for any non-empty subset $B \subset X$, $f_{(B)_\varepsilon}(x) \lesssim f_{(R_B)_\varepsilon}(y)$, then there are a_i, b_j and $c_{ij} \in \mathbb{Z}_+$, $i = 1, \cdots, m$ and $j = 1, \cdots, n$, such that

(3) $a_i = \sum\limits_{j=1}^{n} c_{ij} > 0$, $b_j = \sum\limits_{i=1}^{m} c_{ij} > 0$ and $\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n} c_{ij} \leq mn$ for all i, j;

(4) $c_{ij} = 0$ unless $(\lambda_i, \eta_j) \in R$;

(5) *for any subset* $O \subset \{1, \cdots, m'\}$, *set* $X_O = \bigcup_{k\in O} X_k$ *and* $O' = \{l : 1 \le l \le n', R_{X_O} \cap Y_l \neq \emptyset\}$. *If* $\sum_{k\in O}[\chi_{(X_k)_\varepsilon}(x)] = \sum_{l\in O'}[\chi_{(Y_l)_\varepsilon}(y)]$ *in* $K_0(A)$, *then* $\sum_{k\in O}\sum_{\lambda_i\in X_k} a_i = \sum_{l\in O'}\sum_{\eta_j\in Y_l} b_j$.

Proof It is trivial to check that (1) and (2). Then we prove (3), (4) and (5).

Let $R' = \{(k,l) : R_{X_k} \cap Y_l \neq \emptyset\} \subset \{1, \cdots, m'\} \times \{1, \cdots, n'\}$. Then we have for any $O \subset \{1, \cdots, m'\}$, $\sum_{k\in O}[\chi_{(X_k)_\varepsilon}(x)] \le \sum_{l\in R'_O}[\chi_{(Y_l)_\varepsilon}(y)]$. Since $\sum_{k=1}^{m'}[\chi_{(X_k)_\varepsilon}(x)] = \sum_{l=1}^{n'}[\chi_{(Y_l)_\varepsilon}(y)]$, it follows from Theorem 3.1 that there are $d_{kl} \in (K_0(A))_+$, $k = 1, \cdots, m'$ and $l = 1, \cdots, n'$, such that $[\chi_{(X_k)_\varepsilon}(x)] = \sum_{l=1}^{n'} d_{kl}$, $[\chi_{(Y_l)_\varepsilon}(y)] = \sum_{k=1}^{m'} d_{kl}$ and $d_{kl} = 0$ unless $(k,l) \in R'$. Denote by $|X|$ the number of elements in X. Put $c'_{kl} = |X_k| \cdot |Y_l|$ when $d_{kl} \neq 0$ and $c'_{kl} = 0$ when $d_{kl} = 0$, $k = 1, \cdots, m'$ and $l = 1, \cdots, n'$. Let $a'_k = \sum_{l=1}^{n'} c'_{kl}$ and $b'_l = \sum_{k=1}^{m'} c'_{kl}$. Then for any $O \subset \{1, \cdots, m'\}$, we have $O' = R'_O$ and

$$\sum_{k\in O} a'_k = \sum_{k\in O}\sum_{l=1}^{n'} c'_{kl} = \sum_{k\in O}\sum_{l\in R'_{\{k\}}} c'_{kl} \le \sum_{l\in R'_O}\sum_{k=1}^{m'} c'_{kl} = \sum_{l\in R'_O} b'_l. \tag{4.53}$$

Moreover, if $\sum_{k\in O}[\chi_{(X_k)_\varepsilon}(x)] = \sum_{l\in R'_O}[\chi_{(Y_l)_\varepsilon}]$ in $K_0(A)$, then

$$\sum_{k\in O}\sum_{l\in R'_{\{k\}}} d_{kl} = \sum_{k\in O}[\chi_{(X_k)_\varepsilon}(x)] = \sum_{l\in R'_O}[\chi_{(Y_l)_\varepsilon}] = \sum_{l\in R'_O}\sum_{k=1}^{m'} d_{kl}. \tag{4.54}$$

Therefore

$$\sum_{l\in R'_O}\sum_{k\notin O} d_{kl} = \sum_{l\in R'_O}\sum_{k=1}^{m'} d_{kl} - \sum_{k\in O, l\in R'_O} d_{kl} = \sum_{l\in R'_O}\sum_{k=1}^{m'} d_{kl} - \sum_{k\in O, l\in R'_{\{k\}}} d_{kl} = 0. \tag{4.55}$$

So $d_{kl} = 0$ when $l \in R'_O$, $k \notin O$. This implies $c'_{kl} = 0$ when $l \in R'_O$, $k \notin O$. Then

$$\sum_{k\in O} a'_k = \sum_{k\in O}\sum_{l\in R'_{\{k\}}} c'_{kl} = \sum_{k\in O}\sum_{l\in R'_O} c'_{kl} = \sum_{l\in R'_O}\sum_{k=1}^{m'} c'_{kl} = \sum_{l\in R'_O} b'_l. \tag{4.56}$$

Suppose $I_k = \{i : \lambda_i \in X_k\}$ and $J_l = \{j : \eta_j \in Y_l\}$ for all k, l. It follows from Lemma 4.7, for any pair (k,l) which satisfy $c'_{kl} \neq 0$, there are $a_{ikl}, b_{jkl} > 0$ and $c_{ijkl} \in \mathbb{Z}_+$, $i \in I_k$ and $j \in J_l$, such that $\sum_{i\in I_k}\sum_{j\in J_l} c_{ijkl} = c'_{kl}$, $a_{ikl} = \sum_{j\in J_l} c_{ijkl} > 0$, $b_{jkl} = \sum_{i\in I_k} c_{ijkl} > 0$ and $c_{ijkl} = 0$ unless $(\lambda_i, \eta_j) \in R$. Let $a_{ikl} = b_{jkl} = c_{ijkl} = 0$ if they haven't been defined from above, $i = 1, \cdots, m$, $j = 1, \cdots, n$, $k = 1, \cdots, m'$ and $l = 1, \cdots, n'$. Let $a_i = \sum_{k=1}^{m'}\sum_{l=1}^{n'} a_{ikl}$, $b_j = \sum_{k=1}^{m'}\sum_{l=1}^{n'} b_{jkl}$ and $c_{ij} = \sum_{k=1}^{m'}\sum_{l=1}^{n'} c_{ijkl}$. Then we may check that (3) and (4) hold for a_i, b_j, c_{ij}. To see (5), let $O \subset \{1, \cdots, m'\}$. Suppose $\sum_{k\in O}[\chi_{(X_k)_\varepsilon}(x)] = \sum_{l\in R'_O}[\chi_{(Y_l)_\varepsilon}(y)]$ in $K_0(A)$. Note that for each $k = 1, \cdots, m'$,

$$\sum_{i\in I_k} a_i = \sum_{i\in I_k}\sum_{l\in R'_{\{k\}}} a_{ikl} = \sum_{i\in I_k}\sum_{l\in R'_{\{k\}}}\sum_{j\in J_l} c_{ijkl} = \sum_{l\in R'_{\{k\}}} c'_{kl}. \tag{4.57}$$

For each $l = 1, \cdots, n'$,

$$\sum_{j\in J_l} b_j = \sum_{j\in J_l}\sum_{k\in R'^{\{l\}}} b_{jkl} = \sum_{j\in J_l}\sum_{k\in R'^{\{l\}}}\sum_{i\in I_k} c_{ijkl} = \sum_{k\in R'^{\{l\}}} c'_{kl}. \tag{4.58}$$

Therefore, by (4.56)–(4.58),

$$\sum_{k\in O}\sum_{i\in I_k} a_i = \sum_{k\in O}\sum_{l\in R'_{\{k\}}} c'_{kl} = \sum_{l\in R'_O}\sum_{k=1}^{m'} c'_{kl} = \sum_{l\in R'_O}\sum_{k\in R'^{\{l\}}} c'_{kl} = \sum_{l\in R'_O}\sum_{j\in J_l} b_j. \tag{4.59}$$

Lemma 4.9 *Let A be a unital simple C^*-algebra with $TR(A) \le 1$. Let $x, y \in A$ be two self-adjoint elements and $e \in A$ be a projection. Let $\varepsilon > 0$, $d > D_c(x, y)$. Suppose $X = \{\lambda_1, \cdots, \lambda_m\} \subset sp(x)$ is a ε-dense subset of $sp(x)$ and $Y = \{\eta_1, \cdots, \eta_n\} \subset sp(y)$ is a ε-dense subset of $sp(y)$. Let $X_1, \cdots, X_{m'} \subset X$ and $Y_1, \cdots, Y_{n'} \subset Y$ as above lemma. Then there exist self-adjoint elements $x_1, y_1 \in A$, two sets of mutually orthogonal non-zero projections $\{p_1, \cdots, p_m\}$ and $\{q_1, \cdots, q_n\}$ and a unitary $u \in A$ such that*

(1) *$x_1, y_1 \in A_1$, where $A_1 = \big(1 - \sum\limits_{i=1}^m p_i\big)A\big(1 - \sum\limits_{i=1}^m p_i\big)$,*

(2) *$u^* \sum\limits_{i=1}^m p_i u = \sum\limits_{j=1}^n q_j$ and $\big[\sum\limits_{i=1}^m p_i\big] = \big[\sum\limits_{j=1}^n q_j\big] \le [e]$ in $K_0(A)$,*

(3) *$\big\|x - \big(x_1 + \sum\limits_{i=1}^m \lambda_i u^* p_i u\big)\big\| < \varepsilon$ and $\big\|y - \big(y_1 + \sum\limits_{j=1}^n \eta_j q_j\big)\big\| < \varepsilon$,*

(4) *$sp_A(x) \subset (sp_{A_1}(x_1))_\varepsilon$ and $sp_A(y) \subset (sp_{A_2}(y_1))_\varepsilon$,*

(5) *$D_c\big(u^* \sum\limits_{i=1}^m \lambda_i p_i u, \sum\limits_{j=1}^n \eta_j q_j\big) < d + \varepsilon$ in A_2 where $A_2 = \big(\sum\limits_{j=1}^n q_j\big)A\big(\sum\limits_{j=1}^n q_j\big)$,*

(6) *suppose $O \subset \{1, \cdots, m'\}$, $X_O = \bigcup\limits_{k\in O} X_k$, if $(X_O)_{d+\varepsilon} \cap Y = \bigcup\limits_{l\in O'} Y_l$ for some $O' \subset \{1, \cdots, n'\}$ and $\sum\limits_{k\in O} [\chi_{(X_k)_\varepsilon}(x)] = \sum\limits_{l\in O'} [\chi_{(Y_l)_\varepsilon}(y)]$ in $K_0(A)$, then*

$$\sum_{k\in O}\sum_{\lambda_i\in X_k} [u^* p_i u] = \sum_{l\in O'}\sum_{\eta_j\in Y_l} [q_j] \tag{4.60}$$

in $K_0(A_2)$.

