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# Projection Body and Isoperimetric Inequalities for $s$-Concave Functions*

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**Abstract** For a positive integer $s$, the projection body of an $s$-concave function $f : \mathbb{R}^n \to [0, +\infty)$, a convex body in the $(n+s)$-dimensional Euclidean space $\mathbb{R}^{n+s}$, is introduced. Associated inequalities for $s$-concave functions, such as, the functional isoperimetric inequality, the functional Petty projection inequality and the functional Loomis-Whitney inequality are obtained.

**Keywords** Isoperimetric inequality, $s$-Concave functions, Projection body, The Petty projection inequality
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## 1 Introduction

The classical isoperimetric problem is to determine a plane figure of the largest possible area with perimeter of a given length and it was known in Ancient Greece. However, the first mathematically rigorous proof was obtained only in the 19th century by Weierstrass based on works of Bernoulli, Euler, Lagrange and others. The isoperimetric problem is characterized by the isoperimetric inequality.

The higher dimensional generalization of the classical isoperimetric problem is to determine a geometric subject of the maximum volume with boundary of a fixed surface area in the Euclidean space $\mathbb{R}^n$ $(n \geq 2)$.

Let $K$ be a compact convex set in $\mathbb{R}^n$. Then the surface area $S(K)$ and volume $V(K)$ of $K$ satisfy

$$S(K)^n \geq n^n \omega_n V(K)^{n-1}, \tag{1.1}$$

where $\omega_n$ is the volume of the unit ball in $\mathbb{R}^n$. The inequality (1.1) holds as an equality if and only if $K$ is a standard ball.

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The classical isoperimetric inequality is the oldest and most beautiful geometric inequality in mathematics that has an extraordinary variety of connections and applications to a number of areas in mathematics and physics.

The isoperimetric inequality (1.1) for sufficiently smooth domains is equivalent to the Sobolev inequality with optimal constant:

$$\int_{\mathbb{R}^n} \|\nabla f(x)\| \mathrm{d}x \geq n\omega_n^{\frac{1}{n}} \Big( \int_{\mathbb{R}^n} |f(x)|^{\frac{n}{n-1}} \mathrm{d}x \Big)^{\frac{n-1}{n}} \tag{1.2}$$

for all $f \in W^{1,1}(\mathbb{R}^n)$, the usual Sobolev space of real-valued functions of $\mathbb{R}^n$ with $L_1$ partial derivatives. Here $\|\nabla f(x)\|$ is the norm of the gradient of $f$. The extremal functions for (1.2) are the characteristic functions of balls.

The equivalence between the classical isoperimetric inequality and the Sobolev inequality has a profound effect on convex geometry. A typical example is the affine Sobolev inequality which is not only equivalent to the generalized Petty projection inequality but also stronger than the classical isoperimetric inequality (see [45]). The affine Sobolev inequality is the characterization of the affine Sobolev-type inequality. Extensions and analogues are obtained by Lutwak, Yang and Zhang [33], Cianchi, Lutwak, Yang and Zhang [15], Haberl and Schuster [24], Wang [43] and Lin [30]. By Lorentz integrals of the $L_p$ convexification of level sets, Ludwig, Xiao and Zhang obtained the sharp convex Lorentz-Sobolev inequality (see [32]). Later Fang, Xu, Zhou and Zhu obtained a sharp convex mixed Lorentz-Sobolev inequality (see [18]). A new approach to the affine Sobolev type inequalities was presented by Haddad, Jiménez and Montenegro [25].

The geometry on the set of functions is a new field and closely related to convex geometric analysis. In this paper, we will investigate the geometric properties and geometric problems on the set of $s$-concave functions.

A function $f : \mathbb{R}^n \to [0, +\infty)$ is $s$-concave (where $-\infty \leq s \leq \infty$) if

- $f$ is supported on convex set $\Omega \subset \mathbb{R}^n$;
- for every $x, y \in \Omega$ and $0 \leq \lambda \leq 1$,

$$f(\lambda x + (1-\lambda)y) \geq (\lambda f(x)^{\frac{1}{s}} + (1-\lambda) f(y)^{\frac{1}{s}})^s. \tag{1.3}$$

If $s = -\infty, 0, +\infty$, we understand (1.3) in the limit sense. The case $s = +\infty$ is important and deserve a special name log-concave.

There are surprisingly analogies between the theory of convex bodies and the theory of $s$-concave functions. The seed of this process is the Prékopa-Leindler inequality which recognized as the functional version of the Brunn-Minkowski inequality (see [29, 38–39]). More connections between the Prékopa-Leindler inequality and convex geometry are very well described in the survey paper (see [23]).

The interplay between the geometry of $s$-concave functions and the geometry of convex sets becomes increasingly important. The initial work can be found in Ball's paper. Ball [8] obtained the functional Blaschke-Santaló inequality in the even case. Artstein, Klartag and Milman [5] defined a convex body $K_s(f) \subset \mathbb{R}^n \times \mathbb{R}^s$ associated with an $s$-concave function $f$ as follows:

$$K_s(f) := \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^s : f(x) > 0, \|y\| \le f^{\frac{1}{s}}(x)\}, \tag{1.4}$$

where $s$ is a positive integer. By the known Blaschke-Santaló inequality and the convex body $K_s(f)$ (defined in (1.4)), Artstein, Klartag and Milman [5] obtained the general functional version of the Blaschke-Santaló inequality. Aside from proofs by Fradelizi and Meyer[21], there are also proofs by Lehec [27–28]. Various functional forms of the reverse Blaschke-Santaló inequality were investigated by Klartag and Milman [26], Fradelizi and Meyer [22]. Barthe, Böröczky and Fradelizi [10] obtained the stability versions of the functional Blaschke-Santaló inequality. Artstein-Avidan, Klartag, Schütt and Werner [6] provided a functional affine isoperimetric inequality for log-concave functions which can be viewed as an inverse logarithmic Sobolev inequality for entropy. Mixed integrals, quermassintegrals, the Brunn-Minkowski inequality and the Alexandrov-Fenchel inequality for $s$-concave functions were studied by Milman and Rotem [35]. For more information of connections of convex geometry and the theory of $s$-concave functions, one can refer to [1–7, 12–14, 16–20, 26–28, 30, 34, 40–41].

Inspired by ideas and works of Artstein, Klartag and Milman [5], the projection body of an $s$-concave function $f$ (where $s$ is a positive interger), a convex body in the $(n+s)$-dimensional Euclidean space, is defined. Associated inequalities for $s$-concave functions, such as, the functional isoperimetric inequality, the functional Petty projection inequality and the functional Loomis-Whitney inequality are obtained. One of our main results is the following analytic inequality (see Theorem 3.1).

