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Bochner-Martinelli Formula for Higher Spin Operators of Several $\mathbb{R}^6$ Variables*

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Abstract The higher spin operator of several $\mathbb{R}^6$ variables is an analogue of the $\overline{\partial}$-operator in theory of several complex variables. The higher spin representation of $\mathfrak{so}_6(\mathbb{C})$ is $\odot^k\mathbb{C}^4$ and the higher spin operator $\mathcal{D}_k$ acts on $\odot^k\mathbb{C}^4$-valued functions. In this paper, the authors establish the Bochner-Martinelli formula for higher spin operator $\mathcal{D}_k$ of several $\mathbb{R}^6$ variables. The embedding of $\mathbb{R}^{6n}$ into the space of complex $4n \times 4$ matrices allows them to use two-component notation, which makes the spinor calculus on $\mathbb{R}^{6n}$ more concrete and explicit. A function annihilated by $\mathcal{D}_k$ is called k-monogenic. They give the Penrose integral formula over $\mathbb{R}^{6n}$ and construct many k-monogenic polynomials.

Keywords Higher spin operator, k-Monogenic, Bochner-Martinelli formula, Penrose integral formula
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1 Introduction

After Pertici [17] proved the Hartogs' phenomenon for several quaternionic variables, people began to study the regular function theory of several quaternionic variables, generalizing the theory of several complex variables. So far the k-Cauchy-Fueter complex, the quaternionic counterpart of Dolbeault complex is known explicitly (see [1–2, 5–6] for $k = 1$, [23] and references there in).

More generally, people are interested in constructing the corresponding differential complex for several $\mathbb{R}^n$ variables. Such complexes exist in the stable case, i.e., the number of variables is less than or equal to $\frac{n}{2}$, see [7, 15, 20]. It is interesting to develop the function of regular functions of several $\mathbb{R}^n$ variables. Ren and Wang [19] investigated several octonionic variables.

On the other hands, physicists are interested in 6-dimensional physics, in particular, the superconformal field theory over space-time $\mathbb{R}^{5,1}$ and the 6-dimensional theory of self-dual three-forms, the reduction of which gives us the self-dual string equations in 4-dimensions (see e.g. [14, 16, 21] and references there in). It is well known that the orthogonal Lie algebra is isomorphic to the special linear algebra $\mathfrak{sl}_4(\mathbb{C})$, i.e.,

$$\mathfrak{so}_6(\mathbb{C}) \cong \mathfrak{sl}_4(\mathbb{C}), \tag{1.1}$$

see [11, P. 327]. Then the basic spin representation of $\mathfrak{so}_6(\mathbb{C})$ is isomorphic to $\mathbb{C}^4$ as a $\mathfrak{sl}_4(\mathbb{C})$-module and $\odot^k\mathbb{C}^4$ is the higher spin representation of $\mathfrak{so}_6(\mathbb{C})$. So Kang and Wang [13] defined

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the elliptic version of the massless field operators on $\mathbb{R}^6$ by the imbedding

$$\tau : \mathbb{R}^6 \hookrightarrow \wedge^2\mathbb{C}^4 \subseteq \mathbb{C}^{4\times 4},$$
$$\mathbf{x} = (x^0, x^1, \cdots, x^5) \mapsto \tau_0(\mathbf{x}) := \begin{pmatrix} 0 & \mathbf{i}x^1 + x^6 & x^4 + \mathbf{i}x^5 & x^2 + \mathbf{i}x^3 \\ -\mathbf{i}x^1 - x^6 & 0 & x^2 - \mathbf{i}x^3 & -x^4 + \mathbf{i}x^5 \\ -x^4 - \mathbf{i}x^5 & -x^2 + \mathbf{i}x^3 & 0 & -\mathbf{i}x^1 + x^6 \\ -x^2 - \mathbf{i}x^3 & x^4 - \mathbf{i}x^5 & \mathbf{i}x^1 - x^6 & 0 \end{pmatrix}. \tag{1.2}$$

This embedding is the generalization of the embedding of quaternionic space $\mathbb{H} \hookrightarrow {}^{2\times 2}$, which is based on

$$\mathfrak{so}_4(\mathbb{C}) \cong \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C}). \tag{1.3}$$

Sämann and Wolf [21] and Mason et al. [16] defined the massless field operators on Lorentzian space $\mathbb{R}^{5,1}$ by an embedding $\mathbb{R}^{5,1}$ into $\mathbb{C}^{4\times 4}$, which is the generalization of the embedding of Minkowski space $\mathbb{R}^{3,1}$ into 2×2-Hermitian matrix space. The massless field operators are higher spin operators.

By embedding $\mathbb{H}^n$ into $\mathbb{C}^{2n\times 2}$, Wang defined the k-Cauchy-Fueter operator over $\mathbb{H}^n$ and gave the k-Cauchy-Fueter complex and twistor transform over $\mathbb{H}^n$ by using the twistor method with the advantage of two-component station, see [23]. Motivated by the embedding (1.2), we give an embedding $\tau : \mathbb{R}^{6n} \hookrightarrow \mathbb{C}^{4n\times 4}$:

$$(z^{\alpha B}) := \tau(\mathbf{x}) := \begin{pmatrix} \tau_0(\mathbf{x}_0) \\ \tau_0(\mathbf{x}_1) \\ \cdots \\ \tau_0(\mathbf{x}_{n-1}) \end{pmatrix}, \tag{1.4}$$

$\alpha = 0, 1, \cdots, 4n-1$, $B = 0, 1, 2, 3$, where we denote the element of $\mathbb{R}^{6n}$ by $\mathbf{x} = (\mathbf{x}_0, \mathbf{x}_1, \cdots, \mathbf{x}_{n-1})$ with $\mathbf{x}_l = (x_{6l+1}, x_{6l+2}, \cdots, x_{6l+6})$. In this paper, we use the notation $A, B, C, D \in \{0, 1, 2, 3\}$, $\alpha, \beta \in \{0, 1, \cdots, 4n-1\}$, $j, l, m \in \{0, 1, \cdots, n-1\}$. We denote by $\odot^k\mathbb{C}^4$ the k-th symmetric power of $\mathbb{C}^4$, whose element is 4^k-tuple $(\phi_{A_1\cdots A_k})$ with $\phi_{A_1\cdots A_k} \in \mathbb{C}$, which are invariant under the permutations of subscripts $A_1, \cdots, A_k = 0, 1, 2, 3$.

k-Cauchy-Fueter operator is valued in the higher spin representations of $\mathfrak{so}_4(\mathbb{C})$. Recently, there are many works to construct higher spin operators on higher dimensional Euclidean space and to investigate its function theory (see [8–10] and references there in).

