Chin. Ann. Math. Ser. B
44(4), 2023, 501–516
DOI: 10.1007/s11401-023-0028-7

Chinese Annals of Mathematics, Series B

Equicontinuity and Sensitivity of Group Actions*

Shaoting XIE[1] Jiandong YIN[1]

Abstract Let (X, G) be a dynamical system (G-system for short), that is, X is a topological space and G is an infinite topological group continuously acting on X. In the paper, the authors introduce the concepts of Hausdorff sensitivity, Hausdorff equicontinuity and topological equicontinuity for G-systems and prove that a minimal G-system (X, G) is either topologically equicontinuous or Hausdorff sensitive under the assumption that X is a T_3-space and they provide a classification of transitive dynamical systems in terms of equicontinuity pairs. In particular, under the condition that X is a Hausdorff uniform space, they give a dichotomy theorem between Hausdorff sensitivity and Hausdorff equicontinuity for G-systems admitting one transitive point.

Keywords Hausdorff sensitivity, Hausdorff equicontinuity, Topological equicontinuity, Even continuity
2000 MR Subject Classification 17B40, 17B50

1 Introduction

For a discrete dynamical system (X, f), which means that X is a compact metric space with some metric d and $f : X \to X$ is a continuous mapping, the concept of equicontinuity is frequently used in the description of the complexity of discrete dynamical systems (see [19, 24]). Recall that (X, f) is equicontinuous if for each $\varepsilon > 0$, there exists $\delta >$ such that $d(f^n(x), f^n(y)) < \varepsilon$ for all $x, y \in X$ with $d(x, y) < \delta$. From the definition of equicontinuity, it is easy to see that every equicontinuous system admits simple orbit behaviors. Sensitivity as the opposite-side of equicontinuity is also an important concept of discrete dynamical systems as it usually becomes a kernel component of some chaos such as Devaney's chaos (see [8, 12–13, 18, 20, 22, 31]). The Auslander-Yorke Dichotomy Theorem connecting these two concepts is as follows.

Theorem 1.1 (Auslander-Yorke Dichotomy Theorem) *A minimal discrete dynamical system* (*the orbit of each point is dense in the underlying space*) *is either equicontinuous or sensitive.*

Up to now, there are many other forms of equicontinuity and sensitivity, such as mean equicontinuity, density-equicontinuity, density-sensitivity and multi-sensitivity and various cor-

Manuscript received January 20, 2022. Revised June 8, 2022.
[1]Department of Mathematics, Nanchang University, Nanchang 330031, China.
E-mail: xst15970503289@163.com yjdaxf@163.com
*This work was supported by the National Natural Science Foundation of China (Nos. 12061043, 11661054).

responding analogues of the Auslander-Yorke Dichotomy Theorem have been proved (see [14, 17, 20–21]).

Recently, there has appeared a trend to study the complexity of dynamical systems in the setting that the underlying space is unnecessarily metrizable or compact, namely, it is just a topological space. The notion of uniformity on a nonempty set which induces a topology attracts widespread attention of many authors and there are various results on equicontinuity and sensitivity of a continuous mapping on a topological space with a compatible uniformity (see [10, 28] and see Section 3 for the details of uniformity). For example, the authors in [16] introduced the notion of even continuity and obtained that if $\mathcal{F}$ is an evenly continuous family of mappings from a topological space X to a regular space Y and $\mathfrak{C}$ is the topology of pointwise convergence, then the $\mathfrak{C}$-closure $\overline{\mathcal{F}}$ of $\mathcal{F}$ is evenly continuous and $\mathfrak{C}$ is jointly continuous of $\overline{\mathcal{F}}$ (see [6, 23, 26] for more results on even continuity). The notion of topological equicontinuity was introduced in [27] and the authors proved that if $\mathcal{F}$ is a family of mappings from a topological space X to a topological space Y and is topologically equicontinuous at $x \in X$, then, given $K \subset O \subset Y$ with K compact and O open, there is a neighborhood U of x such that $f(U) \subset O$ whenever $f \in \mathcal{F}$ and $f(U)$ meets K. And in [6], the authors pointed out that for topological spaces X and Y with a fixed compatible quasi-uniformity Q in Y and for a family $\mathcal{F}$ of mappings from X to Y, the notions of even continuity, topological equicontinuity and Q-equicontinuity (i.e., equicontinuity with respect to the topology of X and Q) are compared. Moreover, it was shown that Q-equicontinuity implies even continuity, and if Q is locally symmetric, it implies topological equicontinuity too. The definitions of equicontinuity, topological equicontinuity and even continuity were proved to be equivalent in [6, 27] if the involved space is compact. In [10], the authors made a classification of even continuity and topological equicontinuity of a dynamical system and introduced the concept of eventual sensitivity under the condition that the underlying space is a compact Hausdorff space. Besides, the authors gave a dichotomy between eventual sensitivity and equicontinuity, that is, a transitive dynamical system is either equicontinuous or eventually sensitive. The concept of Hausdorff sensitivity was introduced by Good and Macías in [11] and it was proved that sensitivity and Hausdorff sensitivity coincide if the underlying space is a compact Hausdorff space. Moreover, the authors proved in [10] that a minimal dynamical system (X, f) is either topological equicontinuous or Hausdorff sensitive if X is a T_3-space. The concept of Hausdorff equicontinuity was introduced by Wang in [28] and it was proved that the notions of Hausdorff equicontinuity and equicontinuity, Hausdorff sensitivity and sensitivity are mutually equivalent if X is a compact Hausdorff space.