**The isoperimetric inequality for $s$-concave functions.** For a positive integer $s$, if $f$ is an $s$-concave function and twice continuously differentiable, then

$$\int_{\mathbb{R}^n} f^{1-\frac{1}{s}}(1+\|\nabla f^{\frac{1}{s}}\|^2)^{\frac{1}{2}}\mathrm{d}x \ge c_{n,s}\Big(\int_{\mathbb{R}^n} f\mathrm{d}x\Big)^{1-\frac{1}{n+s}}, \tag{1.5}$$

where $c_{n,s} = \frac{(n+s)}{s}\big(\frac{\omega_{n+s}}{\omega_s}\big)^{\frac{1}{n+s}}$. Equality holds if and only if $f = (a-\|x-b\|^2)_+^{\frac{s}{2}}$ with $a>0, b \in \mathbb{R}^n$, $t \in \mathbb{R}$, $t_+ = \max\{t, 0\}$.

When $s = +\infty$, the projection body of $s$-concave functions (i.e., log-concave functions) was defined in [19]. We will investigate the projection body of $s$-concave functions when $s$ is a positive integer. In Section 4, we investigate the affine isoperimetric inequality. For an $s$-concave function $f$, the projection body $\Pi^{(s)}f$ of $f$ is defined by

$$\Pi^{(s)}f = \Pi K_s(f).$$

Here $\Pi K$ denotes the projection body of $K$ (defined in Section 2). Since $\Pi K_s(f)$ is a convex body in $\mathbb{R}^{n+s}$, it will be proved that $\Pi^{(s)}f$ inherits almost all properties of the projection body for convex bodies, such as, continuity, affine invariance, valuation and etc..

Another main theorem of our results is the following Petty projection inequality for s-concave functions (see Theorem 4.1): Let $s\in\mathbb{N}$, and let $f:\mathbb{R}^n\to[0,\infty)$ be an $s$-concave function and twice continuously differentiable. If

$$(x,\widetilde{x})=(x_1,\cdots,x_n,x_{n+1},\cdots,x_{n+s})\in\mathbb{R}^n\times\mathbb{R}^s$$

satisfies $x_{n+s}^2=f(x)^{\frac{2}{s}}-\sum\limits_{i=n+1}^{n+s}x_i^2$, then

$$\begin{aligned}&\int_{\mathbb{S}^{n+s-1}}\Big[\int_{\mathbb{R}^{n+s-1}}|\langle u,(f^{\frac{1}{s}}\nabla f^{\frac{1}{s}},-\widetilde{x})\rangle|\frac{\mathrm{d}x_1\cdots\mathrm{d}x_{n+s-1}}{|x_{n+s}|}\Big]^{-(n+s)}\mathrm{d}u\\&\le\widetilde{c}_{n,s}\Big(\int_{\mathbb{R}^n}f\mathrm{d}x\Big)^{1-s-n},\end{aligned}\tag{1.6}$$

where $\widetilde{c}_{n,s}=(n+s)\omega_s\big(\frac{\omega_{n+s}}{\omega_{n+s-1}\omega_s}\big)^{n+s}$. Equality holds if and only if $f=(a+\langle b,x\rangle-\langle\phi x,x\rangle)_+^{\frac{s}{2}}$ for $a>0$, $b\in\mathbb{R}^n$ and a positive definite matrix $\phi$.

Finally, we will deduce a reverse inequality of (1.6). Zhang projection inequality for $s$-concave functions ((4.18) in Corollary 4.2) is obtained. The Loomis-Whitney inequality of 1-concave functions ((4.20) in Theorem 4.2) is discussed in Section 4.

## 2 Preliminaries

We work in $n$-dimensional Euclidean space $\mathbb{R}^n$, endowed with the usual scalar product $\langle x,y\rangle$ and norm $\|x\|$. Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with nonempty interiors) in the Euclidean space $\mathbb{R}^n$. We write $\mathcal{K}_o^n$ for the set of convex bodies that contain the origin in their interiors. Let $B_n=\{x\in\mathbb{R}^n:\|x\|\le1\}$ denote the standard unit ball in $\mathbb{R}^n$ and $\mathbb{S}^{n-1}=\{x\in\mathbb{R}^n:\|x\|=1\}$ denote the unit sphere in $\mathbb{R}^n$. Let $V_n(K)$ denote the $n$-dimensional volume of $K$. Volumes of the unit ball $B_n$ and the unit sphere $\mathbb{S}^{n-1}$ can be expressed, respectively, as

$$V_n(B_n)=\omega_n=\frac{\pi^{\frac{n}{2}}}{\Gamma(1+\frac{n}{2})}\quad\text{and}\quad V_{n-1}(\mathbb{S}^{n-1})=n\omega_n=\frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}.\tag{2.1}$$

Here $\Gamma(\cdot)$ is the Gamma function.

We write $\mathrm{GL}(n)$ for the group of general linear transformations in $\mathbb{R}^n$. For $\phi\in\mathrm{GL}(n)$, we write $\phi^t$ for the transpose of $\phi$ and $\phi^{-t}$ for the inverse of the transpose (contragradient) of $\phi$. Let $\det\phi$ denote the determinant of $\phi$. For $K\in\mathcal{K}^n$, let $h(K;\cdot)=h_K:\mathbb{R}^n\to\mathbb{R}$ denote the support function of $K$, i.e.,

$$h(K;x)=\max\{\langle x,y\rangle:y\in K\}.$$

Let $N_K(x)$ be the unit outer normal at $x\in\partial K$. Then

$$h_K(N_K(x))=\langle N_K(x),x\rangle,\quad\forall x\in\partial K.\tag{2.2}$$

For $\lambda > 0$, the support function of the convex body $\lambda K = \{\lambda x : x \in K\}$ satisfies

$$h_{\lambda K}(\cdot) = \lambda h_K(\cdot). \tag{2.3}$$

For $\phi \in \mathrm{GL}(n)$ the support function of the image $\phi K = \{\phi y : y \in K\}$ is given by

$$h_{\phi K}(x) = h_K(\phi^t x), \quad \forall x \in \mathbb{R}^n. \tag{2.4}$$

For $K \in \mathcal{K}_o^n$, the polar body $K^*$ of $K$ is defined by

$$K^* = \{x \in \mathbb{R}^n : \langle x, y\rangle \le 1 \text{ for all } y \in K\}.$$

Let $\rho(K;\cdot) = \rho_K : \mathbb{R}^n \setminus \{0\} \to [0, +\infty)$ denote the radial function of $K \in \mathcal{K}_o^n$, i.e.,

$$\rho_K(x) = \max\{\lambda > 0 : \lambda x \in K\}.$$

It is not hard to verify that

$$h_{K^*}(\cdot) = \frac{1}{\rho_K(\cdot)} \quad \text{and} \quad \rho_{K^*}(\cdot) = \frac{1}{h_K(\cdot)}. \tag{2.5}$$

The $n$-dimensional volume of a convex body $K$ is given by

$$V_n(K) = \frac{1}{n}\int_{\mathbb{S}^{n-1}} \rho_K(u)^n \mathrm{d}u, \tag{2.6}$$

and the surface area of a convex body $K$ is given by

$$S(K) = \int_{\partial K} \mathrm{d}\mu_K(x), \tag{2.7}$$

where $\mathrm{d}\mu_K$ denotes the surface area measure of $K$.