Motivated by the higher spin operators on $\mathbb{R}^6$ (see [13]), we introduce higher spin operators $\mathcal{D}_k$ of several $\mathbb{R}^6$ variables as

$$\mathcal{D}_k : C^\infty(\mathbb{R}^{6n}, \odot^k\mathbb{C}^4) \to C^\infty(\mathbb{R}^{6n}, \mathbb{C}^4 \otimes \odot^{k-1}\mathbb{C}^4), \tag{1.5}$$

with

$$(\mathcal{D}_k\phi)^{\alpha}_{A_2\cdots A_k} := \sum_{B=0}^{3} Z^{\alpha B}\phi_{A_2\cdots A_k B} \tag{1.6}$$

for $k = 1, 2, \cdots$, where $Z^{\alpha B}$ are complex vector fields and the matrix $(Z^{\alpha B})$ is just the embedding matrix (1.2)–(1.4) with the coordinate x^j replaced by ∂_{x^j}, see (2.2) for details. A $\odot^k\mathbb{C}^4$-valued function ϕ on a domain U of $\mathbb{R}^{6n}$ is called k-monogenic on U if

$$\mathcal{D}_k\phi = 0.$$

In this way, we could begin to study regular functions of several $\mathbb{R}^6$ variables.

Integral representation formula, i.e., Bochner-Martinelli integral formula is a fundamental and powerful tool to study holomorphic functions of several complex variables, see [18–19, 24–26] and references there in. For quaternion Siegel upper half-space, we know a better kernel, Cauchy-Szegö kernel reproduces regular functions, see [3–4]. In [23], Wang gave the Bochner-Martinelli type formula for k-Cauchy-Fueter operator. Wang and Ren [22] established the explicit Bochner-Martinelli kernel by using technique of Clifford analysis. They also established Bochner-Martinelli type formula for several octonionic variables, see [19]. Kang and Wang [13] gave Bochner-Martinelli type formula for higher spin massless field operators on $\mathbb{R}^6$. In this paper, we get the Bochner-Martinelli type formula for higher spin operator $\mathcal{D}_k$ on $\mathbb{R}^{6n}$.

Theorem 1.1 (Bochner-Martinelli type formula) *Let U be an open bounded set of $\mathbb{R}^{6n}$ with C^1 boundary. For $\phi \in C(\overline{U}, \odot^k\mathbb{C}^4) \cap C^1(U, \odot^k\mathbb{C}^4)$, $k = 1, 2, \cdots$ and any fixed $\mathbf{x} \in U$, we have*

$$\begin{aligned}\phi_{A_1\cdots A_k}(\mathbf{x}) = \sum_{\alpha}\sum_{B}\int_{\partial U} \mathcal{E}_{\alpha(A_1}(\mathbf{x}-\mathbf{y})\phi_{A_2\cdots A_k)B}(\mathbf{y})\mathrm{d}\sigma^{\alpha B}(\mathbf{y})\\ -\sum_{\alpha}\int_U \mathcal{E}_{\alpha(A_1}(\mathbf{x}-\mathbf{y})\big(\mathcal{D}_k\phi(\mathbf{y})\big)^{\alpha}_{A_2\cdots A_k)}\mathrm{d}v(\mathbf{y}).\end{aligned} \tag{1.7}$$

Wang and Kang introduced the Penrose integral formula and series expansion of k-regular functions over $\mathbb{H}^n$ (see [12]). Sämann and Wolf [21] constructed the Penrose integral formula over the twistor space $\mathbb{CP}^3$. We construct lots of k-monogenic functions by writing down the Penrose integral formula explicitly in terms of nonhomogeneous coordinates.

Let (ω, p) be the coordinates of $\mathbb{C}^{4n+3}$, $\omega \in \mathbb{C}^3, p \in \mathbb{C}^{4n}$. The Penrose type contour integral transformation is

$$\begin{aligned}&(\mathcal{P}f)_{A_1\cdots A_k}(\mathbf{x})\\ &:= \int_{|\omega_1|=1}\mathrm{d}\omega_1\int_{|\omega_2|=1}\mathrm{d}\omega_2\int_{|\omega_3|=1}\mathrm{d}\omega_3 \cdot \omega_{A_1}\cdots\omega_{A_k} f\Big(\omega_2, \omega_3, \omega_4, \cdots, \sum_B z^{\alpha B}\omega_B, \cdots\Big)\end{aligned} \tag{1.8}$$

for f holomorphic on $\mathbb{C}^{4n+3}\backslash\{\omega_1\omega_2\omega_3 = 0\}$. We prove that the Penrose-type integral formula produces k-monogenic functions and find many nontrivial k-monogenic polynomials by choosing suitable holomorphic functions in the formula. Abundance of k-monogenic functions show the theory of several $\mathbb{R}^6$ variables is interesting.

Based on isomorphism (1.1) and (1.3), we can use two-component notation, which makes the spinor calculus on $\mathbb{R}^{4n}$ and $\mathbb{R}^{6n}$ more concrete and explicit. There is no such isomorphism for $\mathfrak{so}_n(\mathbb{C})$ with $n > 6$. That is why we only consider several $\mathbb{R}^6$ variables.

2 The Higher Spin Massless Field Operator $\mathcal{D}_k$ Over $\mathbb{R}^{6n}$

In $\mathbb{R}^6$ (see [13]), we use complex vector fields

$$(\nabla^{AB}) := \begin{pmatrix} 0 & \mathbf{i}\partial_{x^0}+\partial_{x^5} & \partial_{x^3}+\mathbf{i}\partial_{x^4} & \partial_{x^1}+\mathbf{i}\partial_{x^2}\\ -\mathbf{i}\partial_{x^0}-\partial_{x^5} & 0 & \partial_{x^1}-\mathbf{i}\partial_{x^2} & -\partial_{x^3}+\mathbf{i}\partial_{x^4}\\ -\partial_{x^3}-\mathbf{i}\partial_{x^4} & -\partial_{x^1}+\mathbf{i}\partial_{x^2} & 0 & -\mathbf{i}\partial_{x^0}+\partial_{x^5}\\ -\partial_{x^1}-\mathbf{i}\partial_{x^2} & \partial_{x^3}-\mathbf{i}\partial_{x^4} & \mathbf{i}\partial_{x^0}-\partial_{x^5} & 0\end{pmatrix}, \tag{2.1}$$

to define higher spin operators. Here we consider complex vector fields

$$(Z^{\alpha B})_{4n\times 4} := \begin{pmatrix} \vdots \\ Z^{(4l+A)B} \\ \vdots \end{pmatrix}, \tag{2.2}$$

where $l=0,1,\cdots,n-1$, $A,B=0,1,2,3$ and

$$(Z^{(4l+A)B})_{4\times 4} := \begin{pmatrix} 0 & \mathbf{i}\partial_{x_{6l+1}}+\partial_{x_{6l+6}} & \partial_{x_{6l+4}}+\mathbf{i}\partial_{x_{6l+5}} & \partial_{x_{6l+2}}+\mathbf{i}\partial_{x_{6l+3}} \\ -\mathbf{i}\partial_{x_{6l+1}}-\partial_{x_{6l+6}} & 0 & \partial_{x_{6l+2}}-\mathbf{i}\partial_{x_{6l+3}} & -\partial_{x_{6l+4}}+\mathbf{i}\partial_{x_{6l+5}} \\ -\partial_{x_{6l+4}}-\mathbf{i}\partial_{x_{6l+5}} & -\partial_{x_{6l+2}}+\mathbf{i}\partial_{x_{6l+3}} & 0 & -\mathbf{i}\partial_{x_{6l+1}}+\partial_{x_{6l+6}} \\ -\partial_{x_{6l+2}}-\mathbf{i}\partial_{x_{6l+3}} & \partial_{x_{6l+4}}-\mathbf{i}\partial_{x_{6l+5}} & \mathbf{i}\partial_{x_{6l+1}}-\partial_{x_{6l+6}} & 0 \end{pmatrix}. \tag{2.3}$$

