Chin. Ann. Math. Ser. B
44(4), 2023, 481–488
DOI: 10.1007/s11401-023-0026-9

Chinese Annals of Mathematics, Series B

Difference Independence of the Euler Gamma Function*

Qiongyan WANG[1] Xiao YAO[2]

Abstract In this paper, the authors established a sharp version of the difference analogue of the celebrated Hölder's theorem concerning the differential independence of the Euler gamma function Γ. More precisely, if P is a polynomial of $n+1$ variables in $\mathbb{C}[X, Y_0, \cdots, Y_{n-1}]$ such that

$$P(s, \Gamma(s+a_0), \cdots, \Gamma(s+a_{n-1})) \equiv 0$$

for some $(a_0, \cdots, a_{n-1}) \in \mathbb{C}^n$ and $a_i - a_j \notin \mathbb{Z}$ for any $0 \leq i < j \leq n-1$, then they have

$$P \equiv 0.$$

Their result complements a classical result of algebraic differential independence of the Euler gamma function proved by Hölder in 1886, and also a result of algebraic difference independence of the Riemann zeta function proved by Chiang and Feng in 2006.

Keywords Algebraic difference independence, Euler gamma function, Algebraic difference equations
2000 MR Subject Classification 11M06, 39A05

1 Introduction

A classical theorem of Hölder [8] states that the Euler gamma function

$$\Gamma(s) = \int_0^{+\infty} t^{s-1} \mathrm{e}^{-t} \mathrm{d}t, \quad \Re s > 0,$$

which can be analytically continued to the whole complex plane $\mathbb{C}$, does not satisfy any nontrivial algebraic differential equation whose coefficients are polynomials in $\mathbb{C}$. We state it in the following.

Theorem A *Let P be a polynomial of $n+1$ variables in $\mathbb{C}[X, Y_0, \cdots, Y_{n-1}]$. Assume that*

$$P(s, \Gamma(s), \cdots, \Gamma^{(n-1)}(s)) \equiv 0,$$

then we have

$$P \equiv 0.$$

Manuscript received June 25, 2022. Revised November 28, 2022.
[1]School of Mathematical Sciences, Peking University, Beijing 100871, China.
E-mail: qiongyanwang@aliyun.com
[2]Corresponding author. School of Mathematical Sciences, Nankai University, Tianjin 300071, China.
E-mail: yaoxiao@nankai.edu.cn
*This work was supported by the National Natural Science Foundation of China (No. 11901311).

To the best of our knowledge, the Euler gamma function Γ seems to be the first known example which satisfies the algebraic differential independence property in the literature. It is well known that the Riemann zeta function ζ is associated with Γ by the famous Riemann functional equation

$$\zeta(1-s)=2^{1-s}\pi^{-s}\cos\frac{\pi s}{2}\Gamma(s)\zeta(s). \tag{1.1}$$

Motivated by the Riemann functional equation, it is natural to consider the algebraic differential independence property for the Riemann zeta function. The study of the algebraic differential independence of the Riemann zeta function ζ can be dated back to Hilbert. In [7], he conjectured that Hölder's result can be extended to the Riemann zeta function ζ. Later, this conjecture was verified by Ostrowski in [14].

Bank and Kaufman [2–3] made the following celebrated generalizations of Hölder's result.

Theorem B *Let P be a polynomial in $K[X,Y_0,\cdots,Y_{n-1}]$, where K is the field of all meromorphic functions such that the Nevanlinna's characteristic $T(r,f)=o(r)$ as r goes to infinity for any f in K. Assume that*

$$P(s,\Gamma(s),\cdots,\Gamma^{(n-1)}(s))\equiv 0,$$

then we have

$$P\equiv 0.$$

For the Nevanlinna characteristic $T(r,f)$, we refer to Hayman's book (see [6]) for a detailed introduction. Since Γ and ζ appeared very naturally in Riemann functional equation (1.1), Markus in [13] posted an open problem to study the joint algebraic differential independence of Γ and ζ. We refer the readers to the references [9–12] for the recent developments in this direction.