Suppose that $\mu$ is a probability measure on a space $X$ and $g : X \to I \subset \mathbb{R}$ is a $\mu$-integrable function, where $I$ is a possibly infinite interval. Jensen's inequality states that if $F : I \to \mathbb{R}$ is a convex function, then

$$\int_X F(g(x))\mathrm{d}\mu(x) \ge F\Big(\int_X g(x)\mathrm{d}\mu(x)\Big). \tag{2.8}$$

If $F$ is strictly convex, equality holds if and only if $g(x)$ is constant for $\mu$-almost all $x \in X$.

The classical projection body was introduced at the turn of the previous century by Minkowski. Let $K \in \mathcal{K}^n$. The projection body $\Pi K$ of $K$ is defined as an origin-symmetric convex body in $\mathbb{R}^n$ whose support function is

$$h(\Pi K, u) = V_{n-1}(K|u^\perp)$$

for $u \in \mathbb{S}^{n-1}$. Here $V_{n-1}(K|u^\perp)$ is the $n-1$ dimensional volume of $K$ projecting to the hyperplane that passing through the origin with the normal direction $u$. The support function $h(\Pi K, u)$ can be rewritten as

$$h_{\Pi K}(u) = \frac{1}{2}\int_{\partial K} |\langle u, N_K(y)\rangle| \mathrm{d}\mu_K(y). \tag{2.9}$$

Interest in projection body was rekindled by three highly influential articles, which appeared in the latter half of the 60's, by Bolker [11], Petty [36] and Schneider [42].

The fundamental inequality for projection body in the field of affine isoperimetric inequalities is the following Petty projection inequality (see [37]): If $K \in \mathcal{K}^n$, then

$$V_n(K)^{n-1}V_n(\Pi^* K) \leq \Big(\frac{\omega_n}{\omega_{n-1}}\Big)^n \tag{2.10}$$

with equality if and only if $K$ is an ellipsoid. Here $\Pi^* K$ denotes the polar body of the projection body $\Pi K$ rather than $(\Pi K)^*$.

The reverse Petty projection inequality reads as

$$\frac{1}{n^n}\binom{2n}{n} \leq V_n(K)^{n-1}V_n(\Pi^* K) \tag{2.11}$$

with equality if and only if $K$ is a simplex.

The inequality (2.11) was conjectured by Ball [9] and first proved by Zhang [44]. The inequality (2.11) is also known as Zhang projection inequality.

One reason that the operator $\Pi$ is so useful in these areas is that projection body of affinely equivalent convex bodies are affinely equivalent. Specifically

$$\Pi(\phi K) = |\det\phi|\phi^{-t}\Pi K \quad \text{and} \quad \Pi(K+x) = \Pi K \tag{2.12}$$

for every $K \in \mathcal{K}^n$, $\phi \in \mathrm{GL}(n)$ and $x \in \mathbb{R}^n$.

Let $s, n \in \mathbb{N}$, and $\mathrm{supp}(f) = \{x \in \mathbb{R}^n : f(x) \neq 0\}$. $f : \mathbb{R}^n \to [0,\infty)$ is $s$-concave, and we denote $f \in \mathrm{Conc}_s(\mathbb{R}^n)$, if $f$ is upper semi-continuous, the closure $\overline{\mathrm{supp}(f)}$ of $\mathrm{supp}(f)$ is a convex body and $f^{\frac{1}{s}}$ is concave on $\overline{\mathrm{supp}(f)}$. The class $\mathrm{Conc}_s^{(2)}(\mathbb{R}^n)$ shall consist of such $f \in \mathrm{Conc}_s(\mathbb{R}^n)$ which are twice continuously differentiable in the interior of their support.

As in [5–6], associated with a function $f \in \mathrm{Conc}_s(\mathbb{R}^n)$, the convex body $K_s(f)$ in $\mathbb{R}^n \times \mathbb{R}^s$ is given by

$$K_s(f) := \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^s : x \in \mathrm{supp}(f), \|y\| \leq f^{\frac{1}{s}}(x)\}. \tag{2.13}$$

A special function in the class $\mathrm{Conc}_s(\mathbb{R}^n)$, which will play the role of the Euclidean ball in convexity, is

$$g_s(x) = (1-\|x\|^2)_+^{\frac{s}{2}}, \tag{2.14}$$

where, for $a \in \mathbb{R}$, $a_+ = \max\{a, 0\}$. It follows immediately from the definition that $K_s(g_s(x)) = B_2^{n+s}$. By Fubini's theorem, we have that for all $f \in \mathrm{Conc}_s(\mathbb{R}^n)$,

$$V_{n+s}(K_s(f)) = V_s(B_2^s)\int_{\mathbb{R}^n} f(x)\mathrm{d}x. \tag{2.15}$$

By (2.13), the boundary of $K_s(f)$ is given by $\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^s : \|y\| = f^{\frac{1}{s}}(x)\}$, and the boundary of $K_s(f)$ is union of the graphs of the two mappings

$$(x_1,\cdots,x_n,x_{n+1},\cdots,x_{n+s-1}) \mapsto (x_1,\cdots,x_n,x_{n+1},\cdots,x_{n+s-1},+x_{n+s})$$

and

$$(x_1,\cdots,x_n,x_{n+1},\cdots,x_{n+s-1})\mapsto(x_1,\cdots,x_n,x_{n+1},\cdots,x_{n+s-1},-x_{n+s})$$

with $x=(x_1,\cdots,x_n)\in\mathbb{R}^n$, and

$$x_{n+s}=\Big(f^{\frac{2}{s}}(x)-\sum_{i=n+1}^{n+s-1}x_i^2\Big)^{\frac{1}{2}}. \tag{2.16}$$

Because of symmetry, it is enough to consider only the "positive" part of $\partial K_s(f)$, in which the last coordinate is non-negative. Then the surface area element of $K_s(f)$ is

$$\mathrm{d}\mu_{K_s(f)}=\frac{f^{\frac{1}{s}}(1+\|\nabla f^{\frac{1}{s}}\|^2)^{\frac{1}{2}}}{|x_{n+s}|}\mathrm{d}x_1\cdots\mathrm{d}x_{n+s-1}. \tag{2.17}$$

In [6], the authors proved the following results.