Let $\varepsilon_{ABCD}=\varepsilon^{ABCD}$ be the sign of the permutation from $(0,1,2,3)$ to (A,B,C,D). Obviously, we have $\sum\limits_{C,D}\varepsilon_{ABCD}\varepsilon^{CDEF}=2(\delta_A^E\delta_B^F-\delta_A^F\delta_B^E)$. We will use $\varepsilon_{ABCD}=\varepsilon^{ABCD}$ to lower and raise indices:

$$Z_{\alpha B} := \frac{1}{2}\sum_{C,D}\varepsilon_{ABCD}Z^{(4l+C)D}, \quad Z^{\alpha B} := \frac{1}{2}\sum_{C,D}\varepsilon^{ABCD}Z_{(4l+C)D}, \tag{2.4}$$

when $\alpha=4l+A$. By (2.2) and (2.4), we have

$$(Z_{\alpha B}) = (\overline{Z^{\alpha B}}). \tag{2.5}$$

The following three propositions generalize the corresponding results in 6 dimensional case in [13].

Proposition 2.1 *For $z^{\alpha B}$ defined by (1.4) and the operator $Z_{\beta D}$ defined by (2.5), we have*

$$Z_{(4m+C)D}z^{(4l+A)B} = 2\delta_m^l\big(\delta_C^A\delta_D^B-\delta_C^B\delta_D^A\big). \tag{2.6}$$

Proof When $m\neq l$, $Z_{(4m+C)D}z^{(4l+A)B}=0$ for any $A,B,C,D\in\{0,1,2,3\}$, by using (1.2)–(1.4) and (2.3)–(2.5). Now we consider the case $m=l$. It is direct to check that

$$\begin{aligned} Z_{(4l)1}z^{(4l)1} &= (-\mathbf{i}\partial_{x^{6l+1}}+\partial_{x^{6l+6}})(\mathbf{i}x^{6l+1}+x^{6l+6}) = 2, \\ Z_{(4l)1}z^{(4l+2)3} &= (-\mathbf{i}\partial_{x^{6l+1}}+\partial_{x^{6l+6}})(-\mathbf{i}x^{6l+1}+x^{6l+6}) = 0. \end{aligned}$$

So we have $Z_{(4l)1}z^{(4l+1)0}=-2$ and $Z_{(4l)1}z^{(4l+3)2}=0$ by $z^{(4l+A)B}=-z^{(4l+B)A}$. And we have $Z_{(4l)1}z^{(4l+A)B}=0$ for other case, since z^{AB} is independent of x^{6l+1} and x^{6l+6}. Hence we have

$$Z_{(4l)1}z^{(4l+A)B} = 2(\delta_0^A\delta_1^B-\delta_0^B\delta_1^A),$$

i.e., (2.6) holds for $(C,D)=(0,1)$. Similarly, when $(C,D)\neq(0,1)$, (2.6) also holds. The proposition is proved.

Proposition 2.2 *For operator $Z^{\alpha B}$ and $Z_{\alpha B}$ defined by (2.2)–(2.3) and (2.5), we have*

$$\sum_{\alpha} Z_{\alpha B_1}Z^{\alpha B_2} = \delta_{B_1}^{B_2}\Delta, \tag{2.7}$$

where $\Delta=\sum\limits_{i=1}^{6n}\partial_{x_i}^2$ is the Lapalace operator on $\mathbb{R}^{6n}$.

Proof When $B_1 \neq B_2$, we have

$$\sum_{\alpha} Z_{\alpha B_1} Z^{\alpha B_2} = \frac{1}{2} \sum_{l} \sum_{A} \sum_{C,D} \varepsilon_{AB_1CD} Z^{(4l+C)D} Z^{(4l+A)B_2}$$

by definition (2.4). Note that $Z^{(4l+A)B_2} = 0$ if $A = B_2$. Thus $\varepsilon_{AB_1CD} \nabla^{(4l+A)B_2} \neq 0$ if and only if C or $D = B_2$. So we have

$$\begin{aligned}\sum_{\alpha} Z_{\alpha B_1} Z^{\alpha B_2} &= \frac{1}{2} \sum_{l} \sum_{A} \Big(\sum_{C} \varepsilon_{AB_1CB_2} Z^{(4l+C)B_2} + \sum_{D} \varepsilon_{AB_1B_2D} Z^{(4l+B_2)D} \Big) Z^{(4l+A)B_2} \\ &= \frac{1}{2} \sum_{l} \sum_{A} \Big(\sum_{C} \varepsilon_{AB_1CB_2} Z^{(4l+C)B_2} + \sum_{D} \varepsilon_{AB_1DB_2} Z^{(4l+D)B_2} \Big) Z^{(4l+A)B_2} \\ &= \sum_{l} \sum_{A} \sum_{C} \varepsilon_{AB_1CB_2} Z^{(4l+C)B_2} Z^{(4l+A)B_2} = 0\end{aligned}$$

by $\varepsilon_{AB_1B_2D} = -\varepsilon_{AB_1DB_2}$ and $Z^{B_2D} = -Z^{DB_2}$. When $B_1 = B_2$, we have

$$\sum_{\alpha} Z_{\alpha B} Z^{\alpha B} = \sum_{\alpha} \overline{Z^{\alpha B}} Z^{\alpha B}$$

by using (2.5). It is directed to check that

$$\sum_{A} \overline{Z^{(4l+A)B}} Z^{(4l+A)B} = \partial^2_{x^{6l+1}} + \partial^2_{x^{6l+6}} + \partial^2_{x^{6l+4}} + \partial^2_{x^{6l+5}} + \partial^2_{x^{6l+2}} + \partial^2_{x^{6l+3}}$$

for fixed $l \in \{0, 1, \cdots, n-1\}$ and $B \in \{0, 1, 2, 3\}$. So we have

$$\sum_{\alpha} \overline{Z^{\alpha B}} Z^{\alpha B} = \Delta.$$

The proposition is proved.