The study of group actions on metric spaces (sometimes, topological spaces) is an extremely interesting topic of topological dynamical systems and ergodic theory. For instance, in [25], the author introduced the concepts of sensitivity and equicontinuity for a group action on a compact metric space and proved that such an action is transitive if it admits an ergodic invariant probability measure with full support and it is either minimal and equicontinuous or sensitive. In [7], the authors gave some applications of topological equicontinuity and even continuity to topologized semigroups and topologized groupoids acting on compact metric spaces. It was turned out in [7] that a semitopological group is a paratopological group if and only if it has the evenly continuous family of right translations (see [7, Theorem 5.2]). In this paper, we prove that the main results of [10] are also true under the framework that G is an infinite topological group which continuously acts on a Hausdorff uniform space X whose topology is induced by a uniformity on X.

The paper is organized as follows. In Section 2, we give some basic notions of this paper. In Section 3, we introduce the notions of Hausdorff equicontinuity and Hausdorff sensitivity of group actions and we give a dichotomy theorem about Hausdorff equicontinuity and Hausdorff sensitivity. Finally, in Section 4, we study the topological equicontinuity and even continuity and obtain some properties about them. Besides, we give the Auslander-Yorke Dichotomy Theorem between topological equicontinuity and Hausdorff sensitivity and a classification of topologically transitive dynamical systems in terms of equicontinuity pairs.

2 Preliminaries of G-Systems

Throughout the paper, by a dynamical system (X, G) (G-system for short) we mean that X is a topological space, G is an infinite topological group and $\Gamma : G \times X \to X$ defined as $(g, x) \mapsto gx$ for all $g \in G$ and $x \in X$, is a continuous mapping satisfying:

(1) $\Gamma(e, x) = x$ for each $x \in X$, where e is the identity element of G;

(2) $\Gamma(g_1, \Gamma(g_2, x)) = \Gamma(g_1 g_2, x)$ for all $g_1, g_2 \in G$ and $x \in X$.

Each $g \in G$ can be regarded as a homeomorphism from X to itself if there is no confusion. For more details of G-systems, we refer the readers to see [2].

Let (X, G) be a G-system and $x \in X$. The orbit of x under the action of G is denoted by $Gx = \{gx : g \in G\}$. A subset Λ of X is G-invariant if $gx \in \Lambda$ for each $x \in \Lambda$ and each $g \in G$, i.e., $G\Lambda = \Lambda$. Let $U, V \subset X$, define the hitting time set of U and V by

$$N(U, V) = \{g \in G : gU \cap V \neq \emptyset\}$$

and the recurrence time set of x entering U by

$$N(x, U) = \{g \in G : gx \in U\}.$$

A point $y \in X$ is called an ω-limit point of x if $N(x, U)$ is infinite for every neighborhood U of y. The collection of all ω-limit points of x is called the ω-limit set of x and we denote it by $\omega_G(x)$. It is easy to show that $\omega_G(x)$ is G-invariant for each $x \in X$ and $\omega_G(x) = \omega_G(gx)$ for each $g \in G$. Moreover, if X is compact, then $\omega_G(x) \neq \emptyset$ for each $x \in X$ by the Cantor's intersection theorem (see [27]).

For $x \in X$, the closure of the orbit Gx of x under G in X, denoted by $\overline{Gx}$, is the union of Gx and $\omega_G(x)$, i.e., $\overline{Gx} = Gx \cup \omega_G(x)$. The interior of a subset A of X is denoted by $\mathrm{int}_X(A)$.

A point $x \in X$ is called a transitive point of (X, G) if $\overline{Gx} = X$. We denote the set of transitive points of (X, G) by $\mathrm{Trans}(X, G)$. The non-wandering set of x, denoted by $\Omega_G(x)$, is defined as: $y \in \Omega_G(x)$ if and only if for each neighborhood U of x and each neighborhood V of y and every $F \in \mathrm{Fin}(G)$, there exists $g \in G - F := \{g : g \in G, g \notin F\}$, such that $gU \cap V \neq \emptyset$, where $\mathrm{Fin}(G)$ denotes the collection of all finite subsets of G. Clearly, for each $x \in X$, $\omega_G(x) \subseteq \Omega_G(x)$.

A G-system(X, G) is called

(i) transitive if $N(U, V) \neq \emptyset$ for each pair of nonempty open sets U and V;

(ii) minimal if $\overline{Gx} = X$ for every $x \in X$; equivalently, there is no proper nonempty closed G-invariant subset of X.

A point $x \in X$ is called minimal if the subset system $(\overline{Gx}, G)$ of (X, G) is minimal.

Lemma 2.1 (see [5]) *Let (X, G) be a G-system, where X is a Hausdorff space without isolated points. Then (X, G) is transitive if and only if $N(U, V)$ is infinite for each pair of nonempty open subsets U and V of X.*

Under the assumption that X is a Hausdorff space without isolated points, for a transitive system (X, G), it is easy to derive that $\Omega_G(x) = X$ for every $x \in X$ by Lemma 2.1.

For sake of convenience, we denote by $\mathcal{N}_x$ the collection of all neighborhoods of x in X.