It is interesting to study the algebraic difference independence of ζ or Γ. Feng and Chiang proved the following result.

Theorem C *Let P be a polynomial of $n+1$ variables in $\mathbb{C}[X,Y_0,\cdots,Y_{n-1}]$ and $s_0,\cdots,s_{n-1}$ be n distinct numbers in $\mathbb{C}$. Assume that*

$$P(s,\zeta(s+s_0),\cdots,\zeta(s+s_{n-1}))\equiv 0,$$

then we have

$$P\equiv 0.$$

Chiang and Feng's result extended a result of Ostrowski in [14] where the assumption that $s_0,\cdots,s_{n-1}$ are n distinct real numbers is needed. Indeed, Chiang and Feng proved that Theorem C also holds under the same assumption in Theorem B, we refer the interested readers to [4] for the details. Here, we also mention two remarkable universality results due to Voronin in 1970s for the differential case in [16] and the difference case in [17]. We refer to [15] for the detailed introduction of the recent developments in this direction.

To the best of our knowledge, the topic of the algebraic difference independence of the Euler gamma function was first addressed by Hardouin in [5] in the framework of difference Galois theory. Motivated by the multiplication theorem of Euler gamma function

$$\Gamma(ns) = n^{ns-\frac{1}{2}}(2\pi)^{\frac{1-n}{2}}\prod_{j=0}^{n-1}\Gamma\Big(s+\frac{j}{n}\Big), \tag{1.2}$$

Hardouin proved the following result.

Theorem D (see [5]) *Let $a_0, \cdots, a_{n-1}$ be n complex numbers in $\mathbb{C}$, $b_0, \cdots, b_{m-1} (\geq 2)$ be m integers such that $\{a_j (\mathrm{mod}\ 1)\}_{j=0}^{n-1}$ and $\Big\{\sum\limits_{l=0}^{b_j-1}\frac{l}{b_j}(\mathrm{mod}\ 1)\Big\}_{j=0}^{m-1}$ are $\mathbb{Z}$-linearly independent. Assume that*

$$P(s, \Gamma(s+a_0), \cdots, \Gamma(s+a_{n-1}), \Gamma(b_0 s), \cdots, \Gamma(b_{m-1}s)) \equiv 0$$

for some polynomial P, then we have

$$P \equiv 0.$$

Hardouin's proof relies on Kolchin's type theorem in an essential way. See also in [1] for a detailed discussion of Kolchin's type theorem and several powerful applications in algebraic independence problems.

Our starting point is another well known difference equation of Γ,

$$\Gamma(s+1) = s\Gamma(s). \tag{1.3}$$

This may be the obvious obstruction for us to study the algebraic difference independence of the Euler gamma function Γ. One can not expect to obtain Theorem B for Γ directly. While in this paper, we will show that the machinery exhibited in (1.3) is the only obstruction to get the algebraic difference independence of Γ. Now, we state our main result in the following. In this paper, we will use an elementary method inspired by [8, 14] to prove our main result, which avoids the advanced difference Galois theory. This may be of independent interest.

We define

$$\mathcal{H} := \{(a_0, \cdots, a_{n-1}) \in \mathbb{C}^n : a_i - a_j \notin \mathbb{Z} \text{ for any } 0 \leq i < j \leq n-1\}. \tag{1.4}$$

Now, we state our main result in the following.