Define the inner products

$$(\phi, \psi) := \sum_{A_1, A_2, \cdots, A_k} \int_{\mathbb{R}^{6n}} \phi_{A_1A_2\cdots A_k}(\mathbf{y}) \overline{\psi_{A_1A_2\cdots A_k}(\mathbf{y})} \mathrm{d}v(\mathbf{y}), \quad \phi, \psi \in C_0^\infty(\mathbb{R}^{6n}, \mathbb{C}^{4n} \otimes \odot^{k-1}\mathbb{C}^4)$$

and

$$(\psi, \Psi) := \sum_{\alpha} \sum_{A_2, \cdots, A_k} \int_{\mathbb{R}^{6n}} \psi^{\alpha}_{A_2\cdots A_k}(\mathbf{y}) \overline{\Psi^{\alpha}_{A_2\cdots A_k}(\mathbf{y})} \mathrm{d}v(\mathbf{y}), \quad \psi, \Psi \in C_0^\infty(\mathbb{R}^{6n}, \mathbb{C}^{4n} \otimes \odot^{k-1}\mathbb{C}^4).$$

Let $L^2(\mathbb{R}^{6n}, \odot^k\mathbb{C}^4)$ and $L^2(\mathbb{R}^{6n}, \mathbb{C}^{4n} \otimes \odot^{k-1}\mathbb{C}^4)$ be the completion of $C_0^\infty(\mathbb{R}^{6n}, \odot^k\mathbb{C}^4)$ and $C_0^\infty(\mathbb{R}^{6n}, \mathbb{C}^{4n} \otimes \odot^{k-1}\mathbb{C}^4)$, respectively. Let $\mathcal{D}_k^*$ be the adjoint of the operator $\mathcal{D}_k$ in (1.5) as

$$(\mathcal{D}_k\phi, \psi) = (\phi, \mathcal{D}_k^*\psi)$$

for any $\phi \in C_0^\infty(\mathbb{R}^{6n}, \odot^k\mathbb{C}^4), \psi \in C_0^\infty(\mathbb{R}^{6n}, (\mathbb{C}^4)^* \otimes \odot^{k-1}\mathbb{C}^4)$.

We will use primed symmetrization of indices

$$\phi_{\cdots(A_1\cdots A_k)\cdots} := \frac{1}{k!} \sum_{\sigma \in S_k} \phi_{\cdots A_{\sigma(1)}\cdots A_{\sigma(k)}\cdots}, \tag{2.8}$$

where S_k is the group of permutations of k letters.

For any $\phi,\psi\in L^2(\mathbb{R}^{6n},\odot^k\mathbb{C}^4)$, by their symmetrization, we have

$$\sum_{A_1,\cdots,A_k}\phi_{A_1A_2\cdots A_k}\overline{\psi_{A_1A_2\cdots A_k}}=\sum_{A_1,\cdots,A_k}\phi_{A_1A_2\cdots A_k}\overline{\psi_{(A_1A_2\cdots A_k)}},\quad \text{a.e..} \tag{2.9}$$

Proposition 2.3 *For any $\phi\in C_0^\infty(\mathbb{R}^{6n},\odot^k\mathbb{C}^4)$, $k=1,2,\cdots$, we have*

$$\mathcal{D}_k^*\mathcal{D}_k\phi=-\Delta\phi,$$

where Δ is the Laplace operator on $\mathbb{R}^{6n}$.

Proof For $\phi\in C_0^\infty(\mathbb{R}^{6n},\odot^k\mathbb{C}^4)$ and $\psi\in C_0^\infty(\mathbb{R}^{6n},\mathbb{C}^{4n}\otimes\odot^{k-1}\mathbb{C}^4)$, we have

$$\begin{aligned}(\mathcal{D}_k\phi,\psi)&=\sum_{\alpha}\sum_{A_2,\cdots,A_k}\int_{\mathbb{R}^{6n}}\sum_{A_1}Z^{\alpha A_1}\phi_{A_2\cdots A_kA_1}\cdot\overline{\psi^{\alpha}_{A_2\cdots A_k}}\mathrm{d}v(\mathbf{y})\\&=-\sum_{A_1,\cdots,A_k}\int_{\mathbb{R}^{6n}}\phi_{A_1A_2\cdots A_k}\cdot\overline{\sum_{\alpha}Z_{\alpha A_1}\psi^{\alpha}_{A_2\cdots A_k}}\mathrm{d}v(\mathbf{y})\\&=-\sum_{A_1,\cdots,A_k}\int_{\mathbb{R}^{6n}}\phi_{A_1A_2\cdots A_k}\cdot\overline{\sum_{\alpha}Z_{\alpha(A_1}\psi^{\alpha}_{A_2\cdots A_k)}}\mathrm{d}v(\mathbf{y})=(\phi,\mathcal{D}_k^*\psi),\end{aligned}$$

which follows from Stokes' formula. Here we use (2.9) in the third identity. Note that

$$(\mathcal{D}_k^*\psi)_{A_1\cdots A_k}=-\sum_{\alpha}Z_{\alpha(A_1}\psi^{\alpha}_{A_2\cdots A_k)}.$$

We get

$$\begin{aligned}(\mathcal{D}_k^*\mathcal{D}_k\phi)_{A_1\cdots A_k}&=-\frac{1}{k!}\sum_{\alpha}\sum_{\sigma\in S_k}Z_{\alpha A_{\sigma(1)}}\Big(\sum_{B}Z^{\alpha B}\phi_{\alpha_{\sigma(2)}\cdots A_{\sigma(k)}B}\Big)\\&=-\frac{1}{k!}\sum_{\sigma\in S_k}\sum_{B}\delta^{B}_{A_{\sigma(1)}}\Delta\phi_{A_{\sigma(2)}\cdots A_{\sigma(k)}B}=-\frac{1}{k!}\sum_{\sigma\in S_k}\Delta\phi_{A_{\sigma(2)}\cdots A_{\sigma(k)}A_{\sigma(1)}}\\&=-\Delta\phi_{A_1A_2\cdots A_k}\end{aligned}$$

by using (1.6) and (2.7). Then $(\mathcal{D}_k^*\mathcal{D}_k)\phi=-\Delta\phi$. The proposition is proved.

3 The Bochner-Martinelli Integral Formula

It is directed to check that $K(\mathbf{y}):=\frac{C}{|\mathbf{y}|^{6n-2}}$ is the fundamental solution of the Laplacian operator Δ on $\mathbb{R}^{6n}$, where $C=\frac{(3n)!}{12n(1-3n)\pi^{3n}}$ and $|\mathbf{y}|=(y_1^2+y_2^2+\cdots+y_{6n}^2)^{\frac12}$. Define the Bochner-Martinelli kernel

$$\mathcal{E}_{\alpha B}(\mathbf{y}):=\overline{Z^{\alpha B}}K(\mathbf{y})=Z_{\alpha B}K(\mathbf{y}),$$

which is a $\mathbb{C}^{4n}\otimes\odot^k\mathbb{C}^4$ valued function. Define the $(6n-1)$-form

$$\mathrm{d}\widehat{y^{\alpha}}:=i_{\partial_{y^\alpha}}\mathrm{d}v(\mathbf{y}),\quad \mathrm{d}\sigma^{\alpha B}(\mathbf{y}):=i_{Z^{\alpha B}}\mathrm{d}v(\mathbf{y}), \tag{3.1}$$

where i is the interior product and $\mathrm{d}v(\mathbf{y}):=\mathrm{d}y^1\wedge\mathrm{d}y^2\wedge\cdots\wedge\mathrm{d}y^{6n}$ is the standard volume form on $\mathbb{R}^{6n}$. Obviously, we have

$$i_{(c_1\partial_{y^{\beta_1}}+c_2\partial_{y^{\beta_2}})}\mathrm{d}v(\mathbf{y})=c_1\mathrm{d}\widehat{y^{\beta_1}}+c_2\mathrm{d}\widehat{y^{\beta_2}},\quad \beta_1,\beta_2\in\{1,\cdots,6n\}.$$