Proof Let $M > 0$ such that $\|x\|, \|y\| \le M$ and $\varepsilon < M$. Denote by $f \in C[-2M, 2M]$ the function that $f(t) = 1$ if $|t| \le \frac{\varepsilon}{8}$, $f(t) = 0$ if $|t| \ge \frac{\varepsilon}{4}$ and $f(t)$ is linear in $\big[-\frac{\varepsilon}{4}, -\frac{\varepsilon}{8}\big]$ and $\big[\frac{\varepsilon}{8}, \frac{\varepsilon}{4}\big]$. Let $a \in A$ be a self-adjoint element with $\|a\| \le M$ and $\lambda \in sp(a)$. Let $a' = a - \lambda$. Then $0 \in sp(a')$. Since $TR(A) \le 1$, there is a non-zero projection $p' \in \mathrm{Her}(f(a'))$. Since $f(a')f_\varepsilon(a')a' = 0$, $p' f_\varepsilon(a')a' = 0$. Set $b = f_\varepsilon(a')a' + \lambda(1 - p')$. Then

$$\|(b + \lambda p') - a\| < \varepsilon \quad \text{and} \quad b \cdot \lambda p' = 0. \tag{4.61}$$

The proof implies that there are two sets of mutually orthogonal non-zero projections $\{p'_1, \cdots, p'_m\}$ and $\{q'_1, \cdots, q'_n\}$ and two self-adjoint elements $x'_1 \in \big(1 - \sum\limits_{i=1}^m p'_i\big)A\big(1 - \sum\limits_{i=1}^m p'_i\big), y'_1 \in \big(1 - \sum\limits_{i=1}^m q'_i\big)A\big(1 - \sum\limits_{i=1}^m q'_i\big)$ such that

$$\Big\|x - \Big(x'_1 + \sum_{i=1}^m \lambda_i p'_i\Big)\Big\| < \varepsilon \quad \text{and} \quad \Big\|y - \Big(y'_1 + \sum_{j=1}^n \eta_j q'_j\Big)\Big\| < \varepsilon. \tag{4.62}$$

By [6, Lemma 3.5.7], there is a non-zero projection $e_1 \in A$ such that $(mn+1)[e_1] \leq [e], [p_i']$ and $[q_j']$ in $K_0(A)$ for all i, j. Suppose $R = \{(\lambda_i, \eta_j) : \mathrm{dist}(\lambda_i, \eta_j) < d+\varepsilon\}$. Therefore, R satisfies the condition of Lemma 4.8. It follows from [7, Theorem 6.11], A has Riesz interpolation property. So by Lemma 4.8, there are $c_{ij} \in \mathbb{Z}_+$ such that

(I) $a_i = \sum\limits_{j=1}^{n} c_{ij} > 0$, $b_j = \sum\limits_{i=1}^{m} c_{ij} > 0$ and $\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n} c_{ij} \leq mn$ for all i, j,

(II) $c_{ij} = 0$ unless $(i, j) \in R$,

(III) suppose $O \subset \{1, \cdots, m'\}$, $X_O = \bigcup\limits_{k\in O} X_k$, $O' = \{l : 1 \leq l \leq n', R_{X_O} \cap Y_l \neq \emptyset\}$, if $\sum\limits_{k\in O} [\chi_{(X_k)_\varepsilon}(x)] = \sum\limits_{l\in O'} [\chi_{(Y_l)_\varepsilon}(y)]$ in $K_0(A)$, then $\sum\limits_{k\in O} \sum\limits_{\lambda_i\in X_k} a_i = \sum\limits_{l\in O'} \sum\limits_{\eta_j\in Y_l} b_j$.

There exist two sets $\{p_1, \cdots, p_m\}$ and $\{q_1, \cdots, q_n\}$ of non-zero mutually orthogonal projections such that $p_i \leq p_i'$ and $q_j \leq q_j'$ for all i, j, $[p_i] = \sum\limits_{j=1}^{n} c_{ij}[e_1]$ and $[q_j] = \sum\limits_{i=1}^{m} c_{ij}[e_1]$. Then there exists $u \in A$ such that $u^* \sum\limits_{i=1}^{m} p_i u = \sum\limits_{j=1}^{n} q_j$. Let $x_1 = u^*\big(x_1' + \sum\limits_{i=1}^{m} \lambda_i(p_i' - p_i)\big)u$ and $y_1 = y_1' + \sum\limits_{j=1}^{n} \eta_j(q_j' - q_j)$. It is obvious that these projections p_i, q_j ($i = 1, \cdots, m$ and $j = 1, \cdots, n$), self-adjoint elements x_1, y_1 and unitary u satisfy (1)–(4).

To see (5), let O_1 be a open set. Suppose $O_1' = O_1 \cap \{\lambda_1, \cdots, \lambda_m\} = \{\lambda_{i_1}, \cdots, \lambda_{i_k}\}$. Let $O_2 = R_{O_1'}$. Then

$$
\begin{aligned}
\Big[\chi_{O_1}^{A_2}\Big(u^* \sum_{i=1}^{m} \lambda_i p_i u\Big)\Big] &= \sum_{l=1}^{k} \sum_{j=1}^{n} c_{i_l j}[e_1] \\
&= \sum_{l=1}^{k} \sum_{j\in R_{\{i_l\}}} c_{i_l j}[e_1] \\
&\leq \sum_{l=1}^{k} \sum_{j\in O_2} c_{i_l j}[e_1] \\
&\leq \sum_{j\in O_2} \sum_{i=1}^{m} c_{ij}[e_1] \\
&\leq \Big[\chi_{(O_1)_{d+\varepsilon}}^{A_2}\Big(\sum_{j=1}^{n} \eta_j q_j\Big)\Big]
\end{aligned}
\tag{4.63}
$$

in $K_0(A)$. Suppose $v \in A$ such that $vv^* = \chi_{O_1}^{A_2}\big(u^* \sum\limits_{i=1}^{m} \lambda_i p_i u\big)$ and $v^*v \leq \chi_{(O_1)_{d+\varepsilon}}^{A_2}\big(\sum\limits_{j=1}^{n} \eta_j q_j\big)$. Since $v = vv^*v = v\big(\sum\limits_{j=1}^{n} q_j\big) = \big(\sum\limits_{j=1}^{n} q_j\big)v$, $v = \big(\sum\limits_{j=1}^{n} q_j\big)v\big(\sum\limits_{j=1}^{n} q_j\big) \in A_2$. In other words, $\big[\chi_{O_1}^{A_2}\big(u^* \sum\limits_{i=1}^{m} \lambda_i p_i u\big)\big] \leq \big[\chi_{(O_1)_{d+\varepsilon}}^{A_2}\big(\sum\limits_{j=1}^{n} \eta_j q_j\big)\big]$ in $K_0(A_2)$. By the arbitrariness of O_1,

$$
D_c\Big(u^* \sum_{i=1}^{m} \lambda_i p_i u, \sum_{j=1}^{n} \eta_j q_j\Big) < d + \varepsilon \tag{4.64}
$$

in A_2.

To see (6), suppose $O \subset \{1, \cdots, m'\}$, $X_O = \bigcup_{k\in O} X_k$, $O' = \{l : 1 \le l \le n', R_{X_O} \cap Y_l \ne \emptyset\}$, if $\sum_{k\in O} [\chi_{(X_k)_\varepsilon}(x)] = \sum_{l\in O'} [\chi_{(Y_l)_\varepsilon}(y)]$ in $K_0(A)$, it follows from (III) that $\sum_{k\in O} \sum_{\lambda_i\in X_k} a_i = \sum_{l\in O'} \sum_{\eta_j\in Y_l} b_j$. Therefore, $\sum_{k\in O} \sum_{\lambda_i\in X_k} [u^*p_iu] = \sum_{l\in O'} \sum_{\eta_j\in Y_l} [q_j]$ in $K_0(A)$, and furthermore in $K_0(A_2)$. It shows that this lemma holds.

Lemma 4.10 *Let A be a unital simple C^*-algebra with $TR(A) \le 1$. Let $x, y \in A$ be two self-adjoint elements and $e \in A$ be a projection. Suppose $\varepsilon > 0$, $d > D_c(x, y)$. If $X = \{\lambda_1, \cdots, \lambda_m\} \subset sp(x)$ is a ε-dense subset of $sp(x)$ and $Y = \{\eta_1, \cdots, \eta_n\} \subset sp(y)$ is a ε-dense subset of $sp(y)$, then there exist self-adjoint elements $x_1, y_1 \in A$, two sets of mutually orthogonal non-zero projections $\{p_1, \cdots, p_m\}$ and $\{q_1, \cdots, q_n\}$ and a unitary $u \in A$ satisfying the following:*

(1) $u^* \sum_{i=1}^{m} p_iu = \sum_{j=1}^{n} q_j$ *and* $\big[\sum_{j=1}^{n} q_j\big] \le [e]$ *in* $K_0(A)$,

(2) $x_1, y_1 \in A_1$, *where* $A_1 = \big(1 - \sum_{j=1}^{n} q_j\big)A\big(1 - \sum_{j=1}^{n} q_j\big)$,

(3) $\big\|u^*xu - \big(x_1 + u^* \sum_{i=1}^{m} \lambda_ip_iu\big)\big\| < 2\varepsilon$ *and* $\big\|y - \big(y_1 + \sum_{j=1}^{n} \eta_jq_j\big)\big\| < 2\varepsilon$,

(4) $sp_A(x) \subset (sp_{A_1}(x_1))_\varepsilon$ *and* $sp_A(y) \subset (sp_{A_1}(y_1))_\varepsilon$,

(5) $D_c(x_1, y_1) < d + 16\varepsilon$ *in* A_1,

(6) $D_c\big(u^* \sum_{i=1}^{m} \lambda_ip_iu, \sum_{j=1}^{n} \eta_jq_j\big) < d + 3\varepsilon$ *in* $\big(\sum_{j=1}^{n} q_j\big)A\big(\sum_{j=1}^{n} q_j\big)$.

Proof We may assume that $\varepsilon < d - D_c(x, y)$. Let Ω be the closure of $(sp(x))_{2\varepsilon} \cup (sp(y))_{2\varepsilon}$. Let $O_1, \cdots, O_k \subset \Omega$ be open sets in Lemma 4.4. Let $f_{(O_i)_{2\varepsilon}}(t)$ be those functions in Lemma 4.4, $i = 1, \cdots, k$.

Suppose $X_1, \cdots, X_{m'} \subset X$ and $Y_1, \cdots, Y_{n'} \subset Y$ are these subsets as in Lemma 4.8. We take two condition for open set O:

(I) $O \cap X = \emptyset$,

(II) $\overline{O} \cap sp(x) = O_{2\varepsilon} \cap sp(x)$, $\overline{O_d} \cap sp(y) = O_{d+2\varepsilon} \cap sp(y)$ and $[\chi_{(O)_\varepsilon}(x)] = [\chi_{O_{d+\varepsilon}}(y)]$ in $K_0(A)$.

Without loss of generality, we may assume that $(O_i)_{6\varepsilon}$, $i = 1, \cdots, l$ don't satisfy condition (I) and (II), and $(O_i)_{6\varepsilon}$, $i = l + 1, \cdots, k$ satisfy condition (I) or (II).

In the first step, we consider the case that $i \le l$.

Put

$$\sigma_i = \inf\{d_\tau(f_{(O_i)_{d+10\varepsilon}}(y)) - d_\tau(f_{(O_i)_{6\varepsilon}}(x)) : \tau \in T(A)\} \tag{4.65}$$

for $i = 1, \cdots, l$ and

$$\sigma = \min\{\sigma_1, \cdots, \sigma_l\}. \tag{4.66}$$

We claim that $\sigma > 0$.

To show the claim, we consider $i \le l$ into three cases. In the first case, we consider the case that $\overline{(O_i)_{6\varepsilon}} \cap sp(x) \ne (O_i)_{8\varepsilon} \cap sp(x)$. We choose $f_{(O_i)_{9\varepsilon}}(t)$ such that $f_{(O_i)_{9\varepsilon}}(t) = 1$ when $t \in (O_i)_{8\varepsilon}$, $i = 1, \cdots, l$. We can choose $g \in C(\Omega)$ as a non-zero function such that $0 \le g \le 1$ and the support of g is contained in $(O_i)_{8\varepsilon} \backslash \overline{(O_i)_{6\varepsilon}}$ and $g(x) \ne 0$. Then $\tau(f_{(O_i)_{9\varepsilon}}(x)) - d_\tau(f_{(O_i)_{6\varepsilon}}(x)) \ge \tau(g(x)) > 0$ since A is simple. Note that $D_T(x, y) \le D_c(x, y) < d$. So

$$d_\tau(f_{(O_i)_{d+10\varepsilon}}(y)) - d_\tau(f_{(O_i)_{6\varepsilon}}(x)) \ge \tau(f_{(O_i)_{9\varepsilon}}(x)) - d_\tau(f_{(O_i)_{6\varepsilon}}(x)) \ge \tau(g(x)) \tag{4.67}$$

for each $\tau \in T(A)$. Since $T(A)$ is weak*-compact, $\sigma_i > 0$.

In the second case, if $\overline{(O_i)_{d+6\varepsilon}} \cap sp(y) \neq (O_i)_{d+8\varepsilon} \cap sp(y)$. We choose $f_{(O_i)_{d+9\varepsilon}}(t)$ such that $f_{(O_i)_{d+9\varepsilon}}(t) = 1$ when $t \in (O_i)_{d+8\varepsilon}$, $i = 1, \cdots, l$. Therefore,

$$d_\tau(f_{(O_i)_{d+10\varepsilon}}(y)) - d_\tau(f_{(O_i)_{6\varepsilon}}(x)) \geq \tau(f_{(O_i)_{d+9\varepsilon}}(y)) - d_\tau(f_{(O_i)_{d+6\varepsilon}}(y)) > 0 \tag{4.68}$$

for each $\tau \in T(A)$. Such as (4.67), $\sigma_i > 0$.