3 Hausdorff Equicontinuity and Hausdorff Sensitivity of *G*-Systems

3.1 Uniform spaces

Let X be a nonempty set. Denote the diagonal of $X \times X$ by $\triangle = \{(x, x) : x \in X\}$. Given two subsets A and B of $X \times X$, the inverse of A is denoted by $A^{-1} = \{(y, x) : (x, y) \in A\}$ and the composition of A and B is defined as $A \circ B = \{(x, z) :$ There exists $y \in X$ such that $(x, y) \in A$ and $(y, z) \in B\}$. Use nA to denote $\overbrace{A \circ A \circ \cdots \circ A}^{n \text{ times}}$. If $A \subseteq X \times X$ contains the diagonal $\triangle$, then we call the set A an entourage of the diagonal $\triangle$.

Definition 3.1 (see [30]) *A uniformity $\mathcal{D}$ on a nonempty set X is a collection of entourages of the diagonal satisfying the following conditions*:

(1) $D_1, D_2 \in \mathcal{D} \Rightarrow D_1 \cap D_2 \in \mathcal{D}$.

(2) $D \in \mathcal{D}, D \subseteq E \Rightarrow E \in \mathcal{D}$.

(3) $D \in \mathcal{D} \Rightarrow E \circ E \subseteq D$ *for some* $E \in \mathcal{D}$.

(4) $D \in \mathcal{D} \Rightarrow E^{-1} \subseteq D$ *for some* $E \in \mathcal{D}$.

Meanwhile, we call the pair $(X, \mathcal{D})$ a uniform space. $\mathcal{D}$ is separating if $\bigcap\limits_{D \in \mathcal{D}} D = \triangle$, at the same time, we say that X is separated. A sub-collection $\mathcal{E}$ of $\mathcal{D}$ is called a base for $\mathcal{D}$ if for every $D \in \mathcal{D}$, there exists $E \in \mathcal{E}$ such that $E \subseteq D$. Obviously, if $\mathcal{D}$ is separating, then $\bigcap\limits_{E \in \mathcal{E}} E = \triangle$ for each base $\mathcal{E}$ of $\mathcal{D}$. An entourage D of the diagonal $\triangle$ is called symmetric if $D = D^{-1}$.

Clearly, each base $\mathcal{E}$ of a uniformity $\mathcal{D}$ has the following properties:

(1) If $D_1, D_2 \in \mathcal{D}$, then there exists $E \in \mathcal{E}$ such that $E \subseteq D_1 \cap D_2$.

(2) $D \in \mathcal{D} \Rightarrow E \circ E \subseteq D$ for some $E \in \mathcal{E}$.

(3) $D \in \mathcal{D} \Rightarrow E^{-1} \subseteq D$ for some $E \in \mathcal{E}$.

For an entourage $D \in \mathcal{D}$ and a point $x \in X$, write $D[x] = \{y \in X : (x, y) \in D\}$. For $x \in X$, the collection $\mathcal{U}_x = \{D[x] : D \in \mathcal{D}\}$ is a neighborhood base at x which induces a topology of X. This topology can be also induced by a base $\mathcal{E}$ of $\mathcal{D}$ in the same manner. The topology is Hausdorff if and only if $\mathcal{D}$ is separating.

Lemma 3.1 (see [9]) *For a compact Hausdorff space X, there is a unique uniformity $\mathcal{D}$ which induces the topology of X.*

In the following, when we say that (X, G) is a G-system, it always means that X is a Hausdorff space and G is an infinite group continuously acting on X. Let α, β be two open covers of X. We say that β refines α if for each $B \in \beta$, there exists $A \in \alpha$ such that $B \subset A$.

Lemma 3.2 (see [11]) *Let X be a compact Hausdorff space and let $\mathcal{D}$ be the unique uniformity of X that induces the topology of X. Then for every open cover $\mathcal{U}$ of X, there exists a symmetric entourage $V \in \mathcal{D}$ such that $\mathcal{C}_V := \{V[x] : x \in X\}$ refines $\mathcal{U}$.*

Lemma 3.3 (see [11]) *Let X be a compact Hausdorff space and let $\mathcal{D}$ be the unique uniformity of X which induces the topology of X. If $\mathcal{U}$ is an open cover of X, then $\bigcup\limits_{U \in \mathcal{U}} U \times U \in \mathcal{D}$.*

Definition 3.2 (see [29]) *Let X be a compact Hausdorff space and let $\mathcal{D}$ be the unique uniformity of X which induces the topology of X. A G-system (X, G) is said to have sensitive dependence on initial conditions (sensitivity for short) if there exists a symmetric $D \in \mathcal{D}$ such that*

$$N_D(U) = \{g \in G : \textit{There exist } x, y \in U \textit{ such that } (gx, gy) \notin D\} \neq \emptyset$$

for each nonempty open $U \subseteq X$. We refer to such a D as a sensitivity entourage.

Definition 3.3 (see [3]) *Let (X, G) be a G-system, where X is a compact Hausdorff space whose topology is induced by the unique uniformity $\mathcal{D}$ of X. A point $x \in X$ is called an equicontinuity point of (X, G) if for each symmetric entourage $D \in \mathcal{D}$, there exists $U \in \mathcal{N}_x$ such that $(gx, gy) \in D$ for every $g \in G$ and every $y \in U$. Meanwhile, we say that (X, G) is equicontinuous at x.*

Denote the set of all equicontinuity points of (X, G) by $Eq(X, G)$. If (X, G) is equicontinuous at every point of X, then we say that (X, G) is equicontinuous, i.e., for all symmetric $E \in \mathcal{D}$, there exists a symmetric $D \in \mathcal{D}$ such that for all $x, y \in X$, if $(x, y) \in D$, then $(gx, gy) \in E$ for each $g \in G$. Therefore, a G-system (X, G) is equicontinuous if and only if $Eq(X, G) = X$. $x \in X$ is called a sensitive point if $x \notin Eq(X, G)$.