Then $\mathrm{d}\sigma^{\alpha B}(\mathbf{y})$ in (3.1) can be rewritten as

$$(\mathrm{d}\sigma^{\alpha B}(\mathbf{y})) = \begin{pmatrix} \vdots \\ \mathrm{d}\sigma^{(4l+A)B}(\mathbf{y}_l) \\ \vdots \end{pmatrix},$$

where $\mathbf{y}_l \in \mathbb{R}^6$, $l = 0, 1, \cdots, n-1$, $A, B = 0, 1, 2, 3$ and

$$(\mathrm{d}\sigma^{(4l+A)B}(\mathbf{y}_l)) = \begin{pmatrix} 0 & \mathbf{i}\widehat{\mathrm{d}y^{6l+1}}+\widehat{\mathrm{d}y^{6l+6}} & \widehat{\mathrm{d}y^{6l+4}}+\mathbf{i}\widehat{\mathrm{d}y^{6l+5}} & \widehat{\mathrm{d}y^{6l+2}}+\mathbf{i}\widehat{\mathrm{d}y^{6l+3}} \\ -\mathbf{i}\widehat{\mathrm{d}y^{6l+1}}-\widehat{\mathrm{d}y^{6l+6}} & 0 & \widehat{\mathrm{d}y^{6l+2}}-\mathbf{i}\widehat{\mathrm{d}y^{6l+3}} & -\widehat{\mathrm{d}y^{6l+4}}+\mathbf{i}\widehat{\mathrm{d}y^{6l+5}} \\ -\widehat{\mathrm{d}y^{6l+4}}-\mathbf{i}\widehat{\mathrm{d}y^{6l+5}} & -\widehat{\mathrm{d}y^{6l+2}}+\mathbf{i}\widehat{\mathrm{d}y^{6l+3}} & 0 & -\mathbf{i}\widehat{\mathrm{d}y^{6l+1}}+\widehat{\mathrm{d}y^{6l+6}} \\ -\widehat{\mathrm{d}y^{6l+2}}-\mathbf{i}\widehat{\mathrm{d}y^{6l+3}} & \widehat{\mathrm{d}y^{6l+4}}-\mathbf{i}\widehat{\mathrm{d}y^{6l+5}} & \mathbf{i}\widehat{\mathrm{d}y^{6l+1}}-\widehat{\mathrm{d}y^{6l+6}} & 0 \end{pmatrix}.$$

This matrix is just (1.2) with x^i replaced by $\widehat{\mathrm{d}y^{6l+i}}$.

Lemma 3.1 *For a domain U in $\mathbb{R}^{6n}$ and $u, w \in C^1(U, \mathbb{C})$, we have*

$$\mathrm{d}(u \cdot w \mathrm{d}\sigma^{\alpha B}(\mathbf{y})) = (Z^{\alpha B} u \cdot w + u \cdot Z^{\alpha B} w) \mathrm{d}v(\mathbf{y}). \tag{3.2}$$

Proof For fixed $j_1, j_2 \in \{1, \cdots, 6n\}$, we have

$$\sum_{i=1}^{6n} \partial_{y^i} h \cdot \mathrm{d}y^i \wedge (i_{c_1 \partial_{y^{j_1}} + c_2 \partial_{y^{j_2}}} \mathrm{d}v(\mathbf{y})) = (c_1 \partial_{y^{j_1}} + c_2 \partial_{y^{j_2}}) h \cdot \mathrm{d}v(\mathbf{y}). \tag{3.3}$$

Thus for $h(\mathbf{y}) \in C^1(U)$, we have

$$\mathrm{d}(h \cdot \mathrm{d}\sigma^{\alpha B}) = \sum_{i=1}^{6n} \partial_{y^i} h \cdot \mathrm{d}y^i \wedge \mathrm{d}\sigma^{\alpha B} = \sum_{i=1}^{6n} \partial_{y^i} h \cdot \mathrm{d}y^i \wedge i_{Z^{\alpha B}} \mathrm{d}v(\mathbf{y}) = Z^{\alpha B} h \cdot \mathrm{d}v(\mathbf{y})$$

by (3.3). Then (3.2) follows.

Lemma 3.2 *Let U be an bounded domain of $\mathbb{R}^{6n}$ with C^1 boundary ∂U and $\Phi \in C(\overline{U}, \odot^k \mathbb{C}^4)$ $\cap C^1(U, \odot^k \mathbb{C}^4)$. For a fixed point $\mathbf{x} \notin U$ and fixed $A_1, \cdots, A_k \in \{0, 1, 2, 3\}$, we have*

$$\begin{aligned} &\int_{\partial U} \sum_\alpha \sum_B \mathcal{E}_{\alpha A_1}(\mathbf{x}-\mathbf{y}) \Phi_{A_2 \cdots A_k B}(\mathbf{y}) \mathrm{d}\sigma^{\alpha B}(\mathbf{y}) \\ &= \sum_\alpha \int_U \mathcal{E}_{\alpha A_1}(\mathbf{x}-\mathbf{y}) (\mathcal{D}_k \Phi(\mathbf{y}))^\alpha_{A_2 \cdots A_k} \mathrm{d}v(\mathbf{y}). \end{aligned} \tag{3.4}$$

Proof Apply Lemma 3.1 to $u(\cdot) = Z_{AA_1} K(\mathbf{x} - \cdot), w(\cdot) = \Phi_{A_2 \cdots A_k B}(\cdot)$ to get

$$\begin{aligned} &\int_{\partial U} \mathrm{d}_{\mathbf{y}} \Big[\sum_\alpha \sum_B \mathcal{E}_{\alpha A_1}(\mathbf{x}-\mathbf{y}) \Phi_{A_2 \cdots A_k B}(\mathbf{y}) \mathrm{d}\sigma^{\alpha B}(\mathbf{y}) \Big] \\ &= \sum_\alpha \sum_B \int_U [-Z^{\alpha B} Z_{\alpha A_1} K(\mathbf{x}-\mathbf{y}) \cdot \Phi_{A_2 \cdots A_k B}(\mathbf{y}) + Z_{\alpha A_1} K(\mathbf{x}-\mathbf{y}) \cdot Z^{\alpha B} \Phi_{A_2 \cdots A_k B}(\mathbf{y})] \mathrm{d}v(\mathbf{y}) \\ &= \sum_B \int_U -\delta^B_{A_1} \Delta K(\mathbf{x}-\mathbf{y}) \cdot \Phi_{A_2 \cdots A_k B}(\mathbf{y}) \mathrm{d}v(\mathbf{y}) + \sum_\alpha \int_U \mathcal{E}_{\alpha A_1}(\mathbf{x}-\mathbf{y}) (\mathcal{D}_k \Phi(\mathbf{y}))^\alpha_{A_2 \cdots A_k} \mathrm{d}v(\mathbf{y}) \\ &= \sum_\alpha \int_U \mathcal{E}_{\alpha A_1}(\mathbf{x}-\mathbf{y}) (\mathcal{D}_k \Phi(\mathbf{y}))^\alpha_{A_2 \cdots A_k} \mathrm{d}v(\mathbf{y}) \end{aligned}$$