**Lemma 2.1** *Let $f\in\mathrm{Conc}_s^{(2)}(\mathbb{R}^n)$. Then for all $(x,\widetilde{x})=(x_1,\cdots,x_n,x_{n+1}\cdots,x_{n+s})\in\partial K_s(f)$ with $x=(x_1,\cdots,x_n)\in\mathrm{int}(\mathrm{supp}(f))$,*

$$N_{K_s(f)}(x,\widetilde{x})=\frac{(f^{\frac{1}{s}}\nabla f^{\frac{1}{s}},-x_{n+1},\cdots,-x_{n+s})}{f^{\frac{1}{s}}(1+\|\nabla f^{\frac{1}{s}}\|^2)^{\frac{1}{2}}} \tag{2.18}$$

*and*

$$\kappa_{K_s(f)}(x,\widetilde{x})=\Big|\frac{\det(\nabla^2 f^{\frac{1}{s}})}{f^{\frac{s-1}{s}}(1+\|\nabla f^{\frac{1}{s}}\|^2)^{\frac{n+s+1}{2}}}\Big|. \tag{2.19}$$

*Here $f$ is evaluated at $x=(x_1,\cdots,x_n)\in\mathbb{R}^n$.*

## 3 The Isoperimetric Inequality of $s$-Concave Functions

In this section, we obtain the isoperimetric inequality for $s$-concave functions.

**Theorem 3.1** *Let $s\in\mathbb{N}$ and $f\in\mathrm{Conc}_s^{(2)}(\mathbb{R}^n)$. Then*

$$\int_{\mathbb{R}^n}f^{1-\frac{1}{s}}(1+\|\nabla f^{\frac{1}{s}}\|^2)^{\frac{1}{2}}\mathrm{d}x\geq c_{n,s}\Big(\int_{\mathbb{R}^n}f\mathrm{d}x\Big)^{1-\frac{1}{n+s}}, \tag{3.1}$$

*where $c_{n,s}=\frac{(n+s)}{s}\big(\frac{\omega_{n+s}}{\omega_s}\big)^{\frac{1}{n+s}}$. Equality holds if and only if $f=(a-\|x-b\|^2)_+^{\frac{s}{2}}$ for $a>0,b\in\mathbb{R}^n$.*

**Proof** By (2.7) and (2.17) we have

$$\begin{aligned}S(K_s(f))&=\int_{\partial K_s f}\mathrm{d}\mu_{K_s(f)}(x)\\&=2\int_{\mathbb{R}^{n+s-1}}\frac{f^{\frac{1}{s}}(1+\|\nabla f^{\frac{1}{s}}\|^2)^{\frac{1}{2}}}{|x_{n+s}|}\mathrm{d}x_1\cdots\mathrm{d}x_{n+s-1}.\end{aligned} \tag{3.2}$$

The last equality follows as the boundary of $K_s(f)$ consists of two, "positive" and "negative", parts. By (2.16) and a direct calculation, we have

$$\int_{\mathbb{R}^{s-1}}\frac{\mathrm{d}x_{n+1}\cdots\mathrm{d}x_{n+s-1}}{|x_{n+s}|}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^{s-1}} f^{-\frac{1}{s}} \Big(1 - \sum_{i=n+1}^{n+s-1} \Big(\frac{x_i}{f^{\frac{1}{s}}}\Big)^2\Big)^{-\frac{1}{2}} \mathrm{d}x_{n+1} \cdots \mathrm{d}x_{n+s-1} \\
&= \int_{\sum\limits_{i=n+1}^{n+s-1} y_i^2 \le 1} \frac{f^{-\frac{s-1}{s}}}{f^{-\frac{1}{s}}} \Big(1 - \sum_{i=n+1}^{n+s-1} y_i^2\Big)^{-\frac{1}{2}} \mathrm{d}y_{n+1} \cdots \mathrm{d}y_{n+s-1} \\
&= \frac{f^{\frac{s-1}{s}}}{f^{\frac{1}{s}}} (s-1)\omega_{s-1} \int_0^1 r^{s-2}(1-r^2)^{-\frac{1}{2}} \mathrm{d}r \\
&= \frac{f^{\frac{s-1}{s}}}{f^{\frac{1}{s}}} \frac{(s-1)\omega_{s-1}}{2} B\Big(\frac{s-1}{2}, \frac{1}{2}\Big) \\
&= \frac{f^{\frac{s-1}{s}}}{f^{\frac{1}{s}}} \frac{\pi^{\frac{s}{2}}}{\Gamma\big(\frac{s}{2}\big)}.
\end{aligned} \tag{3.3}
$$

Together with (3.3), (3.2) means

$$S(K_s(f)) = \frac{2\pi^{\frac{s}{2}}}{\Gamma(\frac{s}{2})} \int_{\mathbb{R}^n} f^{1-\frac{1}{s}} (1 + \|\nabla f^{\frac{1}{s}}\|^2)^{\frac{1}{2}} \mathrm{d}x.$$

By the isoperimetric inequality (1.1), we have

$$
\begin{aligned}
S(K_s(f))^{n+s} &= \Big(\frac{2\pi^{\frac{s}{2}}}{\Gamma(\frac{s}{2})} \int_{\mathbb{R}^n} f^{1-\frac{1}{s}} (1 + \|\nabla f^{\frac{1}{s}}\|^2)^{\frac{1}{2}} \mathrm{d}x\Big)^{n+s} \\
&\ge (n+s)^{n+s} \omega_{n+s} \Big(V_s(B_2^s) \int_{\mathbb{R}^n} f \mathrm{d}x.\Big)^{n+s-1}.
\end{aligned}
$$

Equality holds if and only if $K_s(f)$ is a ball, this means that $f = (a - \|x-b\|^2)_+^{\frac{s}{2}}$ for $a \in \mathbb{R}, b \in \mathbb{R}^n$.

Let $f \in \mathrm{Conc}_s(\mathbb{R}^n)$. If $s = 1$, namely $f$ is a concave function in $\mathbb{R}^n$, we have the following corollary.