In the last case, $\overline{(O_i)_{6\varepsilon}} \cap sp(x) = (O_i)_{8\varepsilon} \cap sp(x)$, $\overline{(O_i)_{d+6\varepsilon}} \cap sp(y) = (O_i)_{d+8\varepsilon} \cap sp(y)$ and $[\chi_{(O_i)_{7\varepsilon}}(x)] \neq [\chi_{(O_i)_{d+7\varepsilon}}(y)]$ in $K_0(A)$. However, $[\chi_{(O_i)_{7\varepsilon}}(x)] \leq [\chi_{(O_i)_{d+7\varepsilon}}(y)]$. This implies that

$$d_\tau(f_{(O_i)_{d+10\varepsilon}}(y)) - d_\tau(f_{(O_i)_{6\varepsilon}}(x)) \geq \tau(\chi_{(O_i)_{d+7\varepsilon}}(y)) - \tau(\chi_{(O_i)_{7\varepsilon}}(x)) > 0 \tag{4.69}$$

for each $\tau \in T(A)$. Therefore, $\sigma_i > 0$.

As above, we obtain $\sigma > 0$. So the claim holds.

Choose

$$f_{(O_i)_{4\varepsilon}} \tag{4.70}$$

such that $f_{(O_i)_{4\varepsilon}}(t) = 1$ when $t \in (O_i)_{3\varepsilon}$, $i = 1, \cdots, l$. Choose $f_{(O_i)_{d+12\varepsilon}}$ such that $f_{(O_i)_{d+12\varepsilon}}(t) = 1$ when $t \in (O_i)_{d+11\varepsilon}$, $i = 1, \cdots, l$. According to the definition of d_τ, we have $\tau(f_{(O_i)_{d+12\varepsilon}}(y)) \geq d_\tau(f_{(O_i)_{d+10\varepsilon}}(y))$ and $d_\tau(f_{(O_i)_{6\varepsilon}}(x)) \geq \tau(f_{(O_i)_{4\varepsilon}}(x))$ for all $i \leq l$ and $\tau \in T(A)$. Therefore,

$$\begin{aligned} &\inf\{\tau(f_{(O_i)_{d+12\varepsilon}}(y)) - \tau(f_{(O_i)_{4\varepsilon}}(x)) : i = 1, \cdots, l, \tau \in T(A)\} \\ \geq &\inf\{d_\tau(f_{(O_i)_{d+10\varepsilon}}(y)) - d_\tau(f_{(O_i)_{6\varepsilon}}(x)) : i = 1, \cdots, l, \tau \in T(A)\} \\ = &\,\sigma. \end{aligned} \tag{4.71}$$

In the second step, we consider $i \geq l + 1$.

Let $\sigma_i' = \sigma_i'' = \varepsilon$ if $(O_i)_{6\varepsilon}$ satisfy condition (I). Otherwise let $\sigma_i' = \inf\{\mathrm{dist}(\lambda, (O_i)_{7\varepsilon}) : \lambda \in sp(x) \backslash (O_i)_{7\varepsilon}\}$ and $\sigma_i'' = \inf\{\mathrm{dist}(\eta, (O_i)_{d+7\varepsilon}) : \eta \in sp(y) \backslash (O_i)_{d+7\varepsilon}\}$. Then $\sigma_i', \sigma_i'' \geq \varepsilon$ for $i = l + 1, \cdots, k$ according to condition (II). Let

$$\sigma' = \min\{\sigma_i', \sigma_i'' : i = l + 1, \cdots, k\}. \tag{4.72}$$

Then $\sigma' \geq \varepsilon$. We choose $f_{(O_i)_{\frac{15\varepsilon}{2}}}$ such that $f_{(O_i)_{\frac{15\varepsilon}{2}}}(t) = 1$ when $t \in (O_i)_{7\varepsilon}$ and $f_{(O_i)_{d+\frac{15\varepsilon}{2}}}$ such that $f_{(O_i)_{d+\frac{15\varepsilon}{2}}}(t) = 1$ when $t \in (O_i)_{d+7\varepsilon}$, $i = l + 1, \cdots, k$.

In the third step, we begin to show the lemma.

Let N be an integer number and $e' \in A$ be a projection such that

$$\frac{1}{N} < \frac{\sigma}{8}, \quad [e'] \leq [e] \quad \text{and} \quad N[e'] \leq [1] \tag{4.73}$$

in $K_0(A)$. It follows from [7, 2.5.11] that, there is a $\varepsilon_1 > 0$ such that for any normal elements a, b with $sp(a), sp(b) \subset \Omega$, if $\|a - b\| < \varepsilon_1$, then

$$\|f_{(O_i)_{4\varepsilon}}(a) - f_{(O_i)_{4\varepsilon}}(b)\| < \frac{\sigma}{8} \quad \text{and} \quad \|f_{(O_i)_{d+12\varepsilon}}(a) - f_{(O_i)_{d+12\varepsilon}}(b)\| < \frac{\sigma}{8} \tag{4.74}$$

for $i = 1, \cdots, l$ and

$$\|f_{(O_i)_{\frac{15\varepsilon}{2}}}(a) - f_{(O_i)_{\frac{15\varepsilon}{2}}}(b)\| < 1 \tag{4.75}$$

for $i = l+1, \cdots, k$. We may assume

$$\varepsilon_1 < \min\Big\{d - D_c(x,y), \frac{\varepsilon}{5}, \delta\Big\}, \tag{4.76}$$

where δ satisfies Lemma 2.3. Suppose $X' = \{\lambda_1', \cdots, \lambda_{m_1}'\} \subset sp(x)$ is a ε_1-dense subset of $sp(x)$ and $Y' = \{\eta_1', \cdots, \eta_{n_1}'\} \subset sp(y)$ is a ε_1-dense subset of $sp(y)$. Suppose $X_1', \cdots, X_{m_1'}' \subset X'$ and $Y_1', \cdots, Y_{n_1'}' \subset Y'$ are these sets corresponding to Lemma 4.9. We apply Lemma 4.9 for e' (replace of e in Lemma 4.9), ε_1 (replace of ε in Lemma 4.9) and $d - \varepsilon_1$ (replace of d in Lemma 4.9). Then there exist self-adjoint elements $x_1, y_1 \in A$, two sets of mutually orthogonal non-zero projections $\{p_1', \cdots, p_{m_1}'\}$ and $\{q_1', \cdots, q_{n_1}'\}$ and a unitary $u \in A$ such that

$$u^* \sum_{i=1}^{m_1} p_i' u = \sum_{j=1}^{n_1} q_j' \quad \text{and} \quad \Big[\sum_{i=1}^{m_1} p_i'\Big] = \Big[\sum_{j=1}^{n_1} q_j'\Big] \le [e'] \le [e] \quad \text{in } K_0(A), \tag{4.77}$$

$$x_1, y_1 \in A_1, \quad \text{where } A_1 = \Big(1 - \sum_{j=1}^{n_1} q_i'\Big) A \Big(1 - \sum_{j=1}^{n_1} q_i'\Big), \tag{4.78}$$

$$sp_A(x) \subset (sp_{A_1}(x_1))_{\varepsilon_1} \quad \text{and} \quad sp_A(y) \subset (sp_{A_1}(y_1))_{\varepsilon_1}, \tag{4.79}$$

$$\Big\| u^* x u - \Big(x_1 + u^* \sum_{i=1}^{m_1} \lambda_i' p_i' u\Big)\Big\| < \varepsilon_1 \quad \text{and} \quad \Big\| y - \Big(y_1 + \sum_{j=1}^{n_1} \eta_j' q_j'\Big)\Big\| < \varepsilon_1, \tag{4.80}$$

$$D_c\Big(u^* \sum_{i'=1}^{m_1} \lambda_{i'}' p_i' u\Big), \sum_{j'=1}^{n_1} \eta_j' q_j'\Big) < d \quad \text{in } A_2, \quad \text{where } A_2 = \Big(\sum_{j=1}^{n_1} q_j'\Big) A \Big(\sum_{j=1}^{n_1} q_j'\Big), \tag{4.81}$$

and suppose $I \subset \{1, \cdots, m_1'\}$, $X_I' = \bigcup_{i' \in I} X_{i'}'$, if $(X_I')_d \cap Y' = \bigcup_{j' \in J} Y_{j'}'$ for some $J \subset \{1, \cdots, n_1'\}$ and $\sum_{i' \in I} [\chi_{(X_{i'}')_{\varepsilon_1}}(x)] = \sum_{j' \in J} [\chi_{(Y_{j'}')_{\varepsilon_1}}(y)]$ in $K_0(A)$, then

$$\sum_{i' \in I} \sum_{\lambda_i \in X_{i'}'} [u^* p_i' u] = \sum_{j' \in J} \sum_{\eta_j \in Y_{j'}'} [q_j'] \tag{4.82}$$

in $K_0(A_2)$.

There are two mutually disjoint index sets $\{B_1, \cdots, B_m\}$ and $\{C_1, \cdots, C_n\}$ such that $\bigcup_{i=1}^{m} B_i = \{1, \cdots, m_1\}$, $\bigcup_{j=1}^{n} C_j = \{1, \cdots, n_1\}$, $\text{dist}(\lambda_{i'}', \lambda_i) < \varepsilon_1$ for each $i' \in B_i$ and $\text{dist}(\eta_{j'}', \eta_j) < \varepsilon_1$ for each $j' \in C_j$, $i = 1, \cdots, m$ and $j = 1, \cdots, n$. Let $p_i = \sum_{i' \in B_i} p_{i'}'$ and $q_j = \sum_{j' \in C_j} q_{j'}'$. Let $x_2 = u^* \sum_{i=1}^{m} \lambda_i p_i u$, $y_2 = \sum_{j=1}^{n} \eta_j q_j$. Then

$$\Big\| x_2 - \sum_{i=1}^{m_1} \lambda_i' p_i' \Big\| < \varepsilon_1 \quad \text{and} \quad \Big\| y_2 - \sum_{j=1}^{n_1} \eta_j' q_j' \Big\| < \varepsilon_1. \tag{4.83}$$

It is immediately that (1)–(4) hold. It follows from Lemma 2.3,

$$D_c\Big(u^* x_2 u, u^* \sum_{i=1}^{m_1} \lambda_i p_i' u\Big) \le \varepsilon \quad \text{and} \quad D_c\Big(y_2, \sum_{j=1}^{n_1} \eta_j q_j'\Big) \le \varepsilon \tag{4.84}$$

in A_2. Therefore (6) holds. It remains to show that (5) holds.

Note that

$$sp_{A_1}(x_1) \subset sp_A\Big(x_1 + u^* \sum_{i=1}^{m_1} \lambda_i' p_i' u\Big) \subset (sp_A(x))_{\varepsilon_1} \subset \Omega. \tag{4.85}$$

In the same way,

$$sp_{A_1}(y_1) \subset (sp_A(y))_{\varepsilon_1} \subset \Omega. \tag{4.86}$$

It follows from (4.74) and (4.80),

$$\Big\| f^A_{(O_i)_{4\varepsilon}}(u^*xu) - \Big(f^{A_1}_{(O_i)_{4\varepsilon}}(x_1) + f^{A_2}_{(O_i)_{4\varepsilon}}\Big(u^* \sum_{i'=1}^{m_1} \lambda_{i'}' p_i' u\Big)\Big)\Big\| \le \frac{\sigma}{8}, \tag{4.87}$$

$$\Big\| f^A_{(O_i)_{d+12\varepsilon}}(y) - \Big(f^{A_1}_{(O_i)_{d+12\varepsilon}}(y_1) + f^{A_2}_{(O_i)_{d+12\varepsilon}}\Big(\sum_{j'=1}^{n_1} \eta_j' q_j'\Big)\Big)\Big\| \le \frac{\sigma}{8} \tag{4.88}$$

for $i = 1, \cdots, l$. Recall that $\frac{1}{N} < \frac{\sigma}{8}$, then

$$\tau\Big(f^{A_2}_{(O_i)_{4\varepsilon}}\Big(u^* \sum_{i'=1}^{m_1} \lambda_{i'}' p_i' u\Big)\Big) \le \tau(e') \le \frac{\sigma}{8}, \tag{4.89}$$

$$\tau\Big(f^{A_2}_{(O_i)_{d+12\varepsilon}}\Big(\sum_{j'=1}^{n_1} \eta_j' q_j'\Big)\Big) \le \tau(e') \le \frac{\sigma}{8}. \tag{4.90}$$