Proposition 3.1 *If X is a compact Hausdorff space and (X, G) is sensitive with a sensitivity entourage $D \in \mathcal{D}$, where $\mathcal{D}$ is the unique compatible uniformity on X, then $N_D(U)$ is infinite for each nonempty open $U \subseteq X$.*

Proof Suppose that there is a nonempty open $U \subseteq X$ such that $N_D(U)$ is finite, say $N_D(U) = \{g_1, g_2, \cdots, g_k\}$. Choose $E \in \mathcal{D}$ such that $2E \subseteq D$ by Definition 3.1(3). Let $x \in U$ and $D' \in \mathcal{D}$ such that for each $y \in X$, if $(x, y) \in D'$, then $(g_i x, g_i y) \in E$ for each $i \in \{1, 2, \cdots, k\}$. Write $W = U \cap D'[x]$, then W is a neighborhood of x. Therefore, $N_D(W) \neq \emptyset$. But $g_i W \subseteq E[g_i x]$ for all $i \in \{1, 2, \cdots, k\}$ which means that if $a, b \in W$, then $(g_i a, g_i b) \in D$ for all $i \in \{1, 2, \cdots, k\}$. Thus there exist $g_n \in G$ and $a_0, b_0 \in W$ such that $(g_n a_0, g_n b_0) \notin D$. Clearly, $g_n \neq g_i$ for each $i \in \{1, 2, \cdots, k\}$ and $g_n \in N_D(W) \subset N_D(U)$. This is contrary to $N_D(U) := \{g_1, g_2, \cdots, g_k\}$.

In the following Definition 3.4, we introduce the concept of Hausdorff sensitivity of G-systems which was given firstly by Good and Macías in [11] for a single continuous self-mapping on a topological space. And we prove that for a G-system, Hausdorff sensitivity and sensitivity are equivalent if the involved space is compact.

Definition 3.4 *A G-system (X, G) is said to be Hausdorff sensitive if there exists a finite open cover $\mathcal{U}$ of X such that for any $x \in X$ and any $V \in \mathcal{N}_x$, there exists $y \in V$ with $x \neq y$ and $g \in G$ such that $\{gx, gy\} \nsubseteq U$ for all $U \in \mathcal{U}$.*

Example 3.1 Let $X = \mathbb{R}$ and $(X, \mathcal{T})$ be a topological space with the topology induced by the usual metric of $\mathbb{R}$. It is easy to verify that $(X, \mathcal{T})$ is a Hausdorff space. Let $A = [0, +\infty)$, $\mathcal{T}_0 = A \cap \mathcal{T}$ and $\mathcal{T}_1 = \mathcal{T} \cup \mathcal{T}_0$. Then $(X, \mathcal{T}_1)$ is also a Hausdorff space since $\mathcal{T} \subset \mathcal{T}_1$. Let $G = \mathbb{R}$ be the additive group of reals. Define $\Gamma : G \times X \to X$ by $\Gamma(r, x) = r + x$ for all $r \in G$, $x \in X$. Then (X, G) is a G-system and it is not hard to see that (X, G) is Hausdorff sensitive if we note that $\mathcal{U} = \{(-\infty, 0), [0, +\infty)\}$ is a finite open cover of X.

Remark 3.1 A G-system (X, G) is Hausdorff sensitive if and only if there exists a finite open cover $\mathcal{U}$ of X such that for each nonempty open set V of X, there exist $x, y \in V$ with $x \neq y$ and $g \in G$ such that $\{gx, gy\} \nsubseteq U$ for all $U \in \mathcal{U}$.

Proposition 3.2 *Under the assumption that X is compact, a G-system (X, G) is sensitive if and only if it is Hausdorff sensitive.*

Proof Suppose that $\mathcal{D}$ is a compatible uniformity of X by Lemma 3.1. Let $V \in \mathcal{D}$ and set $\mathcal{C}_V := \{V[x] : x \in X\}$.

Assume that (X, G) is Hausdorff sensitive. Let $\mathcal{U}$ be a finite open cover of X given by the definition of Hausdorff sensitivity. Since X is compact, by Lemma 3.2, there exists a symmetric $V \in \mathcal{D}$ such that $\mathcal{C}_V$ covering X refines $\mathcal{U}$. Let $x \in X$ and $W \in \mathcal{N}_x$. Then there exist $x' \in W$ and $g \in G$ such that $\{gx, gx'\} \nsubseteq U$ for all $U \in \mathcal{U}$. This implies that $(gx, gx') \notin V$. Otherwise, $\{gx, gx'\} \subset V[gx] \subset U'$ for some $U' \in \mathcal{U}$, a contradiction. Therefore, (X, G) is sensitive.