by $\sum\limits_\alpha Z^{\alpha B} Z_{\alpha A_1} = \sum\limits_\alpha Z_{\alpha A_1} Z^{\alpha B} = \delta^B_{A_1} \Delta$ and $\Delta K(\mathbf{x}-\mathbf{y}) = 0$ for $\mathbf{x} \notin U$.

We also need the identity corresponding to (3.4) for $x \in U$.

Lemma 3.3 *Let U be an bounded domain in $\mathbb{R}^{6n}$ with C^1 boundary ∂U and $\phi \in C(\overline{U}, \odot^k \mathbb{C}^4) \cap C^1(U, \odot^k \mathbb{C}^4)$. Then we have*

$$\int_{\partial U} \sum_{\alpha} \sum_{B} \mathcal{E}_{\alpha(A_1}(\mathbf{x}-\mathbf{y})\phi_{A_2\cdots A_k)B}(\mathbf{x})\mathrm{d}\sigma^{\alpha B}(\mathbf{y}) = \begin{cases} \phi_{A_1\cdots A_k}(\mathbf{x}), & \text{if } \mathbf{x} \in U, \\ 0, & \text{if } \mathbf{x} \notin \overline{U}. \end{cases} \tag{3.5}$$

Proof For a fixed point $\mathbf{x} \notin \overline{U}$, applying Lemma 3.2 to $\Phi(\mathbf{y}) \equiv \phi(\mathbf{x})$ to get

$$\int_{\partial U} \sum_{\alpha} \sum_{B} \mathcal{E}_{\alpha A_1}(\mathbf{x}-\mathbf{y})\phi_{A_2\cdots A_k B}(\mathbf{x})\mathrm{d}\sigma^{\alpha B}(\mathbf{y}) = 0$$

by $\mathcal{D}_k \Phi(\mathbf{y}) \equiv 0$. It implies (3.5) holds for $\mathbf{x} \notin \overline{U}$ by symmetrization.

When $\mathbf{x} \in U$, let $B(\mathbf{x}, \varepsilon) = \{\mathbf{y} \in \mathbb{R}^{6n} : |\mathbf{y}-\mathbf{x}| \leq \varepsilon\} \subseteq U$ with ε sufficiently small. Applying Lemma 3.2 to $\Phi(\mathbf{y}) \equiv \phi(\mathbf{x})$ and the domain $U \setminus B(\mathbf{x}, \varepsilon)$, we have

$$\int_{\partial U \setminus \partial B(\mathbf{x},\varepsilon)} \sum_{\alpha} \sum_{B} \mathcal{E}_{\alpha A_1}(\mathbf{x}-\mathbf{y})\phi_{A_2\cdots A_k B}(\mathbf{x})\mathrm{d}\sigma^{\alpha B}(\mathbf{y}) = 0,$$

i.e.,

$$\begin{aligned} &\int_{\partial U} \sum_{\alpha} \sum_{B} \mathcal{E}_{\alpha A_1}(\mathbf{x}-\mathbf{y})\phi_{A_2\cdots A_k B}(\mathbf{x})\mathrm{d}\sigma^{\alpha B}(\mathbf{y}) \\ &= \int_{\partial B(\mathbf{x},\varepsilon)} \sum_{\alpha} \sum_{B} \mathcal{E}_{\alpha A_1}(\mathbf{x}-\mathbf{y})\phi_{A_2\cdots A_k B}(\mathbf{x})\mathrm{d}\sigma^{\alpha B}(\mathbf{y}). \end{aligned} \tag{3.6}$$

Obviously, we have $Z_{\alpha B}|\mathbf{x}-\mathbf{y}|^2 = -2\overline{\iota(\mathbf{x}-\mathbf{y})}_{\alpha B}$. Then the right hand side of (3.6) equals to

$$\begin{aligned} &\sum_{\alpha} \sum_{B} \phi_{A_2\cdots A_k B}(\mathbf{x}) \int_{\partial B(x,\varepsilon)} \frac{(-3n+1)C}{\varepsilon^{6n}} Z_{\alpha A_1}|\mathbf{x}-\mathbf{y}|^2 \mathrm{d}\sigma^{\alpha B}(\mathbf{y}) \\ &= \sum_{\alpha} \sum_{B} \phi_{A_2\cdots A_k B}(\mathbf{x}) \int_{\partial B(\mathbf{x},\varepsilon)} \frac{(3n)!}{12n\pi^{3n}\varepsilon^{6n}} Z_{\alpha A_1}|\mathbf{x}-\mathbf{y}|^2 \mathrm{d}\sigma^{\alpha B}(\mathbf{y}) \\ &= \sum_{\alpha} \sum_{B} \phi_{A_2\cdots A_k B}(\mathbf{x}) \int_{B(\mathbf{x},\varepsilon)} \frac{(3n)!}{12n\pi^{3n}\varepsilon^{6n}} Z^{\alpha B} Z_{\alpha A_1}|\mathbf{x}-\mathbf{y}|^2 \mathrm{d}v(\mathbf{y}) \\ &= \sum_{B} \phi_{A_2\cdots A_k B}(\mathbf{x}) \int_{B(\mathbf{x},\varepsilon)} \frac{(3n)!}{12n\pi^{3n}\varepsilon^{6n}} \delta_{A_1}^{B} \Delta|\mathbf{x}-\mathbf{y}|^2 \mathrm{d}v(\mathbf{y}) \\ &= \frac{(3n)!}{\pi^{3n}\varepsilon^{6n}} |B(\mathbf{x},\varepsilon)| \phi_{A_2\cdots A_k A_1}(\mathbf{x}) = \phi_{A_1\cdots A_k}(\mathbf{x}) \end{aligned} \tag{3.7}$$

by using Stokes' formula and (3.2) in the first identity, where $|B(\mathbf{x},\varepsilon)| = \frac{\pi^{3n}\varepsilon^{6n}}{(3n)!}$ is the volume of the ball $B(\mathbf{x},\varepsilon)$ in $\mathbb{R}^{6n}$. The lemma is proved.