Therefore, by (4.71) and (4.87)–(4.90),

$$\begin{aligned}
&\tau(f^{A_1}_{(O_i)_{d+12\varepsilon}}(y_1)) - \tau(f^{A_1}_{(O_i)_{4\varepsilon}}(x_1)) \\
\ge{}& \tau\Big(f^{A_1}_{(O_i)_{d+12\varepsilon}}(y_1) + f^{A_2}_{(O_i)_{d+12\varepsilon}}\Big(\sum_{j'=1}^{n_1} \eta_j' q_j'\Big)\Big) \\
&- \tau\Big(f^{A_1}_{(O_i)_{4\varepsilon}}(x_1) - f^{A_2}_{(O_i)_{4\varepsilon}}\Big(u^* \sum_{i'=1}^{m_1} \lambda_{i'}' p_i' u\Big)\Big) - \frac{\sigma}{4} \\
\ge{}& \tau(f^{A}_{(O_i)_{d+12\varepsilon}}(y)) - \tau(f^{A}_{(O_i)_{4\varepsilon}}(x)) - \frac{\sigma}{2} \\
\ge{}& \frac{\sigma}{2}
\end{aligned} \tag{4.91}$$

for all $i \le l$ and $\tau \in T(A)$. Recall that (4.70), then

$$\tau(f^{A_1}_{(O_i)_{d+12\varepsilon}}(y_1)) \le d_\tau(f^{A_1}_{(O_i)_{d+14\varepsilon}}(y_1)) \quad \text{and} \quad d_\tau(f^{A_1}_{(O_i)_{2\varepsilon}}(x_1) \le \tau(f^{A_1}_{(O_i)_{4\varepsilon}}(x_1)). \tag{4.92}$$

Then

$$\begin{aligned}
&\inf\{d_\tau(f^{A_1}_{(O_i)_{d+14\varepsilon}}(y_1)) - d_\tau(f^{A_1}_{(O_i)_{2\varepsilon}}(x_1)) : i = 1, \cdots, l, \tau \in T(A)\} \\
\ge{}& \inf\{\tau(f_{(O_i)_{d+12\varepsilon}}(y_1)) - \tau(f_{(O_i)_{4\varepsilon}}(x_1)) : i = 1, \cdots, l, \tau \in T(A)\} \\
\ge{}& \frac{\sigma}{2}.
\end{aligned} \tag{4.93}$$

Furthermore, it follows from Lemma 2.13, we may write by τ_1 the tracial extension of $\tau \in T(A_1)$. Suppose $\tau_2 = \frac{1}{\|\tau_1\|}\tau_1$. Note that $\tau_2 \in T(A)$ and

$$\tau_1(e) \le \frac{1}{N}\tau_1(1) = \frac{1}{N}\tau_1\Big(\sum_{j=1}^{n} q_j\Big) + \frac{1}{N}\tau_1\Big(1 - \sum_{j=1}^{n} q_j\Big) \le \frac{1}{N}\tau_1(e) + \frac{1}{N}. \tag{4.94}$$

Then $\tau_1(e) \le \frac{1}{N-1}$. Moreover, $1 \le \|\tau_1\| = \tau_1(1) = 1 + \tau_1\big(\sum_{j=1}^{n} q_j\big) \le \frac{N}{N-1}$. So, for any positive $a \in A_1$ with $\|a\| \le 1$,

$$0 \le \tau(a) - \tau_2(a) \le \tau(a) - \frac{N-1}{N}\tau(a) = \frac{1}{N}\tau(a) \le \frac{1}{N} \le \frac{\sigma}{8}. \tag{4.95}$$

This implies that $|d_\tau(a) - d_{\tau_2}(a)| \le \frac{\sigma}{8}$ for any positive element $a \in A_1$ with $\|a\| \le 1$. It follows that

$$\inf\{d_\tau(f^{A_1}_{(O_i)_{d+14\varepsilon}}(y_1)) - d_\tau(f^{A_1}_{(O_i)_{2\varepsilon}}(x_1)) : i = 1, \cdots, l, \tau \in T(A_1)\} \ge \frac{\sigma}{4} > 0. \tag{4.96}$$

This implies that $f^{A_1}_{(O_i)_{2\varepsilon}}(x_1) \lesssim f^{A_1}_{(O_i)_{d+14\varepsilon}}(y_1)$ in A_1 for all $i \le l$, since A has strict comparison for positive elements.

In the following, we will prove that $f^{A_1}_{(O_i)_{2\varepsilon}}(x_1) \lesssim f^{A_1}_{(O_i)_{d+14\varepsilon}}(y_1)$ in A_1 for all $i > l$.

For O_{i_0}, where $i_0 \ge l+1$, if $(O_{i_0})_{6\varepsilon} \cap X = \emptyset$, then $f_{(O_{i_0})_{2\varepsilon}}(x_1) = 0 \lesssim f_{(O_{i_0})_{d+14\varepsilon}}(y_1)$. Otherwise, $\overline{(O_{i_0})_{6\varepsilon}} \cap sp(x) = (O_{i_0})_{8\varepsilon} \cap sp(x)$, $\overline{(O_{i_0})_{d+6\varepsilon}} \cap sp(y) = (O_{i_0})_{d+8\varepsilon} \cap sp(y)$ and $[\chi_{(O_{i_0})_{7\varepsilon}}(x)] = [\chi_{(O_{i_0})_{d+7\varepsilon}}(y)]$ in $K_0(A)$. Note that $(X'_i)_{\varepsilon_1} \cap sp(x) \subset \overline{(O_{i_0})_{6\varepsilon}}$ if $(X'_i)_{\varepsilon_1} \cap \overline{(O_{i_0})_{6\varepsilon}} \ne \emptyset$ since $(X'_i)_{\varepsilon_1} \cap sp(x')$ is ε_1-path connected and $\overline{(O_{i_0})_{6\varepsilon}} \cap sp(x) \cap (X'_i)_{\varepsilon_1} = (O_{i_0})_{8\varepsilon} \cap sp(x) \cap (X'_i)_{\varepsilon_1}$. This implies $\overline{(O_{i_0})_{6\varepsilon}} \cap sp(x) = \bigcup_{i \in I_{i_0}} (X'_i)_{\varepsilon_1} \cap sp(x)$ for some $I_{i_0} \subset \{1, \cdots m'_1\}$. In the same way, $\overline{(O_{i_0})_{d+6\varepsilon}} \cap sp(y) = \bigcup_{j \in J_{i_0}} (Y'_j)_{\varepsilon_1} \cap sp(y)$ for some $J_{i_0} \subset \{1, \cdots n'_1\}$. Note that

$$\sum_{i \in I_{i_0}} [\chi_{(X'_i)_{\varepsilon_1}}(x)] = [\chi_{(O_{i_0})_{7\varepsilon}}(x)] = [\chi_{(O_{i_0})_{d+7\varepsilon}}(y)] = \sum_{j \in J_{i_0}} [\chi_{(Y'_j)_{\varepsilon_1}}(y)] \tag{4.97}$$

and

$$\sum_{i \in I_{i_0}} \chi_{(X'_i)_{\varepsilon_1}}(x) \lesssim f_{\big(\bigcup_{i \in I_{i_0}} (X'_i)_{\varepsilon_1}\big)_{d-\varepsilon_1}}(y) \lesssim \sum_{j \in J_{i_0}} \chi_{(Y'_j)_{\varepsilon_1}}(y) \tag{4.98}$$

in $K_0(A)$. Therefore, $\big(\bigcup_{i \in I_{i_0}} X'_i\big)_d \cap sp(y) = \bigcup_{j \in J_{i_0}} (Y'_j)_{\varepsilon_1} \cap sp(y)$. Furthermore,

$$\Big(\bigcup_{i \in I_{i_0}} X'_i\Big)_d \cap Y' = \bigcup_{j \in J_{i_0}} (Y'_j)_{\varepsilon_1} \cap Y' = \bigcup_{j \in J_{i_0}} Y'_j. \tag{4.99}$$

Now, by (4.97) and (4.99), we can apply (4.82), then

$$\sum_{i \in I} \sum_{i' \in I_i} [u^* p'_i u] = \sum_{j \in J} \sum_{j' \in J_j} [q'_j]. \tag{4.100}$$

Recall that (4.72), (4.76) and (4.85)–(4.86), let $\lambda \in sp(x_1) \backslash (O_{i_0})_{7\varepsilon}$. Then there is $\lambda' \in sp(x)$ such that $|\lambda' - \lambda| < \varepsilon_1$. So $\lambda' \notin sp(x) \cap (O_{i_0})_{6\varepsilon}$,

$$\mathrm{dist}(\lambda, (O_{i_0})_{7\varepsilon} \cap sp(x_1)) \ge \mathrm{dist}(\lambda', (O_{i_0})_{7\varepsilon} \cap (sp(x))_{\varepsilon_1}) - \varepsilon_1 \ge \sigma' - 2\varepsilon_1 > \frac{\varepsilon}{2}. \tag{4.101}$$

This implies that $sp(x_1) \cap ((O_{i_0})_{\frac{15}{2}\varepsilon} \backslash (O_{i_0})_{7\varepsilon}) = \emptyset$. In the same way,

$$sp(y_1) \cap ((O_{i_0})_{d+\frac{15}{2}\varepsilon} \backslash (O_{i_0})_{d+7\varepsilon}) = \emptyset.$$

Recall that the definition of $f_{(O_{i_0})_{\frac{15\varepsilon}{2}}}$ and $f_{(O_{i_0})_{d+\frac{15\varepsilon}{2}}}$, we have

$$f_{(O_{i_0})_{\frac{15\varepsilon}{2}}}(x) = \chi_{(O_{i_0})_{7\varepsilon}}(x), \quad f^{A_1}_{(O_{i_0})_{\frac{15\varepsilon}{2}}}(x_1) = \chi^{A_1}_{(O_{i_0})_{7\varepsilon}}(x_1), \tag{4.102}$$

$$f^{A_2}_{(O_{i_0})_{\frac{15\varepsilon}{2}}}\Big(\sum_{i=1}^{m_1} u^*\lambda' p'_i u\Big) = \chi^{A_2}_{(O_{i_0})_{7\varepsilon}}\Big(\sum_{i=1}^{m_1} u^*\lambda' p'_i u\Big), \tag{4.103}$$

and

$$f_{(O_{i_0})_{d+\frac{15\varepsilon}{2}}}(y) = \chi_{(O_{i_0})_{d+7\varepsilon}}(y), \quad f^{A_1}_{(O_{i_0})_{d+\frac{15\varepsilon}{2}}}(y_1) = \chi^{A_1}_{(O_{i_0})_{d+7\varepsilon}}(y_1), \tag{4.104}$$

$$f^{A_2}_{(O_{i_0})_{d+\frac{15\varepsilon}{2}}}\Big(\sum_{j=1}^{n_1} \eta'_j q'_j\Big) = \chi^{A_2}_{(O_{i_0})_{d+7\varepsilon}}\Big(\sum_{j=1}^{n_1} \eta'_j q'_j\Big). \tag{4.105}$$

Therefore, it follows from (4.75),

$$\begin{aligned}
[\chi^{A_1}_{(O_{i_0})_{7\varepsilon}}(x_1)] + \Big[\chi^{A_2}_{(O_{i_0})_{7\varepsilon}}\Big(\sum_{i=1}^{m_1} u^*\lambda' p'_i u\Big)\Big] &= [\chi^{A}_{(O_{i_0})_{7\varepsilon}}(x)] \\
&= [\chi^{A}_{(O_{i_0})_{d+7\varepsilon}}(y)] \\
&= [\chi^{A_1}_{(O_{i_0})_{d+7\varepsilon}}(y_1)] + \Big[\chi^{A_2}_{(O_{i_0})_{d+7\varepsilon}}\Big(\sum_{j=1}^{n_1} \eta'_j q'_j\Big)\Big]
\end{aligned} \tag{4.106}$$

in $K_0(A)$. It follows from (4.100) that $[\chi^{A_1}_{(O_{i_0})_{7\varepsilon}}(x_1)] = [\chi^{A_1}_{(O_{i_0})_{d+7\varepsilon}}(y_1)]$ in $K_0(A)$ and so does in $K_0(A_1)$. Therefore,

$$f^{A_1}_{(O_{i_0})_{2\varepsilon}}(x_1) \lesssim \chi^{A_1}_{(O_{i_0})_{7\varepsilon}}(x_1) \lesssim \chi^{A_1}_{(O_{i_0})_{d+7\varepsilon}}(y_1) \lesssim f^{A_1}_{(O_{i_0})_{d+14\varepsilon}}(y_1) \tag{4.107}$$

in A_1.

In conclusion, we obtain that $f^{A_1}_{(O_i)_{2\varepsilon}}(x_1) \lesssim f^{A_1}_{(O_i)_{d+14\varepsilon}}(y_1)$ for all i. Then by Lemma 4.4, $D_c(x_1, y_1) \le d + 16\varepsilon$ in A_1.