Suppose that (X, G) is sensitive. Let $V \in \mathcal{D}$ be a sensitivity entourage. Let $V' \in \mathcal{D}$ be symmetric satisfying $2V' \subseteq V$. Since $\mathcal{C}_{V'}$ covers X and X is compact, there exist $x_1, x_2, \cdots, x_n \in X$ such that $\mathcal{U} = \{\mathrm{int}_X(V'[x_j])\}_{j=1}^n$ is a finite open subcover. We show that $\mathcal{U}$ satisfies the definition of Hausdorff sensitivity. Let $x \in X$ and let A be an open subset of X containing x. Then, there exists $W \in \mathcal{D}$ such that $W[x] \subset A$. Hence, there exist $x' \in W[x]$ and $g \in G$ such that $(gx, gx') \notin V$ which indicates that for each $i \in \{1, 2, \cdots, n\}$, $\{gx, gx'\} \nsubseteq V'[x_i]$. Otherwise, there exists $l \in \{1, 2, \cdots, n\}$ such that $\{gx, gx'\} \subseteq V'[x_l]$. Thus $(gx, gx') \in V$, a contradiction. Therefore, for each $i \in \{1, 2, \cdots, n\}$, $\{gx, gx'\} \nsubseteq V'[x_i]$ which shows that (X, G) is Hausdorff sensitive.

In the following Definition 3.5, we introduce the concept of Hausdorff equicontinuity point for G-systems which was given firstly by Wang in [28] for a single continuous self-mapping on a topological space. And we point out that for a G-system, equicontinuity and Hausdorff equicontinuity are equivalent if the involved space is compact.

Definition 3.5 *Let (X, G) be a G-system. A point $x \in X$ is called a Hausdorff equicontinuity point of (X, G) if for each finite open cover $\mathcal{U}$ of X, there exists $V \in \mathcal{N}_x$ such that $\{gx, gy\} \subseteq U$ for some $U \in \mathcal{U}$ whenever $g \in G$ and $y \in V$. In particular, if (X, G) contains a Hausdorff equicontinuity point, then (X, G) is called almost Hausdorff equicontinuous.*

Denote the set of Hausdorff equicontinuity points of (X, G) by $\mathrm{Heq}(X, G)$.

Definition 3.6 *A G-system (X, G) is said to be Hausdorff equicontinuous if for each finite open cover $\mathcal{U}$ of X, there exists a finite open cover $\mathcal{V}$ of X and some $V \in \mathcal{V}$ such that for every $g \in G$ and all $\{x, y\} \subseteq V$, $\{gx, gy\} \subseteq U$ for some $U \in \mathcal{U}$.*

Remark 3.2 If X is compact, then a G-system (X, G) is Hausdorff equicontinuous if and only if each point of X is Hausdorff equicontinuous.

In the following proposition, we prove that for a G-system, equicontinuity and Hausdorff equicontinuity are equivalent if the underlying space X is compact and Hausdorff. But this result is not true if the involved space is not compact, see [28, Example 3.6].

Proposition 3.3 *Under the assumption that X is compact, a G-system (X, G) is equicontinuous if and only if it is Hausdorff equicontinuous.*

Proof Suppose that $\mathcal{D}$ is a compatible uniformity of X by Lemma 3.1.

Assume that (X, G) is equicontinuous. Let $\mathcal{U} = \{U_1, U_2, \cdots, U_m\}$ be a finite open cover of X. Let $D = \bigcup_{i=1}^{m} U_i \times U_i$. Since X is compact, by Lemma 3.3, $D \in \mathcal{D}$. As (X, G) is equicontinuous, there exists a symmetric entourage $D_1 \in \mathcal{D}$ such that $(gx, gy) \in D$ for every $g \in G$ and every $(x, y) \in D_1$. Let $D_2 \in \mathcal{D}$ be symmetric satisfying $2D_2 \subseteq D_1$. Then $\{\mathrm{int}_X(D_2[x]) : x \in X\}$ is an open cover of X. Since X is compact, there exist $x_1, x_1, \cdots, x_n \in X$ such that $\mathcal{U} = \{\mathrm{int}_X(D_2[x_j])\}_{j=1}^{n}$ is a finite subcover of $\{\mathrm{int}_X(D_2[x]) : x \in X\}$. If $\{x, y\} \subseteq \mathrm{int}_X(D_2[x_j])$ for some x_j, then $(x, y) \in D_1$. This implies that $\{gx, gy\} \subseteq U_i$ for some $U_i \in \mathcal{U}$ and every $g \in G$ whenever $\{x, y\} \subseteq \mathrm{int}_X(D_2[x_j])$ for some x_j. Therefore, (X, G) is Hausdorff equicontinuous.

Suppose that (X, G) is Hausdorff equicontinuous. For each symmetric $D \in \mathcal{D}$, there is a symmetric $D_1 \in \mathcal{D}$ satisfying $2D_1 \subseteq D$. Since $\{\mathrm{int}_X(D_1[x]) : x \in X\}$ covers X and X is compact, there exist $x_1, x_1, \cdots, x_n \in X$ such that $\mathcal{U} = \{\mathrm{int}_X(D_1[x_j])\}_{j=1}^{n}$ is a finite subcover of

$\{\mathrm{int}_X(D_1[x]) : x \in X\}$. As (X, G) is Hausdorff equicontinuous, there exists a finite open cover $\mathcal{V}$ of X such that $\{gx, gy\} \subseteq U$ for some $U \in \mathcal{U}$ and every $g \in G$ whenever $\{x, y\} \subseteq V$ for some $V \in \mathcal{V}$. Let $D_2 = \bigcup_{V \in \mathcal{V}} V \times V$. By Lemma 3.3, $D_2 \in \mathcal{D}$. Then whenever $(x, y) \in D_2$, we have $\{gx, gy\} \subseteq U$ for some $U \in \mathcal{U}$. This implies that $(x_j, gx), (x_j, gy) \in D_1$ for some x_j. Hence, $(gx, gy) \in D$ which implies that (X, G) is equicontinuous.