**Corollary 3.1** *Let $f$ be a concave function in $\mathbb{R}^n$. Then*

$$\int_{\mathbb{R}^n} (1 + \|\nabla f\|^2)^{\frac{1}{2}} \mathrm{d}x \ge (n+1)\Big(\frac{\omega_{n+1}}{2}\Big)^{\frac{1}{n+1}} \Big(\int_{\mathbb{R}^n} f \mathrm{d}x\Big)^{\frac{n}{n+1}}$$

*with equality if and only if $f = (a - \|x-b\|^2)_+^{\frac{1}{2}}$ for $a > 0, b \in \mathbb{R}^n$.*

Let $f \in \mathrm{Conc}_s(\mathbb{R}^n)$. When $s = n = 1$, we have the following corollary.

**Corollary 3.2** *Let $f$ be a concave function in $\mathbb{R}$. Then*

$$\int_{\mathbb{R}} (1 + f'^2)^{\frac{1}{2}} \mathrm{d}x \ge 2\Big(\frac{\pi}{2} \int_{\mathbb{R}} f \mathrm{d}x\Big)^{\frac{1}{2}}$$

*with equality if and only if $f(x) = (a - (x-b)^2)_+^{\frac{1}{2}}$ for $a > 0, b \in \mathbb{R}$.*

## 4 The Projection Body of $s$-Concave Functions

We will study the Petty projection body of $s$-concave functions in this section. We give the definition of Petty projection body for $s$-concave functions which is an analogue of the Petty projection body in convex geometry.

**Definition 4.1** *Let $f \in \mathrm{Conc}_s(\mathbb{R}^n)$. The convex body $\Pi^{(s)}f$, the projection body of $s$-concave functions $f$, is defined by*

$$h_{\Pi^{(s)}f}(\cdot) = h_{\Pi K_s(f)}(\cdot). \tag{4.1}$$

Definition 4.1 means that the support function of $\Pi^{(s)}f$ satisfies:

$$h_{\Pi^{(s)}f}(y) = h_{\Pi K_s(f)}(y) = \frac{1}{2}\int_{\partial K_s(f)} |y \cdot N_{K_s(f)}(x)| \mathrm{d}\mu_{K_s(f)}(x), \tag{4.2}$$

where $y = (y_1, \cdots, y_{n+s}) \in \mathbb{R}^n \times \mathbb{R}^s$.

The projection body $\Pi^{(s)}f$ is monotonic.

**Proposition 4.1** *Let $s \in \mathbb{N}$ and $f_1, f_2 \in \mathrm{Conc}_s(\mathbb{R}^n)$. If $f_1 \le f_2$, then*

$$\Pi^{(s)}f_1 \subseteq \Pi^{(s)}f_2. \tag{4.3}$$

**Proof** From (2.13), $K_s(f_1) \subseteq K_s(f_2)$ when $f_1 \le f_2$. Hence $\Pi K_s(f_1) \subseteq \Pi K_s(f_2)$.

**Proposition 4.2** *Let $s \in \mathbb{N}$ and $f_i, f \in \mathrm{Conc}_s^{(2)}(\mathbb{R}^n)$. If $f_i \to f$ as $i \to \infty$, then*

$$\Pi^{(s)}f_i \to \Pi^{(s)}f \quad \textit{as } i \to \infty. \tag{4.4}$$

**Proof** Let $(x, \widetilde{x}) = (x_1, \cdots, x_n, x_{n+1} \cdots, x_{n+s}) \in \partial K_s(f)$ with $x = (x_1, \cdots, x_n) \in \mathrm{int}(\mathrm{supp}(f))$. By (2.2) and (2.19), we have

$$h_{K_s(f)}(x, \widetilde{x}) = \frac{\langle x, f^{\frac{1}{s}} \nabla f^{\frac{1}{s}} \rangle - (x_{n+1}^2 + \cdots + x_{n+s}^2)}{f^{\frac{1}{s}}(1 + \|\nabla f^{\frac{1}{s}}\|^2)^{\frac{1}{2}}}.$$

For convex functions $\varphi, \varphi_i$, if $\varphi_i \to \varphi$, then

$$\nabla\varphi_i(x) \to \nabla\varphi(x) \quad \text{as } i \to +\infty \tag{4.5}$$

for any $x \in \mathbb{R}^n$ in which $\varphi, \varphi_1, \varphi_2, \cdots$ are differentiable (see [40, Theorem 25.7]). Since the functions $f^{\frac{1}{s}}$ and $f_i^{\frac{1}{s}}$ are concave, we have

$$\nabla f_i^{\frac{1}{s}}(x) \to \nabla f^{\frac{1}{s}}(x),$$

when $f_i \to f$. Since a convex function is differentiable almost everywhere, we have $h_{K_s(f_i)}(x, \widetilde{x}) \to h_{K_s(f)}(x, \widetilde{x})$ as $i \to \infty$. Therefore $K_s(f_i) \to K_s(f)$ as $i \to \infty$ and $\Pi^{(s)}f_i \to \Pi^{(s)}f$ as $i \to \infty$.

We have the following integral representation of $h_{\Pi^{(s)}f}$ in $\mathbb{R}^{n+s-1}$.

**Lemma 4.1** *Let $s \in \mathbb{N}$ and $f \in \mathrm{Conc}_s^{(2)}(\mathbb{R}^n)$. The support function of $\Pi^{(s)}f$ can be expressed as the following integral:*

$$h_{\Pi^{(s)}f}(u) = \int_{\mathbb{R}^{n+s-1}} |\langle u, (f^{\frac{1}{s}} \nabla f^{\frac{1}{s}}, -\widetilde{x})\rangle| \frac{\mathrm{d}x_1 \cdots \mathrm{d}x_{n+s-1}}{|x_{n+s}|}, \quad u \in \mathbb{S}^{n+s-1}, \tag{4.6}$$