Theorem 1.1 *Let P be a polynomial of $n+1$ variables in $\mathbb{C}[X, Y_0, \cdots, Y_{n-1}]$. Assume that*

$$P(s, \Gamma(s+a_0), \cdots, \Gamma(s+a_{n-1})) \equiv 0$$

for some $(a_0, \cdots, a_{n-1}) \in \mathcal{H}$, then we have

$$P \equiv 0.$$

We remark that we can also use Theorem D to recover part of the result of Theorem 1.1 under the same condition of $(a_j)_{j=0}^{n-1}$ and also $m = 0$ in Theorem D. While, it can not completely recover Theorem 1.1, since the condition in Theorem 1.1 is sharp. Our result complements the classical result of algebraic differential independence of Euler gamma function proved by Hölder [8] in 1886, and also a result of algebraic difference independence of Riemann zeta function proved by Chiang and Feng [4] in 2006.

Corollary 1.1 *Let P be a polynomial of $n+1$ variables in $\mathbb{C}[X, Y_0, \cdots, Y_{n-1}]$. Assume that*

$$P(s, \Gamma(s), \cdots, \Gamma(s+(n-1)\alpha)) \equiv 0$$

for some $\alpha \notin \mathbb{Q}$, then we have

$$P \equiv 0.$$

Remark 1.1 Theorem 1.1 can be seen as a difference version of the Hölder's theorem. The identity (1.3) shows that the discussion restricted to $\mathcal{H}$ is necessary.

We can also extend Theorem 1.1 to the setting of $K[X, Y_0, \cdots, Y_{n-1}]$ where K is the field of all meromorphic functions such that the Nevanlinna's characteristic $T(r, f) = o(r)$ as r goes to infinity for any f in K. While, we will not address it in this paper.

By Theorem 1.1 and the Euclidean's algorithm, it is not hard to give the following two examples.

Example 1.1 Let $P = P(X, Y, Z)$ be a polynomial of 3 variables in $\mathbb{C}[X, Y, Z]$. Assume that

$$P(s, \Gamma(s+a_0), \Gamma(s+a_1)) \equiv 0,$$

then

$$P \equiv 0,$$

unless $a_1 - a_0 \in \mathbb{Z}$. In the latter case, if $\Re a_0 < \Re a_1$, P can be divided by the polynomial $R(X, Y, Z) = Z - (X+a_0)\cdots(X+a_1-1)Y$.

Example 1.2 Let $P(X, Y, Z, W) = YW - Z^2 - YZ$ in $\mathbb{C}[X, Y, Z, W]$. We have

$$P(s, \Gamma(s), \Gamma(s+1), \Gamma(s+2)) \equiv 0.$$

P belongs to the ideal

$$\langle W - (X+1)Z, Z - XY \rangle$$

generated by $W - (X+1)Z$ and $Z - XY$ in $\mathbb{C}[X, Y, Z, W]$. Furthermore, P can be written by

$$P(X, Y, Z, W) = Y(W - (X+1)Z) + Z(XY - Z).$$

Remark 1.2 Indeed, inspired by Examples 1.1–1.2, we can apply Theorem 1.1 and the Euclidean's algorithm again to give a complete characterization of the following set

$$\mathcal{I} := \{P \in \mathbb{C}[X, Y_0, \cdots, Y_{n-1}] : P(s, \Gamma(s+a_0), \cdots, \Gamma(s+a_{n-1})) \equiv 0\}$$

without any assumption on $a_0, \cdots, a_{n-1}$. While, we will not discuss it in this paper.

2 Proof of Theorem 1.1

In order to prove Theorem 1.1, we need introduce a lexicographic order between any two monomials $Y_0^{i_0}\cdots Y_{n-1}^{i_{n-1}}$ and $Y_0^{j_0}\cdots Y_{n-1}^{j_{n-1}}$ in $\mathbb{C}[Y_0,\cdots,Y_{n-1}]$, which plays an important role in our proof. And this strategty was inspired by Ostrowski's proof of Hölder's classical proof in [14]. It also shares some spirit of Kolchin's type theorem which was used in [1, 5].