Lemma 4.11 *Let A be a unital simple C^*-algebra with $TR(A) \le 1$. Let $x, y \in A$ be two self-adjoint elements. Then*

$$\mathrm{dist}(U(x), U(y)) \le D_c(x, y). \tag{4.108}$$

Proof Let $\varepsilon > 0$ and $d > D_c(x, y)$. Let ε' with $0 < \varepsilon' < \varepsilon$ satisfy Lemma 2.3 (replace of δ by $10\varepsilon'$ in Lemma 2.3). Suppose $\{\lambda_1, \cdots, \lambda_m\} \subset \mathbb{R}$ is a ε'-dense subset of $sp(x)$ and $\{\eta_1, \cdots, \eta_n\} \subset \mathbb{R}$ is a ε'-dense subset of $sp(y)$. Moreover, we may assume $\lambda_1 < \cdots < \lambda_m$ and $\eta_1 < \cdots < \eta_n$. It follows from Lemma 4.10 that there exist self-adjoint elements $x_1, y_1 \in A$, two sets of mutually orthogonal non-zero projections $\{p_1, \cdots, p_m\}$ and $\{q_1, \cdots, q_n\}$ and a unitary $u \in A$ satisfying the following.

Suppose $P = \sum_{j=1}^{n} q_j$, $A_1 = (1-P)A(1-P)$, $A_2 = PAP$, $x_2 = u^* \sum_{i=1}^{m} \lambda_i p_i u$ and $y_2 = \sum_{j=1}^{n} \eta_j q_j$, then

$$u^* \sum_{i=1}^{m} p_i u = \sum_{j=1}^{n} q_j, \tag{4.109}$$

$$x_1, y_1 \in A_1 \quad \text{and} \quad x_2, y_2 \in A_2, \tag{4.110}$$
$$\|u^*xu - (x_1 + x_2)\| < 2\varepsilon' \quad \text{and} \quad \|y - (y_1 + y_2)\| < 2\varepsilon', \tag{4.111}$$
$$D_c(x_1, y_1) < d + 16\varepsilon' \quad \text{in } A_1 \quad \text{and} \quad D_c(x_2, y_2) < d + 3\varepsilon' \quad \text{in } A_2. \tag{4.112}$$

We choose a non-zero projection $e \in A_1$ such that $2^{mn}[e] \leq [p_i], [q_j]$ in $K_0(A)$ for $i = 1, \cdots, m$ and $j = 1, \cdots, n$. Since $TR(A_1) \leq 1$, it follows from Lemma 4.5 that there exists a projection $P_1 \in A_1$, a C^*-subalgebra $C \in I^{(1)}$ of A_1 with $I_C = P_1$, four self-adjoint elements $x_1', y_1' \in C$, $x_2', y_2' \in (1 - P - P_1)A_1(1 - P - P_1)$ such that

$$\|x_1 - (x_1' + x_2')\| < \varepsilon' \quad \text{and} \quad \|y_1 - (y_1' + y_2')\| < \varepsilon', \tag{4.113}$$
$$D_c(x_1', y_1') \leq d + 17\varepsilon' \quad \text{in } C, \tag{4.114}$$
$$D_c(x_2', y_2') \leq d + 17\varepsilon' \quad \text{in } (1 - P - P_1)A_1(1 - P - P_1), \tag{4.115}$$
$$1 - P - P_1 \lesssim e. \tag{4.116}$$

Let $A_3 = (1 - P_1)A(1 - P_1)$. Then $x_2' + x_2, y_2' + y_2 \in A_3$. Since $1 - P - P_1 \lesssim e$ and $2^{mn}[e] \leq [p_i], [q_j]$ for all i, j, there is $v_1 \in A_3$ such that $v_1^* v_1 = 1 - P - P_1$ and $v_1 v_1^* \leq u^* p_1 u$. Suppose $0 < \varepsilon_1 < \frac{\varepsilon'}{2}$ and $P_1' = 1 - P - P_1 + v_1 v_1^*$, $A_1' = P_1' A P_1'$. We can check that $x_2' - \lambda_1(1 - P - P_1) + \varepsilon_1 v_1 + \varepsilon_1 v_1^*$ is invertible in A_1'. Let $x_{21}' = x_2' + \varepsilon_1 v_1 + \varepsilon_1 v_1^* + \lambda_1 v_1 v_1^*$. Then $x_{21}' - \lambda_1 P_1' = x_2' - \lambda_1(1 - P - P_1) + \varepsilon_1 v_1 + \varepsilon_1 v_1^*$ is invertible in A_1'. Therefore, $\lambda_1 \notin sp_{A_1'}(x_{21}')$ and $\|x_{21}' - (x_2' + \lambda_1 v_1 v_1^*)\| < \varepsilon_1$. Let $0 < \varepsilon_2 < \frac{\varepsilon_1}{2}$. Since $[P_1'] \leq 2[e]$, there is $v_2 \in A_3$ such that $v_2^* v_2 = P_1'$ and $v_2 v_2^* \leq u^* p_2 u$. Put $x_{22}' = x_{21}' + \varepsilon_2 v_2 + \varepsilon_2 v_2^* + \lambda_2 v_2 v_2^*$, $P_2' = P_1' + v_2 v_2^*$, $A_2' = P_2' A P_2'$. We also have $\lambda_2 \notin sp_{A_2'}(x_{22}')$ and $\|x_{22}' - (x_{21}' + \lambda_2 v_2 v_2^*)\| < \varepsilon_2$. Since

$$sp_{A_2'}(x_{22}') \subset (sp_{A_2'}(x_{21}' + \lambda_2 v_2 v_2^*))_{\varepsilon_2} = (sp_{A_1'}(x_{21}') \cup \{\lambda_2\})_{\varepsilon_2}, \tag{4.117}$$

if ε_2 is sufficiently small, we may assume $\lambda_1 \notin sp_{A_2'}(x_{22}')$. By induced, we may get $\varepsilon_i > 0$, $v_i \in A_3$, projections $P_i' \in A_3$, C^*-subalgebras A_i' and self-adjoint elements $x_{2i}' \in A_i'$ ($i = 1, \cdots, m$) such that $\varepsilon_{i+1} < \frac{\varepsilon_i}{2}$, $v_i^* v_i = P_i'$, $P_{i+1}' = P_i' + v_i v_i^*$, $v_i v_i^* \leq u^* p_i u$, $A_i' = P_i' A P_i'$, $\{\lambda_1, \cdots, \lambda_i\} \cap sp_{A_i'}(x_{2i}') = \emptyset$, and $\|x_{2,i+1}' - (x_{2i}' + \lambda_{i+1} v_{i+1} v_{i+1}^*)\| < \varepsilon_{i+1}$ for all i. Therefore,

$$\Big\|x_{2m}' - \Big(x_2' + \sum_{i=1}^{m} \lambda_i v_i v_i^*\Big)\Big\| < \varepsilon' \tag{4.118}$$

and $\{\lambda_1, \cdots, \lambda_m\} \cap sp_{A_m'}(x_{2m}') = \emptyset$. Note that $sp_{A_m'}\big(x_2' + \sum_{i=1}^{m} \lambda_i v_i v_i^*\big) \subset (sp_A(x))_{2\varepsilon'}$. So $sp_{A_m'}(x_{2m}') \subset (sp_A(x))_{3\varepsilon'}$. Then $\{\lambda_1, \cdots, \lambda_m\}$ is a $4\varepsilon'$-dense subset of $sp_{A_m'}(x_{2m}')$. Put $\lambda_0 = \lambda_1 - 4\varepsilon'$ and $\lambda_{m+1} = \lambda_m + 4\varepsilon'$. Let $p_i' = \chi_i(x_{2m}')$, where χ_i is a characteristic function associating to $[\lambda_i, \lambda_{i+1}]$, $i = 0, \cdots, m$. Then $p_i' = \chi_i(x_{2m}') \in A_m'$, $i = 0, \cdots, m$ are mutually orthogonal projections satisfying $\sum_{i=0}^{m} p_i' = P_m'$ and

$$\Big\|x_{2m}' - \Big(\lambda_1 p_0' + \sum_{i=1}^{m} \lambda_i p_i'\Big)\Big\| \leq \Big\|x_{2m}' - \sum_{i=0}^{m} \lambda_i p_i'\Big\| + 4\varepsilon' < 8\varepsilon'. \tag{4.119}$$

Now, we suppose

$$x_2'' = \lambda_1 p_0' + \sum_{i=1}^{m} \lambda_i p_i' + \sum_{i=1}^{m} \lambda_i (u^* p_i u - v_i v_i^*) \in A_3. \tag{4.120}$$

Then x_2'' has finitely spectrum. Recall that (4.118)–(4.120), we have

$$\|x_2'' - (x_2' + x_2)\| = \Big\|x_2'' - \Big(x_2' + \sum_{i=1}^{m} \lambda_i u^* p_i u\Big)\Big\| < 9\varepsilon'. \tag{4.121}$$

In the same way, we also have self-adjoint element $y_2'' \in A_3$ with finite spectrum such that

$$\|y_2'' - (y_2' + y_2)\| < 9\varepsilon'. \tag{4.122}$$

Since $D_c(x_2', y_2') < d + 17\varepsilon'$ in $(1 - P - P_1)A(1 - P - P_1)$ and $D_c(x_2, y_2) < d + 3\varepsilon'$ in A_2, we can check that $D_c(x_2' + x_2, y_2' + y_2) < d + 17\varepsilon'$ in A_3. It follows from Lemma 2.3 that we may assume $D_c(x_2'', y_2'') < d + 20\varepsilon$ in A_3. It follows from [7, Theorem 6.11], A_3 has Riesz interpolation property. So by Theorems 3.2 and 4.1, there are unitaries $u_1 \in C$ and $u_2 \in A_3$ such that

$$\|u_1^* x_1' u_1 - y_1'\| < d + 20\varepsilon \quad \text{and} \quad \|u_2^* x_2'' u_2 - y_2''\| < d + 20\varepsilon. \tag{4.123}$$

Let $u' = u_1 + u_2$. Then u' is a unitary of A. Note that $\|u^* x u - (x_1' + x_2'')\| < 21\varepsilon$ and $\|y - (y_1' + y_2'')\| < 21\varepsilon$. In conclusion,

$$\begin{aligned} \|u'^* u x u u' - y\| &< \|u'^*(x_1' + x_2'')u' - (y_1' + y_2'')\| + 42\varepsilon \\ &= \max\{\|u_1^* x_1' u_1 - y_1'\|, \|u_2^* x_2'' u_2 - y_2''\|\} + 42\varepsilon \\ &< d + 62\varepsilon. \end{aligned} \tag{4.124}$$

This implies that $\mathrm{dist}(U(x), U(y)) \le D_c(x, y)$.

Lemma 4.12 *Let A be a unital simple C^*-algebra with $TR(A) \le k$. Then for any $\varepsilon > 0$, any finite subset $B \subset A_+$ and any finite subset $\mathcal{F} \subset A$ with $B \subset \mathcal{F}$, there exists a finite-dimension C^*-subalgebra $C \subset A$ with $I_C = P$ such that for any $x \in \mathcal{F}$ and $b \in B$,*

(1) $\|Px - xP\| < \varepsilon$,

(2) $PxP \in_\varepsilon C$ and $\|PbP\| > \|b\| - \varepsilon$.

Proof We may assume $B = \{b_1, \cdots, b_n\}$ with $\|b_i\| \le 1$ for all i. When $n = 1$, it follows from [7, Corollary 6.4] that lemma holds.

Suppose lemma holds when $n = m$. Let $n = m + 1$. Then for $\{b_1, \cdots, b_m\}$ and $\varepsilon_1 = \frac{\varepsilon}{4}$, there exists a finite-dimension C^*-subalgebra $C \subset A$ with $I_C = P$ such that for any $x \in \mathcal{F}$ and $b_1, \cdots, b_m$,

(1) $\|Px - xP\| < \varepsilon_1$,

(2) $PxP \in_{\varepsilon_1} C$ and $\|Pb_iP\| > \|b_i\| - \varepsilon_1$, $i = 1, \cdots, m$.

If $\|Pb_{m+1}P\| > \|b_{m+1}\| - \varepsilon$, lemma holds when $n = m + 1$.