*where $\widetilde{x} = (x_{n+1}, \cdots, x_{n+s}) \in \mathbb{R}^s$.*

**Proof** Let $u \in \mathbb{S}^{n+s-1}$, $x=(x_1,\cdots,x_n), \widetilde{x}=(x_{n+1},\cdots,x_{n+s})$ such that $(x,\widetilde{x}) \in (\mathbb{R}^n \times \mathbb{R}^s) \cap \partial K_s(f)$. Denote by $\widetilde{\partial} K_s(f)$ the collection of all points $(x_1,\cdots,x_{n+s}) \in \partial K_s(f)$ such that $(x_1,\cdots,x_n) \in \mathrm{int}(\mathrm{supp}(f))$. Since there is no contribution to the integral of $h_{\Pi^{(s)}f}(x)$ from $\partial K_s(f) \backslash \widetilde{\partial} K_s(f)$, by Lemma 2.1 and (2.16)–(2.17), we have

$$\begin{aligned} h_{\Pi^{(s)}f}(u) = h_{\Pi K_s(f)}(u) &= \frac{1}{2}\int_{\partial K_s(f)} |\langle u, N_{K_s(f)}(x)\rangle| \mathrm{d}\mu_{K_s(f)}(x) \\ &= \frac{1}{2}\int_{\widetilde{\partial} K_s(f)} |\langle u, N_{K_s(f)}(x)\rangle| \mathrm{d}\mu_{K_s(f)}(x) \\ &= \int_{\mathbb{R}^{n+s-1}} |\langle u, (f^{\frac{1}{s}}\nabla f^{\frac{1}{s}}, -\widetilde{x})\rangle| \frac{\mathrm{d}x_1\cdots \mathrm{d}x_{n+s-1}}{|x_{n+s}|}, \end{aligned}$$

where $f$ is evaluated at $(x_1,\cdots,x_n) \in \mathbb{R}^n$. The last equality follows as the boundary of $K_s(f)$ consists of two, "positive" and "negative", parts.

For $\phi \in \mathrm{GL}(n)$, set

$$\overline{\phi} = \begin{pmatrix} \phi & 0 \\ 0 & I_{s\times s} \end{pmatrix} \in \mathrm{GL}(n+s), \tag{4.7}$$

where $I_{s\times s}$ is the $s \times s$ identity matrix.

**Proposition 4.3** *Let $s \in \mathbb{N}$ and $f \in \mathrm{Conc}_s(\mathbb{R}^n)$. For $\phi \in GL(n)$, $x_0 \in \mathbb{R}^n$,*

$$\Pi^{(s)}(f\circ\phi) = |\det \overline{\phi}|\overline{\phi}^{-t}\Pi^{(s)}f, \quad \Pi^{(s)}(f(x+x_0)) = \Pi^{(s)}f(x). \tag{4.8}$$

**Proof** By (2.13), we have

$$\overline{\phi} K_s(f) = K_s(f\circ\phi). \tag{4.9}$$

The first equality of (2.12) and Definition 4.1 imply that

$$\Pi(\overline{\phi}K_s(f)) = |\det\overline{\phi}|\overline{\phi}^{-t}\Pi K_s(f). \tag{4.10}$$

The first equality of (4.8) follows from (4.9)–(4.10).

Since $K_s(f(x+x_0)) = K_s(f(x)) + (x_0, 0)$ (here 0 is the null vector in $\mathbb{R}^s$), the translation invariance of $\Pi^{(s)}f$ in (4.8) follows by the second equality of (2.12) and Definition 4.1.

The first formula in (4.8) states that the volume of $\Pi^{(s)}f$ and the polar body of $\Pi^{(s)}f$ are affine invariants, that is, $V_{n+s}(\Pi^{(s)}(f\circ\phi)) = V_{n+s}(\Pi^{(s)}f)$ and $V_{n+s}(\Pi^{(s),*}(f\circ\phi)) = V_{n+s}(\Pi^{(s),*}f)$ for any $\phi \in \mathrm{SL}(n)$.

A Minkowski valuation is a map $\mathrm{Z}: \mathcal{K}_o^n \to \langle \mathcal{K}_o^n, +\rangle$ such that

$$\mathrm{Z}(K\cap L) + \mathrm{Z}(K\cup L) = \mathrm{Z}(K) + \mathrm{Z}(L), \quad K,\ L \in \mathcal{K}_o^n, \tag{4.11}$$

whenever $K \cup L \in \mathcal{K}_o^n$. Here " + " is the Minkowski sum.

The projection operator $\Pi$ is a valuation, i.e.,

$$\Pi K_1 + \Pi K_2 = \Pi(K_1\cup K_2) + \Pi(K_1 \cap K_2), \tag{4.12}$$

whenever $K_1, K_2, K_1 \cup K_2 \in \mathcal{K}^n$.

The operate $\Pi^{(s)}$ is a valuation.

**Proposition 4.4** *Let $s \in \mathbb{N}$, $f_1, f_2 \in \mathrm{Conc}_s(\mathbb{R}^n)$. Then*

$$\Pi^{(s)} f_1 + \Pi^{(s)} f_2 = \Pi^{(s)}(\max\{f_1, f_2\}) + \Pi^{(s)}(\min\{f_1, f_2\}), \tag{4.13}$$

*where " + " is Minkowski addition.*

**Proof** By (4.12) and Definition 4.1, we have

$$\begin{aligned}\Pi^{(s)} f_1 + \Pi^{(s)} f_2 &= \Pi(K_s(f_1)) + \Pi(K_s(f_2)) \\ &= \Pi(K_s(f_1) \cup K_s(f_2)) + \Pi(K_s(f_1) \cap K_s(f_2))\end{aligned}$$

with $K_s(f_1) \cup K_s(f_1)$ is convex. We only need to prove

$$K_s(f_1) \cup K_s(f_2) = K_s(\max\{f_1, f_2\}), \quad K_s(f_1) \cap K_s(f_2) = K_s(\min\{f_1, f_2\}). \tag{4.14}$$

We assume that $(x, y) \in K_s(f_1) \cup K_s(f_2)$. From the definition of $K_s(f)$, there are the following three different possibilities

$$\begin{aligned}&f_2^{\frac{1}{s}}(x) \le \|y\| \le f_1^{\frac{1}{s}}(x), \\ &f_1^{\frac{1}{s}}(x) \le \|y\| \le f_2^{\frac{1}{s}}(x), \\ &\|y\| \le f_1^{\frac{1}{s}}(x) \quad \text{and} \quad \|y\| \le f_2^{\frac{1}{s}}(x).\end{aligned}$$

By these three cases we have

$$\begin{aligned}\|y\| &\le \max\{f_1^{\frac{1}{s}}(x), f_2^{\frac{1}{s}}(x)\} \\ &= (\max\{f_1(x), f_2(x)\})^{\frac{1}{s}},\end{aligned}$$

i.e., $K_s(f_1) \cup K_s(f_2) \subseteq K_s(\max\{f_1, f_2\})$. The reverse inclusion is easy to obtain by the same way. The equality $K_s(f_1) \cap K_s(f_2) = K_s(\min\{f_1, f_2\})$ can be proved by the same way.

We are now in the position to prove the Petty projection inequality for $s$-concave functions.