We first introduce an order for the n symbols $Y_0,\cdots,Y_{n-1}$,

$$Y_0 \prec Y_1 \prec \cdots \prec Y_{n-1}. \tag{2.1}$$

This can be used to induce a lexicographic order between any two monomials $Y_0^{i_0}\cdots Y_{n-1}^{i_{n-1}}$ and $Y_0^{j_0}\cdots Y_{n-1}^{j_{n-1}}$. We still denote it by $\prec$ to simplify the notation. We define it in the following:

Case 1 $Y_0^{i_0}\cdots Y_{n-1}^{i_{n-1}} = Y_0^{j_0}\cdots Y_{n-1}^{j_{n-1}}$ if $i_k = j_k$ for $k = 0,\cdots,n-1$;

Case 2 $Y_0^{i_0}\cdots Y_{n-1}^{i_{n-1}} \prec Y_0^{j_0}\cdots Y_{n-1}^{j_{n-1}}$ if $i_0 < j_0$ or there exists $1 \le k \le n-1$ such that

$$i_0 = j_0,\cdots,i_{k-1} = j_{k-1}, i_k < j_k;$$

Case 3 $Y_0^{j_0}\cdots Y_{n-1}^{j_{n-1}} \prec Y_0^{i_0}\cdots Y_{n-1}^{i_{n-1}}$ can be defined similarly as in Case 2.

For any nonzero polynomial $P = P(X,Y_0,\cdots,Y_{n-1})$ in $\mathbb{C}[X,Y_0,\cdots,Y_{n-1}]$, we write it by

$$P = \sum_{i=(i_0,\cdots,i_{n-1})} \Phi_i(X) Y_0^{i_0}\cdots Y_{n-1}^{i_{n-1}}, \tag{2.2}$$

where $\Phi_i(X) \in \mathbb{C}[X]$ and $\Phi_i(X) \neq 0$. The highest term of P is defined by the maximal element in $\mathcal{T}_P$ with respect to the lexicographic order $\prec$ introduced above, where

$$\mathcal{T}_P := \{Y_0^{i_0}\cdots Y_{n-1}^{i_{n-1}} : \Phi_i(X)\ Y_0^{i_0}\cdots Y_{n-1}^{i_{n-1}} \text{ appeared in (2.2)}\}. \tag{2.3}$$

For any monomial $L = Y_0^{i_0} Y_1^{i_1}\cdots Y_{n-1}^{i_{n-1}}$, we define its degree $\deg(L)$ by

$$\deg(L) := \sum_{k=0}^{n-1} i_k.$$

The height of P is defined by the degree of the highest term of P.

Now, we will prove Theorem 1.1.

Proof Let

$$\mathcal{S} := \{P \in \mathbb{C}[X,Y_0,\cdots,Y_{n-1}] : P(s,\Gamma(s+a_0),\cdots,\Gamma(s+a_{n-1})) \equiv 0\}. \tag{2.4}$$

We will prove Theorem 1.1 by contradiction. We assume that $\mathcal{S} \neq \{0\}$. By our assumption, there exists a nonzero polynomial

$$Q = \sum_{i=(i_0,\cdots,i_{n-1})} \Psi_i(X) Y_0^{i_0}\cdots Y_{n-1}^{i_{n-1}},$$

which is of the lowest height in $\mathcal{S}\backslash\{0\}$ with $\Psi_j(X) Y_0^{j_0}\cdots Y_{n-1}^{j_{n-1}}$ being its highest term for some $j = (j_0,\cdots,j_{n-1})$. Moreover, we also make the following assumption.

Assumption LD The nonzero polynomial $\Psi_j(X)$ appearing in the highest term of Q is also of the lowest degree.