Otherwise, $\|Pb_{m+1}P\| \le \|b_{m+1}\| - \varepsilon$. Since $\|Pb_{m+1}P + (1-P)b_{m+1}(1-P) - b_{m+1}\| < 2\varepsilon_1$, $\|(1 - P)b_{m+1}(1 - P)\| > \|b_{m+1}\| - 2\varepsilon_1$. Let $\mathcal{F}' = \{(1 - P)x(1 - P) : x \in \mathcal{F}\}$. Then for $(1-P)b_{m+1}(1-P)$ and $\mathcal{F}'$, there exists a finite-dimension C^*-subalgebra $C_1 \subset (1-P)A(1-P)$ with $I_{C_1} = P_1$ such that for any $y \in \mathcal{F}'$ and b_{m+1},

(1) $\|P_1 y - yP_1\| < \varepsilon_1$,

(2) $P_1 y P_1 \in_{\varepsilon_1} C_1$ and $\|P_1(1-P)b_{m+1}(1-P)P_1\| > \|(1-P)b_{m+1}(1-P)\| - \varepsilon_1 > \|b_{m+1}\| - 3\varepsilon_1$.

Let $C_2 = C \oplus C_1$ and $P_2 = P + P_1$. Then for any $x \in \mathcal{F}$ and $b_1, \cdots, b_{m+1}$,

(1) $\|P_2x - xP_2\| < 4\varepsilon_1$,

(2) $P_2xP_2 \in_{4\varepsilon_1} C_2$ and $\|P_2b_iP_2\| \geq \|Pb_iP\| > \|b_i\| - \varepsilon_1$, $i = 1, \cdots, m$,

(3) $\|P_2b_{m+1}P_2\| \geq \|P_1(1-P)b_{m+1}(1-P)P_1\| > \|b_{m+1}\| - 3\varepsilon_1$.

This implies that lemma holds for $n = m+1$.

Therefore, lemma holds by induction.

Lemma 4.13 *Let A be a unital simple C^*-algebra with $TR(A) \leq 1$. Let $\{p_1, \cdots, p_n\}$ and $\{q_1, \cdots, q_m\}$ be two sets of mutually orthogonal projections with $\sum\limits_{i=1}^{n} p_i = \sum\limits_{j=1}^{m} q_j = 1$, and $\{\lambda_1, \cdots, \lambda_n\}$, $\{\eta_1, \cdots, \eta_m\}$ be two subsets of $\mathbb{R}$. Suppose $\lambda_1 < \cdots < \lambda_n$ and $\eta_1 < \cdots < \eta_m$. Suppose $x = \sum\limits_{i=1}^{n} \lambda_i p_i$ and $y = \sum\limits_{j=1}^{m} \eta_j q_j$. Let $d = D_c(x, y) > 0$ and $0 < \varepsilon < d$. Then there exists integer $k \leq n$ satisfying one of the following conditions*:

(1) *Let $O_k = \{\lambda_1, \cdots, \lambda_k\}$, $[\chi_{O_k}(x)] \nleq [\chi_{(O_k)_{d-\varepsilon}}(y)]$ in $K_0(A)$,*

(2) *Let $O'_k = \{\lambda_k, \cdots, \lambda_n\}$, $[\chi_{O'_k}(x)] \nleq [\chi_{(O'_k)_{d-\varepsilon}}(y)]$ in $K_0(A)$.*

Proof Let $\Lambda = \{\lambda_1, \cdots, \lambda_n\}$. According to the definition of $D_c(x, y)$, there exists subset $O = \{\lambda_{i_1}, \cdots, \lambda_{i_l}\} \subset \Lambda$ such that $[\chi_O(x)] \nleq [\chi_{O_{d-\varepsilon}}(y)]$ in $K_0(A)$, where $\lambda_{i_1} < \cdots < \lambda_{i_l}$. If there exists η_{j_0} for some j_0 such that $(\eta_{j_0} - (d-\varepsilon), \eta_{j_0} + d - \varepsilon) \cap \Lambda = \emptyset$, then we may assume $\lambda_{k_0} \leq \eta_{j_0} - (d-\varepsilon) < \eta_{j_0} - (d-\varepsilon) \leq \lambda_{k_0+1}$ for some k_0. Let $O_k = \{\lambda_1, \cdots, \lambda_k\}$ and $O'_k = \{\lambda_k, \cdots, \lambda_n\}$. We claim that there exists integer $k \leq n$ such that

$$[\chi_{O_k}(x)] \nleq [\chi_{(O_k)_{d-\varepsilon}}(y)] \tag{4.125}$$

or

$$[\chi_{O'_k}(x)] \nleq [\chi_{(O'_k)_{d-\varepsilon}}(y)] \tag{4.126}$$

in $K_0(A)$. Otherwise, $[1] = [\chi_\Lambda(x)] \leq [\chi_{\Lambda_{d-\varepsilon}}(y)]$ in $K_0(A)$. But it is impossible since $\eta_{j_0} \notin \Lambda_{d-\varepsilon}$. It shows the claim. This implies that lemma holds if $(\eta_{j_0} - (d-\varepsilon), \eta_{j_0} + d - \varepsilon) \cap \Lambda = \emptyset$. So we may assume that for all j,

$$(\eta_j - (d-\varepsilon), \eta_j + d - \varepsilon) \cap \Lambda \neq \emptyset. \tag{4.127}$$

Let $E, F \subset \mathbb{R}$ and $a \in \mathbb{R}$, we write $E < F$ if $\sup E < \inf F$, and $E \leq a$ if $\sup E \leq a$. Let

$$\mathcal{A} = \{A : O \subset A \subset \Lambda, A \leq \lambda_{i_l} = \sup O, [\chi_A(x)] \nleq [\chi_{(A)_{d-\varepsilon}}(y)]\}. \tag{4.128}$$

We write $|A|$ the number of elements of set A. Then there is a set $A_0 \in \mathcal{A}$ such that

$$|A_0| = \sup\{|A| : A \in \mathcal{A}\}. \tag{4.129}$$

It is trivial to check that A_0 satisfyies one of the following cases:

(I) $A_0 = \{\lambda_1, \lambda_2, \cdots, \lambda_{i_l}\}$,

(II) there exist non-empty sets $A_1 < B_1 < \cdots < B_{l'-1} < A_{l'}$ such that $\bigcup\limits_{i=1}^{l'} A_i = A_0$ and $\bigcup\limits_{i=1}^{l'-1} B_i = \{\lambda_1, \lambda_2, \cdots, \lambda_{i_l}\} \backslash A_0$,

(III) there exist non-empty sets $B_1 < A_1 < \cdots < B_{l'} < A_{l'}$ such that $\bigcup_{i=1}^{l'} A_i = A_0$ and $\bigcup_{i=1}^{l'} B_i = \{\lambda_1, \lambda_2, \cdots, \lambda_{i_l}\} \backslash A_0$.

(I) Implies (1) of the lemma holds.

If A_0 satisfies (II), we claim

$$[\chi_{(A_0 \cup B_1)}(x)] \leq [\chi_{(A_0 \cup B_1)_{d-\varepsilon}}(y)] \tag{4.130}$$

in $K_0(A)$. Otherwise, $A_0 \cup B_1 \in \mathcal{A}$, it is contrary with (4.129). It shows the claim. Since

$$[\chi_{(A_0)}(x)] \nleq [\chi_{(A_0)_{d-\varepsilon}}(y)], \tag{4.131}$$

it follows from [8, Theorem 6.8] that there exists $\tau_0 \in T(A)$ such that

$$\tau_0(\chi_{(A_0)}(x)) \geq \tau_0(\chi_{(A_0)_{d-\varepsilon}}(y)). \tag{4.132}$$

Therefore, by (4.130) and (4.132),

$$\tau_0(\chi_{(B_1)}(x)) \leq \tau_0(\chi_{(A_0 \cup B_1)_{d-\varepsilon}}(y)) - \tau_0(\chi_{(A_0)_{d-\varepsilon}}(y)). \tag{4.133}$$

By (4.130)–(4.131),

$$[\chi_{(B_1)}(x)] \neq [\chi_{(A_0 \cup B_1)_{d-\varepsilon}}(y)] - [\chi_{(A_0)_{d-\varepsilon}}(y)] \tag{4.134}$$

in $K_0(A)$. If $[\chi_{A_1}(x)] \nleq [\chi_{(A_1)_{d-\varepsilon}}(y)]$ in $K_0(A)$, then (1) of the lemma holds. Otherwise,

$$[\chi_{A_1}(x)] \leq [\chi_{(A_1)_{d-\varepsilon}}(y)]. \tag{4.135}$$

Note that $\chi_{(A_0 \cup B_1)_{d-\varepsilon}}(y) - \chi_{(A_0)_{d-\varepsilon}}(y)$ and $\chi_{(A_1)_{d-\varepsilon}}(y)$ are mutually orthogonal projections. Then, by (4.133) and (4.135),

$$\tau_0(\chi_{(A_1 \cup B_1)}(x)) \leq \tau_0(\chi_{(A_0 \cup B_1)_{d-\varepsilon}}(y)) - \tau_0(\chi_{(A_0)_{d-\varepsilon}}(y)) + \tau_0(\chi_{(A_1)_{d-\varepsilon}}(y)). \tag{4.136}$$

Furthermore, if $[\chi_{A_1}(x)] = [\chi_{(A_1)_{d-\varepsilon}}(y)]$, then by (4.134),

$$[\chi_{(A_1 \cup B_1)}(x)] \neq [\chi_{(A_0 \cup B_1)_{d-\varepsilon}}(y)] - [\chi_{(A_0)_{d-\varepsilon}}(y)] + [\chi_{(A_1)_{d-\varepsilon}}(y)]. \tag{4.137}$$

If $[\chi_{A_1}(x)] < [\chi_{(A_1)_{d-\varepsilon}}(y)]$, by (4.133),

$$\tau_0(\chi_{(A_1 \cup B_1)}(x)) < \tau_0(\chi_{(A_0 \cup B_1)_{d-\varepsilon}}(y)) - \tau_0(\chi_{(A_0)_{d-\varepsilon}}(y)) + \tau_0(\chi_{(A_1)_{d-\varepsilon}}(y)). \tag{4.138}$$

So (4.137) also holds. Then by (4.136),

$$\begin{aligned} &\tau_0(\chi_{\Lambda \backslash (A_1 \cup B_1)}(x)) \\ &= \tau_0(1) - \tau_0(\chi_{(A_1 \cup B_1)}(x)) \\ &\geq \tau_0(1) - (\tau_0(\chi_{(A_0 \cup B_1)_{d-\varepsilon}}(y)) - \tau_0(\chi_{(A_0)_{d-\varepsilon}}(y)) + \tau_0(\chi_{(A_1)_{d-\varepsilon}}(y))) \\ &= \tau_0(\chi_{(\Lambda \backslash (A_1 \cup B_1))_{d-\varepsilon}}(y)). \end{aligned} \tag{4.139}$$

The last equation (4.139) holds by the following reason.

Suppose $\lambda_{k_1} = \inf A_2$. We consider the case that $\eta_{j_0} \in (\Lambda\backslash(A_1 \cup B_1))_{d-\varepsilon}$ for some j_0, then

$$\eta_{j_0} > \lambda_{k_1} - (d-\varepsilon). \tag{4.140}$$

If $\eta_{j_0} \in (A_1)_{d-\varepsilon}$, then $\chi_{(A_0)_{d-\varepsilon}}(y) = \chi_{(A_0\cup B_1)_{d-\varepsilon}}(y)$. This implies that $A_0 \cup B_1 \in \mathcal{A}$. It is contrary with (4.129). If

$$\eta_{j_0} \in ((A_0 \cup B_1)_{d-\varepsilon}\backslash(A_0)_{d-\varepsilon}) \cap \{\eta_1, \cdots, \eta_m\}, \tag{4.141}$$

we will find that $\eta_{j_0} \leq \lambda_{k_1} - (d-\varepsilon)$. It is contrary with (4.140). Therefore,

$$\chi_{(\Lambda\backslash(A_1\cup B_1))_{d-\varepsilon}}(y) \leq 1 - (\chi_{(A_0\cup B_1)_{d-\varepsilon}}(y) - \chi_{(A_0)_{d-\varepsilon}}(y) + \chi_{(A_1)_{d-\varepsilon}}(y)). \tag{4.142}$$

In reverse, we consider the case that $\eta_{j_0} \notin (\Lambda\backslash(A_1 \cup B_1))_{d-\varepsilon}$ for some j_0, then

$$\eta_{j_0} \leq \lambda_{k_1} - (d-\varepsilon). \tag{4.143}$$

or

$$\eta_{j_0} \geq \lambda_n + (d-\varepsilon). \tag{4.144}$$

Since (4.127), it is impossible for (4.144). Then by (4.127) and (4.143),

$$\lambda_1 - (d-\varepsilon) < \eta_{j_0} < \lambda_{k_1-1} + (d-\varepsilon), \tag{4.145}$$

where $\lambda_{k_1-1} \in B_1$. It shows that $\eta_{j_0} \in (A_1 \cup B_1)_{d-\varepsilon}$. Furthermore, if $\eta_{j_0} \notin (A_1)_{d-\varepsilon}$, then $\eta_{j_0} \in (A_0 \cup B_1)_{d-\varepsilon}\backslash(A_0)_{d-\varepsilon}$. So $\eta_{j_0} \in ((A_0 \cup B_1)_{d-\varepsilon}\backslash(A_0)_{d-\varepsilon}) \cup (A_1)_{d-\varepsilon}$. In other word,

$$\chi_{(\Lambda\backslash(A_1\cup B_1))_{d-\varepsilon}}(y) \geq 1 - (\chi_{(A_0\cup B_1)_{d-\varepsilon}}(y) - \chi_{(A_0)_{d-\varepsilon}}(y) + \chi_{(A_1)_{d-\varepsilon}}(y)). \tag{4.146}$$

By combining (4.142) and (4.146), we have

$$\chi_{(\Lambda\backslash(A_1\cup B_1))_{d-\varepsilon}}(y) = 1 - (\chi_{(A_0\cup B_1)_{d-\varepsilon}}(y) - \chi_{(A_0)_{d-\varepsilon}}(y) + \chi_{(A_1)_{d-\varepsilon}}(y)). \tag{4.147}$$

Therefore, (4.139) holds.