**Theorem 4.1** *Let $s \in \mathbb{N}$, $f \in \mathrm{Conc}_s^{(2)}(\mathbb{R}^n)$, $x = (x_1, \cdots, x_n)$, $\widetilde{x} = (x_{n+1} \cdots, x_{n+s})$. Then for all $(x, \widetilde{x}) \in \partial K_s(f)$ with $x \in \mathrm{int}(\mathrm{supp}(f))$,*

$$\begin{aligned}&\int_{\mathbb{S}^{n+s-1}} \Big[\int_{\mathbb{R}^{n+s-1}} |\langle u, (f^{\frac{1}{s}} \nabla f^{\frac{1}{s}}, -\widetilde{x})\rangle| \frac{\mathrm{d}x_1 \cdots \mathrm{d}x_{n+s-1}}{|x_{n+s}|}\Big]^{-(n+s)} \mathrm{d}u \\ &\le \widetilde{c}_{n,s} \Big(\int_{\mathbb{R}^n} f \mathrm{d}x\Big)^{1-s-n},\end{aligned} \tag{4.15}$$

*where $\widetilde{c}_{n,s} = (n+s)\omega_s \big(\frac{\omega_{n+s}}{\omega_{n+s-1}\omega_s}\big)^{n+s}$. Equality holds if and only if $f = (a + \langle b, x\rangle - \langle \phi x, x\rangle)_+^{\frac{s}{2}}$ for $a > 0, b \in \mathbb{R}^n$ and a positive definite matrix $\phi$.*

**Proof** Let $\Pi^{(s),*}f$ denote the polar body of $\Pi^{(s)}f$. By (2.5)–(2.6), (4.6) and the Petty projection inequality (2.10), we have

$$\begin{aligned}
&V_{n+s}(\Pi^{(s),*}f)\\
&=\frac{1}{n+s}\int_{\mathbb{S}^{n+s-1}} h_{\Pi^{(s)}f}(u)^{-(n+s)}\mathrm{d}u\\
&=\frac{1}{n+s}\int_{\mathbb{S}^{n+s-1}}\Big[\int_{\mathbb{R}^{n+s-1}}|\langle u,(f^{\frac{1}{s}}\nabla f^{\frac{1}{s}},-\widetilde{x})\rangle|\frac{\mathrm{d}x_1\cdots\mathrm{d}x_{n+s-1}}{|x_{n+s}|}\Big]^{-(n+s)}\mathrm{d}u\\
&\le\Big(\frac{\omega_{n+s}}{\omega_{n+s-1}}\Big)^{n+s}\Big(\omega_s\int_{\mathbb{R}^n}f\mathrm{d}x\Big)^{1-s-n}. 
\end{aligned}\tag{4.16}$$

By the equality condition of the Petty projection inequality (2.10), (4.16) holds as an equality if and only if $K_s(f)$ is an ellipsoid. This means that $f=(a+\langle b,x\rangle-\langle\phi x,x\rangle)_+^{\frac{s}{2}}$ for $a\in\mathbb{R},b\in\mathbb{R}^n$ and a positive definite matrix $\phi$.

**Corollary 4.1** *The affine isoperimatric inequality for $s$-concave functions*

$$\int_{\mathbb{S}^{n+s-1}}\Big[\int_{\mathbb{R}^{n+s-1}}|\langle u,(f^{\frac{1}{s}}\nabla f^{\frac{1}{s}},-\widetilde{x})\rangle|\frac{\mathrm{d}x_1\cdots\mathrm{d}x_{n+s-1}}{|x_{n+s}|}\Big]^{-(n+s)}\mathrm{d}u\le\widetilde{c}_{n,s}\Big(\int_{\mathbb{R}^n}f\mathrm{d}x\Big)^{1-s-n},$$

*is stronger than the classical isoperimetric inequality for $s$-concave functions*

$$\int_{\mathbb{R}^n}f^{1-\frac{1}{s}}(1+\|\nabla f^{\frac{1}{s}}\|^2)^{\frac{1}{2}}\mathrm{d}x\ge c_{n,s}\Big(\int_{\mathbb{R}^n}f\mathrm{d}x\Big)^{1-\frac{1}{n+s}}.$$

**Proof** By Jensen's inequality (2.8), Fubini's theorem,

$$|(f^{\frac{1}{s}}\nabla f^{\frac{1}{s}},-\widetilde{x})|=f^{\frac{1}{s}}(1+\|\nabla f^{\frac{1}{s}}\|^2)^{\frac{1}{2}},$$

and (3.3), we have

$$\begin{aligned}
&\Big(\frac{1}{(n+s)\omega_{n+s}}\int_{\mathbb{S}^{n+s-1}}\Big[\int_{\mathbb{R}^{n+s-1}}|\langle u,(f^{\frac{1}{s}}\nabla f^{\frac{1}{s}},-\widetilde{x})\rangle|\frac{\mathrm{d}x_1\cdots\mathrm{d}x_{n+s-1}}{|x_{n+s}|}\Big]^{-(n+s)}\mathrm{d}u\Big)^{-\frac{1}{n+s}}\\
&\le\frac{1}{(n+s)\omega_{n+s}}\int_{\mathbb{S}^{n+s-1}}\int_{\mathbb{R}^{n+s-1}}|\langle u,(f^{\frac{1}{s}}\nabla f^{\frac{1}{s}},-\widetilde{x})\rangle|\frac{\mathrm{d}x_1\cdots\mathrm{d}x_{n+s-1}}{|x_{n+s}|}\mathrm{d}u\\
&=\frac{1}{(n+s)\omega_{n+s}}\int_{\mathbb{R}^{n+s-1}}\int_{\mathbb{S}^{n+s-1}}|\langle u,(f^{\frac{1}{s}}\nabla f^{\frac{1}{s}},-\widetilde{x})\rangle|\mathrm{d}u\frac{\mathrm{d}x_1\cdots\mathrm{d}x_{n+s-1}}{|x_{n+s}|}\\
&=\frac{2\omega_{n+s-1}}{(n+s)\omega_{n+s}}\int_{\mathbb{R}^{n+s-1}}f^{\frac{1}{s}}(1+\|\nabla f^{\frac{1}{s}}\|^2)^{\frac{1}{2}}\frac{\mathrm{d}x_1\cdots\mathrm{d}x_{n+s-1}}{|x_{n+s}|}\\
&=\frac{s\omega_{n+s-1}\omega_s}{(n+s)\omega_{n+s}}\int_{\mathbb{R}^n}f^{1-\frac{1}{s}}(1+\|\nabla f^{\frac{1}{s}}\|^2)^{\frac{1}{2}}\mathrm{d}x.
\end{aligned}\tag{4.17}$$

Therefore, (3.1) follows from (4.15) and (4.17).

We also obtain the Zhang projection inequality for $s$-concave functions.