Proof of Theorem 1.1 For any fixed $x \in U$, applying Lemma 3.2 to $\Phi(\mathbf{y}) = \phi(\mathbf{y})$ and the domain $U \setminus B(\mathbf{x}, \varepsilon)$ to get

$$\begin{aligned} &\Big\{ \int_{\partial U} - \int_{\partial B(\mathbf{x},\varepsilon)} \Big\} \sum_{\alpha} \sum_{B} \mathcal{E}_{\alpha A_1}(\mathbf{x}-\mathbf{y})\phi_{A_2\cdots A_k B}(\mathbf{y})\mathrm{d}\sigma^{\alpha B} \\ &= \sum_{\alpha} \int_{U \setminus B(\mathbf{x},\varepsilon)} \mathcal{E}_{\alpha A_1}(\mathbf{x}-\mathbf{y})(\mathcal{D}_k \phi(\mathbf{y}))^{\alpha}_{A_2\cdots A_k} \mathrm{d}v. \end{aligned}$$

By symmetrizing indices $A_1, \cdots, A_k$, we have

$$\Big\{\int_{\partial U} - \int_{\partial B(\mathbf{x},\varepsilon)}\Big\} \sum_{\alpha}\sum_{B} \mathcal{E}_{\alpha(A_1}(\mathbf{x}-\mathbf{y})\phi_{A_2\cdots A_k)B}(\mathbf{y})\mathrm{d}\sigma^{\alpha B}$$
$$= \sum_{\alpha} \int_{U\setminus B(\mathbf{x},\varepsilon)} \mathcal{E}_{\alpha(A_1}(\mathbf{x}-\mathbf{y})(\mathcal{D}_k\phi(\mathbf{y}))^{\alpha}_{A_2\cdots A_k)}\mathrm{d}v. \tag{3.8}$$

Since $Z^{AB}\phi(\mathbf{y})_{A_2\cdots A_k B}$ is locally bounded and $|\mathcal{E}_{AB}(\mathbf{x}-\mathbf{y})| = O(|\mathbf{x}-\mathbf{y}|^{-6n+1})$, we find that

$$\lim_{\varepsilon\to 0}\sum_{\alpha}\int_{B(\mathbf{x},\varepsilon)} \mathcal{E}_{\alpha A_1}(\mathbf{x}-\mathbf{y})(\mathcal{D}_k\phi(\mathbf{y}))^{\alpha}_{A_2\cdots A_k}\mathrm{d}v(\mathbf{y}) = 0.$$

On the other hand,

$$\lim_{\varepsilon\to 0}\sum_{\alpha}\sum_{B}\int_{\partial B(\mathbf{x},\varepsilon)} \mathcal{E}_{\alpha A_1}(\mathbf{x}-\mathbf{y})[\phi_{A_2\cdots A_k B}(\mathbf{y}) - \phi_{A_2\cdots A_k B}(\mathbf{x})]\mathrm{d}\sigma^{\alpha B}(\mathbf{y}) = 0,$$

since $|f(\mathbf{y}) - f(\mathbf{x})| \le \|f\|_{C^1}|\mathbf{x}-\mathbf{y}|$ for any $\mathbf{y}\in\partial B(\mathbf{x},\varepsilon)$ and $f\in C^1(\overline{B(\mathbf{x},\varepsilon)})$.

This together with Lemma 3.3 and symmetrizing the indices $A_1, \cdots, A_k$ imply that

$$\lim_{\varepsilon\to 0}\int_{\partial B(\mathbf{x},\varepsilon)}\sum_{\alpha}\sum_{B} \mathcal{E}_{\alpha(A_1}(\mathbf{x}-\mathbf{y})\phi_{A_2\cdots A_k)B}(\mathbf{y})\mathrm{d}\sigma^{\alpha B}(\mathbf{y}) = \phi_{A_1\cdots A_k}(\mathbf{x}).$$

The theorem is proved by letting $\varepsilon\to 0$ in (3.8).

4 The *k*-Monogenic Functions and Penrose Integral Formula

Sämann and Wolf [21] have established the Penrose integral formula in terms of homogeneous coordinates of $\mathbb{CP}^7$. We can get the contour integral formula (1.8) over $\mathbb{R}^{6n}$ by taking nonhomogeneous coordinates of $\mathbb{CP}^3$. We check that $\mathcal{P}f$ satisfies the equation $\mathcal{D}_k(\mathcal{P}f) = 0$ by direct differentiation.

Theorem 4.1 *For any holomorphic function f on $\mathbb{C}^{4n+3}\setminus\{\omega_1\omega_2\omega_3 = 0\}$, the $\odot^k\mathbb{C}^4$-valued function $\mathcal{P}f$ on $\mathbb{R}^{6n}$ given by (1.8) is k-monogenic.*

Proof Take $p^{\alpha} := \sum\limits_{B} z^{\alpha B}\omega_B$ with $\omega_0 = 1$. When $\alpha = 4l + A$,

$$Z_{(4m+C)D}p^{\alpha} = \sum_{B} Z_{(4m+C)D}z^{(4l+A)B}\omega_B$$
$$= 2\sum_{B}\delta^l_m(\delta^A_C\delta^B_D - \delta^B_C\delta^A_D)\omega_B = 2\delta^l_m(\delta^A_C\omega_D - \delta^A_D\omega_C) \tag{4.1}$$

by (2.6). For a fixed $A_2, \cdots, A_k, B$, differentiate the contour integral formula (1.8) to get

$$Z_{(4m+C)D}(\mathcal{P}f)_{A_2\cdots A_k B}(\mathbf{x})$$
$$= \int_{|\omega_1|=1}\mathrm{d}\omega_1\int_{|\omega_2|=1}\mathrm{d}\omega_2\int_{|\omega_3|=1}\mathrm{d}\omega_3\cdot\omega_{A_2}\cdots\omega_{A_k}\omega_B\sum_{\alpha}\frac{\partial f}{\partial p^{\alpha}}Z_{(4m+C)D}p^{\alpha}$$
$$= 2\int_{|\omega_1|=1}\mathrm{d}\omega_1\int_{|\omega_2|=1}\mathrm{d}\omega_2\int_{|\omega_3|=1}\mathrm{d}\omega_3\cdot\omega_{A_2}\cdots\omega_{A_k}\omega_B\Big(\frac{\partial f}{\partial p^{4m+C}}\omega_D - \frac{\partial f}{\partial p^{4m+D}}\omega_C\Big) \tag{4.2}$$

by (4.1). For fixed $A_2, \cdots, A_k$, set

$$H_{AB(4m+C)}(\mathbf{x}) := \int_{|\omega_1|=1} \mathrm{d}\omega_1 \int_{|\omega_2|=1} \mathrm{d}\omega_2 \int_{|\omega_3|=1} \mathrm{d}\omega_3 \cdot \omega_{A_2} \cdots \omega_{A_k} \omega_A \omega_B \frac{\partial f}{\partial p^{4m+C}}.$$