Similarly as (4.139), by (4.137) and (4.147), we can also check that $[\chi_{\Lambda\backslash(A_1\cup B_1)}(x)] \neq [\chi_{(\Lambda\backslash(A_1\cup B_1))_{d-\varepsilon}}(y)]$. Then by (4.139), $[\chi_{\Lambda\backslash(A_1\cup B_1)}(x)] \nleq [\chi_{(\Lambda\backslash(A_1\cup B_1))_{d-\varepsilon}}(y)]$. This implies that (2) of the lemma holds.

If A_0 satisfies (III), as (4.130), we have

$$[\chi_{(A_0\cup B_1)}(x)] \leq [\chi_{(A_0\cup B_1)_{d-\varepsilon}}(y)]. \tag{4.148}$$

As (4.133),

$$\tau_0(\chi_{B_1}(x)) \leq \tau_0(\chi_{(A_0\cup B_1)_{d-\varepsilon}}(y)) - \tau_0(\chi_{(A_0)_{d-\varepsilon}}(y)). \tag{4.149}$$

Furthermore, as (4.134),

$$[\chi_{B_1}(x)] \neq [\chi_{(A_0\cup B_1)_{d-\varepsilon}}(y)] - [\chi_{(A_0)_{d-\varepsilon}}(y)]. \tag{4.150}$$

Therefore,

$$\begin{aligned}
&\tau_0(\chi_{\Lambda\setminus B_1}(x))\\
&=\tau_0(1)-\tau_0(\chi_{B_1}(x))\\
&\geq \tau_0(1)-(\tau_0(\chi_{(A_0\cup B_1)_{d-\varepsilon}}(y))-\tau_0(\chi_{(A_0)_{d-\varepsilon}}(y)))\\
&=\tau_0(\chi_{(\Lambda\setminus B_1)_{d-\varepsilon}}(y)).
\end{aligned}\tag{4.151}$$

The last equation of (4.151) holds since $1-(\chi_{(A_0\cup B_1)_{d-\varepsilon}}(y)-\chi_{(A_0)_{d-\varepsilon}}(y))=\chi_{(\Lambda\setminus B_1)_{d-\varepsilon}}(y)$, similarly as (4.147). In the same way, we can check that $[\chi_{\Lambda\setminus B_1}(x)]\neq[\chi_{(\Lambda\setminus B_1)_{d-\varepsilon}}(y)]$. It shows (2) of the lemma holds.

Remark 4.1 Let A be a unital C^*-algebra, $p\in A$ be a non-zero projection and $x\in A$ be self-adjoint element. Suppose $sp(x)\subset[a,b]$, then we will find $sp_{pAp}(pxp)\subset[a,b]$.

Proof We may assume $\sup sp(x)=b$ and $\inf sp(x)=a$. Then $0\leq p(b-x)p\leq(b-a)p$. Therefore, $sp_{pAp}(pxp)\subset[a,b]$.

Lemma 4.14 *Let A be a unital simple C^*-algebra with $TR(A)\leq 1$. Let $x,y\in A$ be self-adjoint elements with finite spectrum. Then*

$$D_c(x,y)\leq \operatorname{dist}(U(x),U(y)).\tag{4.152}$$

Proof Let $x=\sum\limits_{i=1}^{n}\lambda_ip_i$, $y=\sum\limits_{j=1}^{m}\eta_jq_j$, where $\{\lambda_1,\cdots,\lambda_n\},\{\eta_1,\cdots,\eta_m\}\subset\mathbb{R}$, $\{p_1,\cdots,p_n\}$ and $\{q_1,\cdots,q_m\}$ be two sets of mutually orthogonal projections with $\sum\limits_{i=1}^{n}p_i=\sum\limits_{j=1}^{m}q_j=1$. Suppose $\|x\|,\|y\|<M$, $d=D_c(x,y)$. Lemma holds when $d=0$. So we may assume $d>0$. Let $0<\varepsilon<d$. Suppose $\lambda_1<\cdots<\lambda_n$ and $\eta_1<\cdots<\eta_m$. It follows from Lemma 4.13, there exists integer $k\leq n$ satisfying one of the following conditions:

(1) Let $O_k=\{\lambda_1,\cdots,\lambda_k\}$, $[\chi_{O_k}(x)]\not\leq[\chi_{(O_k)_{d-\varepsilon}}(y)]$ in $K_0(A)$.

(2) Let $O'_k=\{\lambda_k,\cdots,\lambda_n\}$, $[\chi_{O'_k}(x)]\not\leq[\chi_{(O'_k)_{d-\varepsilon}}(y)]$ in $K_0(A)$.

We consider case (1). We may assume k is the minimum integer such that case (1) holds. When $(O_k)_{d-\varepsilon}\cap\{\eta_1,\cdots,\eta_m\}=\emptyset$, let $0<\delta<\frac{\varepsilon}{2mnM}$. Since $TR(A)\leq 1$, there exist $C\subset I^{(1)}$ with $I_C=P$ and $\{p_{11},\cdots,p_{1n}\}\subset C$, $\{p_{21},\cdots,p_{2n}\}\subset(1-P)A(1-P)$, $\{q_{11},\cdots,q_{1m}\}\subset C$ and $\{q_{21},\cdots,q_{2m}\}\subset(1-P)A(1-P)$ are four sets of mutually orthogonal projections with $\sum\limits_{i=1}^{n}p_{1i}+\sum\limits_{i=1}^{n}p_{2i}=\sum\limits_{j=1}^{m}q_{1j}+\sum\limits_{j=1}^{m}q_{2j}=1$ such that for all i,j,

$$\|p_{1i}+p_{2i}-p_i\|<\delta\quad\text{and}\quad\|q_{1j}+q_{2j}-q_j\|<\delta.\tag{4.153}$$

Set $x_1=\sum\limits_{i=1}^{n}\lambda_ip_{1i}$, $x_2=\sum\limits_{i=1}^{n}\lambda_ip_{2i}$, $y_1=\sum\limits_{j=1}^{m}\eta_jq_{1j}$ and $y_2=\sum\limits_{j=1}^{m}\eta_jq_{2j}$. Since $(O_k)_{d-\varepsilon}\cap\{\eta_1,\cdots,\eta_m\}=\emptyset$, $D_c(x_1,y_1)\geq d-\varepsilon$ in C. Then

$$d-\varepsilon\leq\|x_1-y_1\|\leq\|x_1+x_2-(y_1+y_2)\|\leq\|x-y\|+\varepsilon.\tag{4.154}$$

This implies lemma holds.

Now we may assume $\eta_{k'}=\max((O_k)_{d-\varepsilon}\cap\{\eta_1,\cdots,\eta_m\})$. If there exists $k'_1<k'$ such that $\eta_{k'_1}\notin(O_k)_{d-\varepsilon}\cap\{\eta_1,\cdots,\eta_m\}$, let $\Lambda=\{\lambda_1,\cdots,\lambda_n\}$ and $\Lambda'=\{\lambda_1,\cdots,\lambda_k\}$. Then

$\mathrm{dist}(\Lambda', \{\eta_{k'_1}\}) \geq d-\varepsilon$ and $\eta_{k'_1} < \eta_{k'} < \lambda_k + (d-\varepsilon)$. Therefore, $\eta_{k'_1} \leq \lambda_k - (d-\varepsilon) < \lambda_{k+1} - (d-\varepsilon)$. This implies that $\mathrm{dist}(\Lambda, \{\eta_{k'_1}\}) \geq d-\varepsilon$. As (4.153)–(4.154), we will obtain $x_1 = \sum\limits_{i=1}^{n} \lambda_i p_{1i}, y_1 = \sum\limits_{j=1}^{m} \eta_j q_{1j} \in C$, where $C \in I^{(1)}$, such that $\|x_1 - y_1\| \leq \|x-y\| + \varepsilon$. Since $\chi_{\{\eta_{k'_1}\}}(y_1) > 0$ and $\chi_{\{\eta_{k'_1}\}_{d-\varepsilon}}(x_1) = 0$, $D_c(x_1, y_1) \geq d-\varepsilon$ in C. So $d-\varepsilon \leq \|x_1 - y_1\| < \|x-y\| + \varepsilon$. This implies lemma holds.

Therefore, furthermore, we may assume

$$(O_k)_{d-\varepsilon} \cap \{\eta_1, \cdots, \eta_m\} = \{\eta_1, \cdots, \eta_{k'}\}. \tag{4.155}$$

Since $[\chi_{O_k}(x)] \nleq [\chi_{(O_k)_{d-\varepsilon}}(y)]$, it follows from [6, Lemma 2.5.2], $\big\|\big(\sum\limits_{i=1}^{k} p_i\big)\big(1 - \sum\limits_{j=1}^{k'} q_j\big)\big\| = 1$. So, $\big\|\big(\sum\limits_{i=1}^{k} p_i\big)\big(1 - \sum\limits_{j=1}^{k'} q_j\big)\big(\sum\limits_{i=1}^{k} p_i\big)\big\| = 1$.

Set

$$\mathcal{F} = \Big\{p_i, q_j, \Big(\sum_{i=1}^{k} p_i\Big)\Big(1 - \sum_{j=1}^{k'} q_j\Big)\Big(\sum_{i=1}^{k} p_i\Big) : i = 1, \cdots, n, j = 1, \cdots, m\Big\}. \tag{4.156}$$

Let $0 < \delta_1 < \delta_2$ with $16\delta_2 + 16\sqrt{2\delta_2} < \frac{\varepsilon}{(n+m)M}$. By Lemma 4.12, there exists a finite-dimension C^*-subalgebra $C' \subset A$ with $I_{C'} = P'$ such that for any $a \in \mathcal{F}$,

(I) $\|P'a - aP'\| < \delta_1$,

(II) $P'aP' \in_{\delta_1} C'$ and $\|P'aP'\| > \|a\| - \delta_1$.