Let

$$T(X, Y_0, \cdots, Y_{n-1}) := Q(X+1, (X+a_0)Y_0, \cdots, (X+a_{n-1})Y_{n-1}). \tag{2.5}$$

Noting that

$$Q(s, \Gamma(s+a_0), \cdots, \Gamma(s+a_{n-1})) \equiv 0,$$

we have

$$T(s, \Gamma(s+a_0), \cdots, \Gamma(s+a_{n-1})) \equiv 0$$

by (1.3). And the highest term of T is $\widehat{\Psi}_j(X)Y_0^{j_0} \cdots Y_{n-1}^{j_{n-1}}$, where

$$\widehat{\Psi}_j(X) := \Psi_j(X+1)(X+a_0)^{j_0} \cdots (X+a_{n-1})^{j_{n-1}}.$$

It follows from the Euclidean's algorithm, there exist two polynomials $R = R(X)$ and $U = U(X)$ in $\mathbb{C}[X]$ such that

$$\widehat{\Psi}_j = R\Psi_j + U,$$

where either $U = 0$ or $0 < \deg U < \deg \Psi_j$. It is easy to see that $\deg R \geq 1$.

We claim that $U = 0$. Otherwise, we know that the polynomial

$$H(X, Y_0, \cdots, Y_{n-1}) := T(X, Y_0, \cdots, Y_{n-1}) - R(X)Q(X, Y_0, \cdots, Y_{n-1})$$

is in $\mathcal{S}$. It follows that the highest term of H is

$$U(X)Y_0^{j_0} \cdots Y_{n-1}^{j_{n-1}}$$

and $0 < \deg U < \deg \Psi_j$. Thus, $H \neq 0$, which contradicts the choice of Q and Assumption LD. Now, we have $U = 0$.

Since $U = 0$, we see that the highest term of H is less than the highest term of Q if $H \neq 0$. This again contradicts our choice of Q. Thus, we get $H = 0$. That is,

$$T(X, Y_0, \cdots, Y_{n-1}) = R(X)Q(X, Y_0, \cdots, Y_{n-1}). \tag{2.6}$$

We first assume that there exists $\beta \notin \Lambda := \{-a_k : 0 \leq k \leq n-1\}$ such that $R(\beta) = 0$. By (2.5)–(2.6), we get

$$Q(\beta+1, (\beta+a_0)Y_0, \cdots, (\beta+a_{n-1})Y_{n-1}) = 0$$

in $\mathbb{C}[Y_0, \cdots, Y_{n-1}]$. This implies that

$$Q(\beta+1, Y_0, \cdots, Y_{n-1}) = \sum_{i=(i_0,\cdots,i_{n-1})} \Psi_i(\beta+1)Y_0^{i_0} \cdots Y_{n-1}^{i_{n-1}} = 0$$

in $\mathbb{C}[Y_0, \cdots, Y_{n-1}]$. Thus, we have

$$\Psi_i(\beta+1) = 0$$

for all i, which implies that each $\Psi_i(X)$ can be divided by $X - \beta - 1$. This contradicts our assumption that Ψ_j is of the lowest degree.

Hence, each root of R lies in Λ. Without loss of generality, we assume that $R(-a_0) = 0$. Thus, we get

$$Q(-a_0 + 1, 0, (a_1 - a_0)Y_1, \cdots, (a_{n-1} - a_0)Y_{n-1}) = 0$$

by (2.5)–(2.6). Recalling that $a_j - a_0 \notin \mathbb{Z}$ for any $j \neq 0$, we have

$$Q(-a_0 + 1, 0, Y_1, \cdots, Y_{n-1}) = 0. \tag{2.7}$$

Taking $X = -a_0 + 1$, $Y_0 = 0$ in (2.5)–(2.6), we get

$$\begin{aligned} &Q(-a_0 + 2, 0, (a_1 - a_0 + 1)Y_1, \cdots, (a_{n-1} - a_0 + 1)Y_{n-1}) \\ = &R(-a_0 + 1)Q(-a_0 + 1, 0, Y_1, \cdots, Y_{n-1}) = 0 \end{aligned}$$

by (2.7). Noting that $a_j - a_0 \notin \mathbb{Z}$ for any $j \neq 0$ again, we obtain