Example 3.2 Let $X = \{a, b, c\}$ be a topological space with the discrete topology. Then X is compact and Hausdorff. Define $f : X \to X$ by $f(a) = b, f(b) = c$ and $f(c) = a$. Clearly, f is a homeomorphism. Let G be the group generated by $\{f, f^{-1}, Id\}$, where Id is the identity map on X. Then (X, G) is a G-system. It is not hard to verify that each point of X is an equicontinuity point of (X, G), by Remark 3.2 and Proposition 3.3, each point of X is a Hausdorff equicontinuity point of (X, G).

Lemma 3.4 *Let (X, G) be a G-system. If there exists a transitive point of (X, G) which is not Hausdorff equicontinuous, then (X, G) is Hausdorff sensitive.*

Proof Suppose that x_0 is a transitive point of (X, G) which is not Hausdorff equicontinuous. Then there is a finite open cover $\mathcal{U}$ of X, for each $V' \in \mathcal{N}_{x_0}$, there exist $g \in G$ and $y \in V'$ such that $\{gx_0, gy\} \nsubseteq U$ for every $U \in \mathcal{U}$. For any $x \in X$ and $V \in \mathcal{N}_x$, there exists $g_1 \in G$ such that $g_1 x_0 \in V$. Noting that $g_1^{-1}V$ is a neighbourhood of x_0. It follows that there exist $g_2 \in G$ and $y \in g_1^{-1}V$ such that $\{g_2 x_0, g_2 y\} \nsubseteq U$ for every $U \in \mathcal{U}$. It can be verified that $\{g_2 g_1^{-1}(g_1 x_0), g_2 g_1^{-1} x\} \nsubseteq U$ or $\{g_2 g_1^{-1} x, g_2 g_1^{-1}(g_1 y)\} \nsubseteq U$. Indeed, suppose on contrary $\{g_2 g_1^{-1}(g_1 x_0), g_2 g_1^{-1} x\} \subseteq U$ and $\{g_2 g_1^{-1} x, g_2 g_1^{-1}(g_1 y)\} \subseteq U$, then $\{g_2 x_0, g_2 y\} \subseteq U$ which is a contradiction. Hence, (X, G) is Hausdorff sensitive.

Theorem 3.1 *Let (X, G) be a G-system. If (X, G) contains a transitive point and (X, G) is almost Hausdorff equicontinuous, then* $\mathrm{Trans}(X, G) = \mathrm{Heq}(X, G)$.

Proof Suppose that $x \in X$ is a transitive point of (X, G) and $z \in X$ is a Hausdorff equicontinuity point of (X, G). For each finite open cover $\mathcal{U}$ of X, there exists $V \in \mathcal{N}_z$ such that $\{gz, gy\} \subseteq U$ for some $U \in \mathcal{U}$ whenever $g \in G$ and $y \in V$. Then, there exists $D \in \mathcal{D}$ such that $D[z] \subset V$. Let $D_1 \in \mathcal{D}$ be symmetric such that $D_1 \circ D_1 \subset D$. As x is a transitive point of (X, G), there exists $g' \in G$ such that $g'x \in D_1[z] \subset D[z]$ which shows that for each $g \in G$, $\{gg'x, gz\} \subset U$. Let $U_1 = (g')^{-1}(D_1[g'x])$, then for any $p \in U_1$, we have $(g'x, g'p) \in D_1$ which together with $(z, g'x) \in D_1$ implies that $(z, g'p) \in D$. Hence, $\{gg'x, gg'p\} \subset U$, which indicates that x is a Hausdorff equicontinuity point.

Suppose that $x \in X$ is a Hausdorff equicontinuity point of (X, G) which is not a transitive point of (X, G), then $\overline{Gx} \subsetneqq X$. Let $x_0 \in X - \overline{Gx}$ be a transitive point of (X, G). As X is regular (see [15, Corollary 8.14]), there exists $V_1 \in \mathcal{N}_{x_0}$ such that $\overline{V_1} \subseteq X - \overline{Gx}$. Then $\mathcal{U} = \{X - \overline{V_1}, X - \overline{Gx}\}$ is a finite open cover of X. As x is a Hausdorff equicontinuity point of (X, G), there exists $V_2 \in \mathcal{N}_x$ such that $\{gx, gy\} \subset X - \overline{V_1}$ or $\{gx, gy\} \subset X - \overline{Gx}$ for all $y \in V_2$ and all $g \in G$. However, for any $g \in G$, $gx \notin X - \overline{Gx}$, then $\{gx, gy\} \nsubseteq X - \overline{Gx}$ for any $y \in V_2$.

As x_0 is a transitive point of (X, G), there exists $g_1 \in G$ such that $g_1 x_0 \in V_2$. It can be verified that $X = \overline{Gx_0} \subseteq X - V_1$ which contradicts that V_1 is nonempty. Hence, x is a transitive point of (X, G).