If δ_1 is sufficiently small, by [6, Lemma 2.5.6], there exist two sets of non-zero projections $\{p'_1, \cdots, p'_n\} \subset C'$ and $\{q'_1, \cdots, q'_m\} \subset C'$ such that $\|P'p_iP' - p'_i\| < \delta_2$ and $\|P'q_jP' - q'_j\| < \delta_2$ for all i, j. Furthermore, we may assume

$$\Big\|\Big(\sum_{i=1}^{k} p'_i\Big)\Big(1 - \sum_{j=1}^{k'} q'_j\Big)\Big(\sum_{i=1}^{k} p'_i\Big) - P'\Big(\sum_{i=1}^{k} p_i\Big)\Big(1 - \sum_{j=1}^{k'} q_j\Big)\Big(\sum_{i=1}^{k} p_i\Big)P'\Big\| < \delta_2. \tag{4.157}$$

Then by (II),

$$\begin{aligned}\Big\|\Big(\sum_{i=1}^{k} p'_i\Big)\Big(1 - \sum_{j=1}^{k'} q'_j\Big)\Big(\sum_{i=1}^{k} p'_i\Big)\Big\| &> \Big\|P'\Big(\sum_{i=1}^{k} p_i\Big)\Big(1 - \sum_{j=1}^{k'} q_j\Big)\Big(\sum_{i=1}^{k} p_i\Big)P'\Big\| - \delta_2 \\ &> 1 - 2\delta_2.\end{aligned} \tag{4.158}$$

Set $r = \big(\sum\limits_{i=1}^{k} p'_i\big)\big(1 - \sum\limits_{j=1}^{k'} q'_j\big)\big(\sum\limits_{i=1}^{k} p'_i\big)$. Since $r \in C'$, $sp_{C'}(r)$ is finitely. Suppose $r = \sum\limits_{i=1}^{l} \alpha_i r_i$, where $r_1, \cdots, r_l$ are mutually orthogonal non-zero projections in C'. By (4.158), without loss of generality, we may assume $\alpha_1 \in (1 - 2\delta_2, 1]$. Note that $r_1 \leq \sum\limits_{i=1}^{k} p'_i$, then

$$\alpha_1 r_1 = r_1 r r_1 = r_1\Big(1 - \sum_{j=1}^{k'} q'_j\Big)r_1. \tag{4.159}$$

Therefore, $\|r_1(1-\sum\limits_{j=1}^{k'} q'_j)r_1 - r_1\| < 2\delta_2$ and then $\|r_1(\sum\limits_{j=1}^{k'} q'_j)r_1\| < 2\delta_2$. So we will find

$$\Big\|(1-r_1)\Big(\sum_{j=1}^{k'} q'_j\Big)(1-r_1) - \Big(\sum_{j=1}^{k'} q'_j\Big)\Big\| < 2\delta_2 + 2\sqrt{2\delta_2}. \tag{4.160}$$

It follows from [6, 2.5.4] that there exists projection $q' \in (1-r_1)C'(1-r_1)$ such that $\|q' - (\sum\limits_{j=1}^{k'} q'_j)\| < 4\delta_2 + 4\sqrt{2\delta_2}$. By [6, 2.5.1], there exists unitary $u \in C'$ such that $q' = u(\sum\limits_{j=1}^{k'} q'_j)u^*$ and $\|P' - u\| < 8\delta_2 + 8\sqrt{2\delta_2}$. Note that $r_1 u(\sum\limits_{j=1}^{k'} q'_j)u^* = r_1 q' = 0$ and $r_1 u(1-\sum\limits_{j=1}^{k'} q'_j)u^* r_1 = r_1$. This implies that $r_1 \le u(1-\sum\limits_{j=1}^{k'} q'_j)u^*$. Therefore, $r_1 u(1-\sum\limits_{j=1}^{k'} q'_j)u^* r_1 = r_1 = r_1(\sum\limits_{i=1}^{k} p'_i)r_1$. Let $x' = \sum\limits_{i=1}^{k} \lambda_i p'_i$ and $y' = u(\sum\limits_{j=k'+1}^{m} \eta_j q'_j)u^*$. By Remark 4.1,

$$sp_{r_1C'r_1}(r_1x'r_1) \subset [\lambda_1, \lambda_k] \quad \text{and} \quad sp_{r_1C'r_1}(r_1y'r_1) \subset [\eta_{k'+1}, \eta_m]. \tag{4.161}$$

Note that $\eta_{k'+1} - \lambda_k \ge d - \varepsilon$ by (4.155). So

$$\chi^{r_1C'r_1}_{[\lambda_1,\lambda_k]}(r_1x'r_1) = r_1 > 0 = \chi^{r_1C'r_1}_{([\lambda_1,\lambda_k])_{d-\varepsilon}}(r_1y'r_1). \tag{4.162}$$

Therefore, $D_c(r_1x'r_1, r_1y'r_1) \ge d - \varepsilon$ in $r_1C'r_1$. Since $r_1C'r_1$ is finite dimension, $d - \varepsilon \le \|r_1x'r_1 - r_1y'r_1\|$. But

$$\begin{aligned}
&\|r_1x'r_1 - r_1y'r_1\| \\
&= \Big\|r_1\Big(\sum_{i=1}^{n} \lambda_i p'_i\Big)r_1 - r_1u\Big(\sum_{j=1}^{m} \eta_j q'_j\Big)u^* r_1\Big\| \\
&\le \Big\|r_1\Big(\sum_{i=1}^{n} \lambda_i p'_i\Big)r_1 - r_1\Big(\sum_{j=1}^{m} \eta_j q'_j\Big)r_1\Big\| + (16\delta_2 + 16\sqrt{2\delta_2})M \\
&\le \Big\|\sum_{i=1}^{n} \lambda_i p'_i - \sum_{j=1}^{m} \eta_j q'_j\Big\| + (16\delta_2 + 16\sqrt{2\delta_2})M \\
&\le \|P'(x-y)P'\| + (n+m)M\delta_2 + (16\delta_2 + 16\sqrt{2\delta_2})M \\
&\le \|x-y\| + 2\varepsilon.
\end{aligned} \tag{4.163}$$

We get $d \le \|x-y\| + 3\varepsilon$.

To consider the case (2) of the lemma, by the same argument, we will also have $d \le \|x-y\| + 3\varepsilon$. Therefore, lemma holds.

Theorem 4.3 *Let A be a unital simple C^*-algebra with $TR(A) \le 1$. Let $x, y \in A$ be self-adjoint elements. Then*

$$D_c(x,y) = \mathrm{dist}(U(x), U(y)). \tag{4.164}$$

Proof It follows from Lemma 4.11, we remain to show $D_c(x,y) \le \mathrm{dist}(U(x), U(y))$.

Let $d = D_c(x, y)$. It holds when $d = 0$.

Now, we may assume $d > 0$. Let $0 < \varepsilon < d$ and $\|x\|, \|y\| \le M$. Let $0 < \varepsilon' < \varepsilon$ satisfy Lemma 2.3 (replace of δ by $10\varepsilon'$ in Lemma 2.3). Suppose $X = \{\lambda_1, \cdots, \lambda_n\}$ is a ε'-dense subset of $sp(x)$ and $Y = \{\eta_1, \cdots, \eta_m\}$ is a ε'-dense subset of $sp(y)$. It follows from Lemma 4.9, there exist two sets of mutually orthogonal projections $\{p_1, \cdots, p_n\}$ and $\{q_1, \cdots, q_m\}$ and self-adjoint elements $x_1, y_1 \in A$ such that $x_1 \sum_{i=1}^{n} p_i = y_1 \sum_{j=1}^{m} q_j = 0$ and $\|x_1 + \sum_{i=1}^{n} \lambda_i p_i - x\| < \varepsilon'$ and $\|y_1 + \sum_{j=1}^{m} \eta_j q_j - y\| < \varepsilon'$.

Set $\mathcal{F} = \{x_1, y_1, p_i, q_j : i = 1, \cdots, n; j = 1, \cdots, m\}$ and $0 < \varepsilon_1' < \varepsilon'$. It follows from Lemma 4.12, there exists finite dimension C^*-subalgebra $C \subset A$ with $I_C = P$ such that for all $a \in \mathcal{F}$,

(1) $\|Pa - aP\| < \varepsilon_1'$,

(2) $PaP \in_{\varepsilon_1'} C$ and $\|PaP\| > \|a\| - \varepsilon_1'$.

Therefore, if ε_1' is sufficiently small, there exist four sets of mutually orthogonal projections $\{p_{k1}, \cdots, p_{kn}\}$ and $\{q_{k1}, \cdots, q_{km}\}$, $k = 1, 2$, such that for all i, j, $p_{1i}, q_{1j} \in C$, $p_{2i}, q_{2j} \in (1 - P)A(1 - P)$, and

$$\|p_{1i} + p_{2i} - p_i\| < \frac{\varepsilon'}{(n+m)M}, \tag{4.165}$$

$$\|q_{1i} + q_{2i} - q_i\| < \frac{\varepsilon'}{(n+m)M}. \tag{4.166}$$

Furthermore, there exist self-adjoint elements $x_2, y_2 \in (1 - P)A(1 - P)$ and $x_3 \in \big(P - \sum_{i=1}^{n} p_{1i}\big)C\big(P - \sum_{i=1}^{n} p_{1i}\big)$, $y_3 \in \big(P - \sum_{j=1}^{m} p_{1j}\big)C\big(P - \sum_{j=1}^{m} p_{1j}\big)$ such that $\|x_2 + x_3 - x_1\| < \varepsilon'$ and $\|y_2 + y_3 - y_1\| < \varepsilon'$. Let $x_4 = x_2 + \sum_{i=1}^{n} \lambda_i p_{2i}$, $y_4 = y_2 + \sum_{j=1}^{m} \eta_j p_{2j}$, $x_5 = x_3 + \sum_{i=1}^{n} \lambda_i p_{1i}$ and $y_5 = y_3 + \sum_{j=1}^{m} \eta_j p_{1j}$. Then $x_4, y_4 \in (1 - P)A(1 - P)$ and $x_5, y_5 \in C$. Furthermore, we have

$$\|x_4 + x_5 - x\| < 3\varepsilon' \quad \text{and} \quad \|y_4 + y_5 - y\| < 3\varepsilon'. \tag{4.167}$$

Since $\|Pp_iP\| \ge \|p_i\| - \varepsilon_1' = 1 - \varepsilon_1'$ and $\|Pq_jP\| \ge \|q_j\| - \varepsilon_1' = 1 - \varepsilon_1'$ for all i, j, p_{1i}, q_{1j} are non-zero projections for all i, j if ε_1' is sufficiently small. Therefore, $sp_C(x_5)$ is a $4\varepsilon'$-dense subset of $(sp(x))_{3\varepsilon'}$ and $sp_C(y_5)$ is a $4\varepsilon'$-dense subset of $(sp(x))_{3\varepsilon'}$.

Set $x_5 = \sum_{i=1}^{k} \alpha_i p_{5i}$ and $y_5 = \sum_{j=1}^{k'} \beta_j q_{5j}$, where $sp_C(x_5) = \{\alpha_1, \cdots, \alpha_k\}$ and $sp_C(y_5) = \{\beta_1, \cdots, \beta_{k'}\}$, $\{p_{51}, \cdots, p_{5k}\}$ and $\{p_{51}, \cdots, p_{5k'}\}$ are two sets of mutually orthogonal projections of C with $\sum_{i=1}^{k} p_{5i} = \sum_{j=1}^{k'} q_{5j} = P$. Let $e \in (1 - P)A(1 - P)$ be a projection such that $2^{mn}[e] \le [p_{5i}], [q_{5j}]$ in $K_0(A)$ for all i, j. Then there exists $C' \in I^{(1)}$ with $I_C = P'$ and $C' \subset (1-P)A(1-P)$, four self-adjoint elements $x_6, y_6 \in C'$ and $x_7, y_7 \in (1 - P - P')A(1 - P - P')$ such that

$$[1 - P - P'] \le [e], \quad \|x_6 + x_7 - x_4\| < \varepsilon' \quad \text{and} \quad \|y_6 + y_7 - y_4\| < \varepsilon'. \tag{4.168}$$

By the same argument of proof of (4.120) in Lemma 4.11, there exist self-adjoint elements $x_8, y_8 \in (1 - P')A(1 - P')$ such that $sp(x_8)$ and $sp(y_8)$ are finitely and

$$\|x_5 + x_7 - x_8\| < 4\varepsilon' \quad \text{and} \quad \|y_5 + y_7 - y_8\| < 4\varepsilon'. \tag{4.169}$$

Therefore, by (4.167)–(4.169),

$$\|x_6 + x_8 - x\| < \|x_5 + x_6 + x_7 - x\| + 4\varepsilon' < \|x_4 + x_5 - x\| + 5\varepsilon' < 8\varepsilon'. \tag{4.170}$$

We also have $\|y_6 + y_8 - y\| < 8\varepsilon'$. Let $d_1 = D_c(x_6, y_6)$ in C' and $d_2 = D_c(x_8, y_8)$ in $(1 - P')A(1 - P')$. By Lemma 2.3, Theorem 4.2 and Lemma 4.14, we get

$$\begin{aligned} d &\le \max\{d_1, d_2\} + 2\varepsilon \\ &\le \max\{\|x_6 - y_6\|, \|x_8 - y_8\|\} + 2\varepsilon \\ &= \|x_6 + x_8 - y_6 - y_8\| + 2\varepsilon \\ &\le \|x - y\| + 10\varepsilon. \end{aligned} \tag{4.171}$$

This implies that theorem holds.

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