$$Q(-a_0 + 2, 0, Y_1, \cdots, Y_{n-1}) = 0$$

in $\mathbb{C}[Y_0, \cdots, Y_{n-1}]$. By induction, we can prove that for any $m \in \mathbb{N}$,

$$Q(-a_0 + m, 0, Y_1, \cdots, Y_{n-1}) = 0$$

in $\mathbb{C}[Y_0, \cdots, Y_{n-1}]$. It follows by the fundamental theorem of algebra, we get

$$Q(X, 0, Y_1, \cdots, Y_{n-1}) = 0$$

in $\mathbb{C}[X, Y_0, \cdots, Y_{n-1}]$. Thus, we proved that Q can be divided by the monomial Y_0, which contradicts the assumption that Q is of the lowest height in $\mathcal{S}$.

Now, we finish the proof of Theorem 1.1.

Acknowledgement The authors would like to thank the referees for the careful reading of the manuscript.

References

[1] Adamczewski, B., Bell, J. P. and Delaygue, E., Algebraic independence of G-functions and congruences à la Lucas, *Ann. Sci. Éc. Norm. Supér.*, **52**(4), 2019, 515–559.

[2] Bank, S. B. and Kaufman, R. P., A note on Hölder's theorem concerning the gamma function, *Math. Ann.*, **232**(2), 1978, 115–120.

[3] Bank, S. B. and Kaufman, R. P., On differential equations and functional equations, *J. Reine Angew. Math.*, **311**(312), 1979, 31–41.

[4] Chiang, Y. M. and Feng, S. J., Difference independence of the Riemann zeta function, *Acta Arith.*, **125**(4), 2006, 317–329.

[5] Hardouin, C., Hypertranscendance des systèmes aux différences diagonaux, *Compos. Math.*, **144**(3), 2008, 565–581.

[6] Hayman, W. K., Meromorphic Functions, Oxford Mathematical Monographs, Clarendon Press, Oxford, 1964.

[7] Hilbert, D., Mathematical problems, *Bull. Amer. Math. Soc.*, **8**(10), 1902, 437–479.

[8] Hölder, O., Über die eigenschaft der γ-funktion, keiner algebraischen differentialgleichung zu genügen, *Math. Ann.*, **28**(10), 1886, 1–13.

[9] Li, B. Q. and Ye, Z., On differential independence of the Riemann zeta function and the Euler gamma function, *Acta Arith.*, **135**(4), 2008, 333–337.

[10] Li, B. Q. and Ye, Z., Algebraic differential equations concerning the Riemann zeta function and the Euler gamma function, *Indiana Univ. Math. J.*, **59**(4), 2010, 1405–1415.

[11] Li, B. Q. and Ye, Z., Algebraic differential equations with functional coefficients concerning ζ and Γ, *J. Differential Equations*, **260**(2), 2016, 1456–1464.

[12] Liao, L. W. and Yang, C. C., On some new properties of the gamma function and the Riemann zeta function, *Math. Nachr.*, **257**, 2003, 59–66.

[13] Markus, L., Differential independence of Γ and ζ, *J. Dynam. Differential Equations*, **19**(1), 2007, 133–154.

[14] Ostrowski, A., Über Dirichletsche Reihen und algebraische Differentialgleichungen, *Math. Z.*, **8**(3–4), 1920, 241–298.

[15] Steuding, J., Value-distribution of L-function, Lecture Notes in Mathematics, **1877**, Springer-Verlag, Berlin, 2007.

[16] Voronin, S. M., The distribution of the nonzero values of the Riemann ζ-function, *Trudy Mat. Inst. Steklov.*, **128**, 1972, 131–150.

[17] Voronin, S. M., Theorem on the "universality" of the Riemann zeta-function, *Izv. Akad. Nauk SSSR Ser. Mat.*, **39**(3), 1975, 475–486.