Theorem 3.2 *If a G-system (X, G) admits at least one transitive point, then one of the following statements exactly holds:*

(1) *If* $\mathrm{Heq}(X, G) \neq \emptyset$*, then* (X, G) *is almost Hausdorff equicontinuous and* $\mathrm{Heq}(X, G) = \mathrm{Trans}(X, G)$;

(2) *if* $\mathrm{Heq}(X, G) = \emptyset$*, then* (X, G) *is Hausdorff sensitive.*

Proof If $\mathrm{Heq}(X, G) \neq \emptyset$ and note that (X, G) has at least one transitive point, then (1) holds by Theorem 3.1. Otherwise, Lemma 3.4 yields that (X, G) is Hausdorff sensitive.

4 Topological Equicontinuity and Even Continuity of *G*-Systems

4.1 Topological equicontinuity of *G*-systems

In this section, we introduce the concept of topological equicontinuity of G-systems, which can be regarded as a generalization of equicontinuity introduced firstly by Royden in [27].

Definition 4.1 *Let (X, G) be a G-system and $x \in X$. $(x, y) \in X \times X$ is called an equicontinuity pair of (X, G) if for each $O \in \mathcal{N}_y$, there exist $U \in \mathcal{N}_x$ and $V \in \mathcal{N}_y$ satisfying: If $g \in N(U, V)$, then $gU \subseteq O$. If $(x, y) \in X \times X$ is an equicontinuity pair of (X, G), then y is called correspondingly an equicontinuity partner of x. (X, G) is called topologically equicontinuous if each element of $X \times X$ is an equicontinuity pair of (X, G).*

Example 4.1 Let $X = \mathbb{R}$ and $(X, \mathcal{T})$ be a topological space with the topology induced by the usual metric of $\mathbb{R}$. It can be verified that $(X, \mathcal{T})$ is a Hausdorff space. Let $G = \mathbb{Z}$ be the integer additive group. Define $\varGamma : G \times X \to X$ by $\varGamma(z, r) = z + r$ for each $z \in G$ and $r \in X$. Then (X, G) is a G-system. It is not hard to verify that each point of $X \times X$ is an equicontinuity pair of (X, G).

Denote the set of equicontinuity pairs of (X, G) by $EqP(X, G)$. Clearly, (X, G) is topologically equicontinuous if and only if $EqP(X, G) = X \times X$. Let $x \in X$. If for all $y \in X$, $(x, y) \in EqP(X, G)$, then we say that (X, G) is topologically equicontinuous at $x \in X$ or x is a topological equicontinuity point of (X, G).

In the rest of this section, for the sake of description, we always assume that X is a Hausdorff space and (X, G) is a G-system.

Proposition 4.1 *If $(x, y) \in EqP(X, G)$, then $(x, gy) \in EqP(X, G)$ for all $g \in G$.*

Proof Let $g \in G$, $O \in \mathcal{N}_{gy}$, then $g^{-1}O \in \mathcal{N}_y$. Since $(x, y) \in EqP(X, G)$, there exist $U \in \mathcal{N}_x$ and $V \in \mathcal{N}_y$ such that for each $g' \in N(U, V)$, $g'U \subseteq g^{-1}O$. Then $gV \in \mathcal{N}_{gy}$. For any $g_1 \in G$ satisfying $g_1 U \cap gV \neq \emptyset$, it has that $g^{-1}g_1 U \cap V \neq \emptyset$. Then $g^{-1}g_1 U \subseteq g^{-1}O$ which means that $g_1 U \subseteq O$. The proof is completed.

Proposition 4.2 *Let $x, y \in X$, $F \in \mathrm{Fin}(G)$, $O \in \mathcal{N}_y$ and $F_1 = \{g \in F : gx = y\}$. Then there exist $U \in \mathcal{N}_x$ and $V \in \mathcal{N}_y$ such that $N(U, V) \cap F = F_1$ and $gU \subseteq V \subseteq O$ for all $g \in F_1$.*

Proof Without loss of generality, assume that $F_1 \neq \emptyset$. If $F_1 = F$, then the result is clear since we can take $V = O$ and $U = \bigcap_{g \in F} g^{-1}O$.

If $F_1 \neq F$, for each $g \in F - F_1$, $gx \neq y$. By the Hausdorff property of X, pick $U_{gx} \in \mathcal{N}_{gx}$ and $V_g \in \mathcal{N}_y$ such that $U_{gx} \cap V_g = \emptyset$. Take

$$V = \Big(\bigcap_{g \in F - F_1} V\Big) \cap O, \quad U = \Big(\bigcap_{g \in F - F_1} g^{-1}U_{gx}\Big) \cap \Big(\bigcap_{g \in F_1} g^{-1}V\Big).$$

Then $U \in \mathcal{N}_x$ and $V \in \mathcal{N}_y$ are desired.

Corollary 4.1 *Let $x, y \in X$ and $F \in \mathrm{Fin}(G)$. Then for every $O \in \mathcal{N}_y$, there exist $U \in \mathcal{N}_x$ and $V \in \mathcal{N}_y$ such that if $g \in N(U, V) \cap F$, then $gU \subseteq O$.*

Proof The result follows obviously from Proposition 4.2.