**Corollary 4.2** *Let $s\in\mathbb{N}$ and $f\in\mathrm{Conc}_s^{(2)}(\mathbb{R}^n)$. If $x=(x_1,\cdots,x_n)$ and $\widetilde{x}=(x_{n+1}\cdots,x_{n+s})$, then for all $(x,\widetilde{x})\in\partial K_s(f)$ with $x=(x_1,\cdots,x_n)\in\mathrm{int}(\mathrm{supp}(f))$,*

$$\int_{\mathbb{S}^{n+s-1}}\Big[\int_{\mathbb{R}^{n+s-1}}|\langle u,(f^{\frac{1}{s}}\nabla f^{\frac{1}{s}},-\widetilde{x})\rangle|\frac{\mathrm{d}x_1\cdots\mathrm{d}x_{n+s-1}}{|x_{n+s}|}\Big]^{-(n+s)}\mathrm{d}u$$

$$\geq \frac{1}{(n+s)^{n+s}}\binom{2n+2s}{n+s}\Big(\omega_s\int_{\mathbb{R}^n} f\mathrm{d}x\Big)^{1-s-n}. \tag{4.18}$$

**Proof** Note that from (2.5)–(2.6), (4.6) and Zhang projection inequality (2.11),

$$\begin{aligned}
&V_{n+s}(\Pi^{(s),*}f)\\
&=\frac{1}{n+s}\int_{\mathbb{S}^{n+s-1}} h_{\Pi^{(s)}f}(u)^{-(n+s)}\mathrm{d}u\\
&=\frac{1}{n+s}\int_{\mathbb{S}^{n+s-1}}\Big[\int_{\mathbb{R}^{n+s-1}}|\langle u,(f^{\frac{1}{s}}\nabla f^{\frac{1}{s}},-\widetilde{x})\rangle|\frac{\mathrm{d}x_1\cdots\mathrm{d}x_{n+s-1}}{|x_{n+s}|}\Big]^{-(n+s)}\mathrm{d}u\\
&\geq\frac{1}{(n+s)^{n+s}}\binom{2(n+s)}{n+s}\Big(\omega_s\int_{\mathbb{R}^n} f\mathrm{d}x\Big)^{1-s-n}.
\end{aligned}$$

For a compact set $K\subseteq\mathbb{R}^n$, let $\{e_1,\cdots,e_n\}$ be the standard Euclidean basis of $\mathbb{R}^n$ and $K|e_i^{\perp}$ denote the orthogonal projection of $K$ on to the one-codimensional subspace $e_i^{\perp}$ perpendicular to $e_i$. Then the following classical Loomis-Whitney inequality (see [31]) states that the $n$-dimensional volume $V_n(K)$ of a compact set $K$ in $\mathbb{R}^n$ is dominated by the geometric mean of $(n-1)$-dimensional volumes $V_{n-1}(K|e_i^{\perp})$ of its coordinate projections on $K|e_i^{\perp}$. That is,

$$V_n(K)^{n-1}\leq\prod_{i=1}^n V_{n-1}(K|e_i^{\perp}) \tag{4.19}$$

with equality if and only if $K$ is a coordinate box (a rectangular parallelepiped whose facets are parallel to the coordinate hyperplanes) in $\mathbb{R}^n$.

We are now ready to prove the following functional Loomis-Whitney inequality.

**Theorem 4.2** *Let $f\in\mathrm{Conc}_1^{(2)}(\mathbb{R}^n)$. If $x=(x_1,\cdots,x_n)$ and $x_{n+1}\in\mathbb{R}$, then for all $(x,x_{n+1})\in\partial K_1(f)$ with $x\in\mathrm{int}(\mathrm{supp}(f))$,*

$$\Big[2\int_{\mathbb{R}^n} f(x)\mathrm{d}x\Big]^n\leq V_n(\overline{\mathrm{supp}(f)})\prod_{i=1}^n\int_{\mathbb{R}^n}|\langle e_i,\nabla f\rangle|\mathrm{d}x, \tag{4.20}$$

*where $\{e_1,\cdots,e_n\}$ is the standard Euclidean basis of $\mathbb{R}^n$.*

**Proof** For the standard Euclidean basis $\{e_1,\cdots,e_n\}$ of $\mathbb{R}^n$, without loss of generality, we assume that $e_1=(1,0,\cdots,0,0,0)$, $\cdots$, $e_n=(0,\cdots,0,1,0)$ and $e_{n+1}=(0,\cdots,0,0,1)$ (since Proposition 4.3).

For $1\leq i\leq n$, by (2.9) and (3.3), we have

$$\begin{aligned}
V_n(K_1(f)|e_i^{\perp})&=\frac{1}{2}\int_{\partial K_1(f)}|\langle e_i,N_{K_1(f)}(x,x_{n+1})\rangle|\mathrm{d}\mu_{K_1(f)}(x,x_{n+1})\\
&=\int_{\mathbb{R}^n} f|\langle e_i,\nabla f\rangle|\frac{1}{|x_{n+1}|}\mathrm{d}x\\
&=\int_{\mathbb{R}^n}|\langle e_i,\nabla f\rangle|\mathrm{d}x.
\end{aligned} \tag{4.21}$$

For $i=n+1$, we have

$$\begin{aligned}V_n(K_1(f)|e_{n+1}^{\perp}) &= \frac{1}{2}\int_{\partial K_1(f)} |\langle e_{n+1}, N_{K_1(f)}(x,x_{n+1})\rangle| \mathrm{d}\mu_{K_1(f)}(x,x_{n+1}) \\ &= \int_{\overline{\mathrm{supp}(f)}} \mathrm{d}x \\ &= V_n(\overline{\mathrm{supp}(f)}). \end{aligned} \tag{4.22}$$

By the definition of $s$-concave function, $\overline{\mathrm{supp}(f)}$ is a convex body in $\mathbb{R}^n$. Hence $V_n(\overline{\mathrm{supp}(f)})$ is finite.

Combining with (2.15), (4.19) and (4.21)–(4.22), we obtain

$$\begin{aligned}\Big[2\int_{\mathbb{R}^n} f(x)\mathrm{d}x\Big]^n &= [V_{n+1}(K_1(f))]^n \\ &\le \prod_{i=1}^{n+1} V_n(K_1(f)|e_i^{\perp}) \\ &= V_n(\overline{\mathrm{supp}(f)})\prod_{i=1}^{n}\int_{\mathbb{R}^n}|\langle e_i, \nabla f\rangle|\mathrm{d}x.\end{aligned}$$

We complete the proof of the functional Loomis-Whitney inequality (4.20).

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