Obviously, we have

$$H_{AB(4m+C)}(\mathbf{x}) = H_{BA(4m+C)}(\mathbf{x}).$$

We can rewrite (4.2) as

$$\nabla_{(4m+C)D}(\mathcal{P}f)_{A_2 \cdots A_k B}(\mathbf{x}) = 2(H_{BD(4m+C)}(\mathbf{x}) - H_{BC(4m+D)}(\mathbf{x})). \tag{4.3}$$

Then we get

$$\begin{aligned}(\mathcal{D}_k(\mathcal{P}f))^{4m+A}_{A_2\cdots A_k} &= \sum_B Z^{(4m+A)B}(\mathcal{P}f)_{A_2\cdots A_k B}(\mathbf{x}) = \sum_{B,C,D} \frac{1}{2}\varepsilon^{ABCD} Z_{(4m+C)D}(\mathcal{P}f)_{A_2\cdots A_k B}(\mathbf{x}) \\ &= \sum_{B,C,D} \varepsilon^{ABCD}(H_{BD(4m+C)}(\mathbf{x}) - H_{BC(4m+D)}(\mathbf{x}))\end{aligned}$$

by raising indices and using (2.4), (4.3) in the second and third identity, respectively. Since ε^{ABCD} is antisymmetric in indices and $H_{AB(4m+C)}$ is symmetric in A and B, we have

$$\sum_{B,D} \varepsilon^{ABCD} H_{BD(4m+C)} = 0 \quad \text{and} \quad \sum_{B,C} \varepsilon^{ABCD} H_{BC(4m+D)} = 0.$$

So $(\mathcal{D}_k(\mathcal{P}f))^{\alpha}_{A_2\cdots A_k} = 0$, i.e., $(\mathcal{P}f)(\mathbf{x})$ is k-monogenic on $\mathbb{R}^{6n}$.

Now we give some concrete k-monogenic functions.

Corollary 4.1 *For fixed nonnegative integers q_1, q_2, q_3 and m_j, $j = 0, 1, \cdots, n-1$, the $\odot^k\mathbb{C}^4$-valued polynomial ϕ with*

$$\phi_{A_1\cdots A_k}(\mathbf{x}) = \begin{cases} \displaystyle\sum_{\substack{\alpha_0+\alpha_1+\cdots+\alpha_{n-1}+a_1=q_1 \\ \beta_0+\beta_1+\cdots+\beta_{n-1}+a_2=q_2 \\ \gamma_0+\gamma_1+\cdots+\gamma_{n-1}+a_3=q_3 \\ \alpha_j+\beta_j+\gamma_j=m_j, 0\le\alpha_j,\beta_j\gamma_j\le m_j}} \prod_{j=0}^{n-1} \frac{m_j!}{\alpha_j!\beta_j!\gamma_j!}(z^{(4j)1})^{\alpha_j}(z^{(4j)2})^{\beta_j}(z^{(4j)3})^{\gamma_j}, & \textit{when } |\mathbf{a}| = |\mathbf{q}| - |\mathbf{m}| \\ 0, \quad \textit{else} \end{cases} \tag{4.4}$$

is k-monogenic, where a_h is the number of h in $\{A_1, \cdots, A_k\}$, $h = 1, 2, 3$, $|\mathbf{a}| = a_1 + a_2 + a_3$, $|\mathbf{q}| = q_1 + q_2 + q_3$, $|\mathbf{m}| = \sum\limits_{j=0}^{n-1} m_j$.

Proof Choose

$$f(\omega_1, \omega_2, \omega_3, \cdots, p^{\alpha}, \cdots) := \frac{\prod\limits_{j=0}^{n-1}(p^{4j})^{m_j}}{\omega_1^{q_1+1}\omega_2^{q_2+1}\omega_3^{q_3+1}} \tag{4.5}$$

holomorphic on $\mathbb{C}^{4n+3}\backslash\{\omega_1\omega_2\omega_3 = 0\}$ for fixed nonnegative integers q_1, q_2, q_3 and m_j, $j = 0, 1, \cdots, n-1$. Note that

$$\int_{|\omega_h|=1} \omega_h^s d\omega_h = \begin{cases} 2\pi\mathbf{i}, & \text{if } s = -1, \\ 0, & \text{otherwise} \end{cases}$$

for $h = 1, 2, 3$. Recall that $\omega_0 \equiv 1$. Substitute f in (4.5) into the Penrose type contour integral formula (1.8) to get

$$
\begin{aligned}
(\mathcal{P}f)_{A_1\cdots A_k}(\mathbf{x}) &= \int_{|\omega_1|=1} \mathrm{d}\omega_1 \int_{|\omega_2|=1} \mathrm{d}\omega_2 \int_{|\omega_3|=1} \mathrm{d}\omega_3 \cdot \omega_{A_1}\cdots\omega_{A_k} \frac{\prod_{j=0}^{n-1}(p^{4j})^{m_j}}{\omega_1^{q_1+1}\omega_2^{q_2+1}\omega_3^{q_3+1}} \\
&= \int_{|\omega_1|=1} \mathrm{d}\omega_1 \int_{|\omega_2|=1} \mathrm{d}\omega_2 \int_{|\omega_3|=1} \mathrm{d}\omega_3 \cdot \omega_1^{a_1}\omega_2^{a_2}\omega_3^{a_3} \prod_{j=0}^{n-1}\Big(\sum_{\substack{\alpha_j+\beta_j+\gamma_j=m_j \\ 0\le\alpha_j,\beta_j,\gamma_j\le m_j}} \frac{m_j!}{\alpha_j!\beta_j!\gamma_j!} \\
&\quad \cdot \frac{\big(z^{(4j)1}\omega_1\big)^{\alpha_j}\big(z^{(4j)2}\omega_2\big)^{\beta_j}\big(z^{(4j)3}\omega_3\big)^{\gamma_j}}{\omega_1^{q_2+1}\omega_2^{q_2+1}\omega_3^{q_3+1}}\Big) \\
&= (2\pi\mathbf{i})^3 \sum_{\substack{\alpha_0+\alpha_1+\cdots+\alpha_{n-1}+a_1=q_1 \\ \beta_0+\beta_1+\cdots+\beta_{n-1}+a_2=q_2 \\ \gamma_0+\gamma_1+\cdots+\gamma_{n-1}+a_3=q_3 \\ \alpha_j+\beta_j+\gamma_j=m_j,0\le\alpha_j,\beta_j\gamma_j\le m_j}} \prod_{j=0}^{n-1} \frac{m_j!}{\alpha_j!\beta_j!\gamma_j!}\big(z^{(4j)1}\big)^{\alpha_j}\big(z^{(4j)2}\big)^{\beta_j}\big(z^{(4j)3}\big)^{\gamma_j}
\end{aligned}
$$

only if $|\mathbf{a}| = |\mathbf{q}| - |\mathbf{m}|$ and $q_j - a_j \ge 0$ $(j = 1, 2, 3)$. Otherwise, it vanishes. By Theorem 4.1, we know $(\mathcal{P}f)(\mathbf{x})$ is k-monogenic on $\mathbb{R}^{6n}$. The corollary is proved.

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