Proposition 4.3 *Let $x, y \in X$. If $y \notin \Omega_G(x)$, then $(x, y) \in EqP(X, G)$.*

Proof Let $O \in \mathcal{N}_y$, as $y \notin \Omega_G(x)$, there exist $U \in \mathcal{N}_x$, $V \in \mathcal{N}_y$ and $F \in \mathrm{Fin}(G)$ such that for each $g \in G - F$, $gU \cap V = \emptyset$. By Corollary 4.1, there exist $U' \in \mathcal{N}_x$ and $V' \in \mathcal{N}_y$ such that for any $g \in F$, if $gU' \cap V' \neq \emptyset$, then $gU' \subseteq O$. Assume $U' \subseteq U$ and $V' \subseteq V \cap O$, then for every $g \in G - F$, $gU' \cap V' = \emptyset$ and thus the result follows from the arbitrariness of $O \in \mathcal{N}_y$.

By Definition 4.1 and Proposition 4.3, if $(x, y) \notin EqP(X, G)$, then there exists $O \in \mathcal{N}_y$ such that for all $U \in \mathcal{N}_x$ and $V \in \mathcal{N}_y$, there exists $g \in N(U, V)$ satisfying $gU \nsubseteq O$. Meanwhile, O is called a splitting neighborhood of y with regard to x.

Definition 4.2 *Let $x, y \in X$ and $(x, y) \in EqP(X, G)$. (x, y) is called a trivial equicontinuity pair of (X, G) if $y \notin \Omega_G(x)$. Otherwise, (x, y) is called a nontrivial equicontinuity pair of (X, G).*

Proposition 4.4 *Each $(x, y) \in EqP(X, G)$ is either a trivial equicontinuity pair of (X, G) or $y \in \omega_G(x)$.*

Proof Assume that (x, y) is a nontrivial equicontinuity pair of (X, G) and $y \notin \omega_G(x)$, then there exist $O \in \mathcal{N}_y$ and $F \in \mathrm{Fin}(G)$ such that for all $g \in G - F$, $gx \notin O$. However, $y \in \Omega_G(x)$, and thus for each $U \in \mathcal{N}_x$ and $V \in \mathcal{N}_y$, there exists $g \in G - F$ such that $gU \cap V \neq \emptyset$. Then there exists $g \in N(U, V)$ such that $gU \nsubseteq O$ which is in contradiction with $(x, y) \in EqP(X, G)$.

Remark 4.1 Proposition 4.4 shows that $(x, y) \in X \times X$ is a nontrivial equicontinuity pair of (X, G) if and only if $(x, y) \in EqP(X, G)$ and $y \in \omega_G(x)$.

Proposition 4.5 *If (X, G) is topologically equicontinuous at $x \in X$, then $\omega_G(x) = \Omega_G(x)$.*

Proof Obviously, $\omega_G(x) \subseteq \Omega_G(x)$. It suffices to prove that $\Omega_G(x) \subseteq \omega_G(x)$. Suppose $y \in \Omega_G(x)$, by the given condition, $(x, y) \in EqP(X, G)$ and therefore from Definition 4.2, (x, y) is a nontrivial equicontinuity pair of (X, G). Then $y \in \omega_G(x)$ by Remark 4.1.

Theorem 4.1 *Suppose that X has no isolated points. Then (X, G) is transitive if and only if there are no trivial equicontinuity pairs of (X, G).*

Proof Firstly, we suppose that (X, G) has no trivial equicontinuity pairs.

Let U and V be nonempty open subsets of X. Pick $x \in U$, $y \in V$.

If $(x, y) \in EqP(X, G)$, then (x, y) is not a trivial equicontinuity pair of (X, G) and thus $y \in \omega_G(x)$ by Remark 4.1 which shows that $N(U, V) \neq \emptyset$.

If $(x, y) \notin EqP(X, G)$, then $y \in \Omega_G(x)$ by Proposition 4.3, and therefore $N(U, V) \neq \emptyset$, i.e. (X, G) is transitive.

Suppose that (X, G) is transitive. Take arbitrarily $(x, y) \in X \times X$, let $U \in \mathcal{N}_x$ and $V \in \mathcal{N}_y$, then $N(U, V)$ is infinite by Lemma 2.1 and $X = \Omega_G(z)$ for all $z \in X$.

If (x, y) is not an equicontinuity pair of (X, G), then the result holds.

If (x, y) is an equicontinuity pair of (X, G), then (x, y) is a trivial equicontinuity pair of (X, G) by the given condition, and so $y \notin \Omega_G(x)$ by Definition 4.2. This contradicts the fact that $y \in \Omega_G(x)$ since $X = \Omega_G(x)$. This contradiction yields the result.

Corollary 4.2 *Suppose that (X, G) is a transitive G-system, where X has no isolated points. If $(x, y) \in EqP(X, G)$, then $y \in \omega_G(x)$.*

Proof By Theorem 4.1, (x, y) is a nontrivial equicontinuity pair of (X, G), then $y \in \omega_G(x)$ by Remark 4.1.

Lemma 4.1 *Let $x, y \in X$. Suppose $(x, y) \notin EqP(X, G)$ and O is a splitting neighborhood of y with regard to x. Then for every $U \in \mathcal{N}_x$ and $V \in \mathcal{N}_y$, $H_{U,V} = \{g \in G : gU \cap V \neq \emptyset$ and $gU \nsubseteq O\}$ is infinite.*

Proof Let $U \in \mathcal{N}_x$, $V \in \mathcal{N}_y$. Then $H_{U,V} \neq \emptyset$ if note $(x, y) \notin EqP(X, G)$. If $H_{U,V}$ is finite, say $H_{U,V} = \{g_1, g_2, \cdots, g_n\}$. By Corollary 4.1, there exist $U' \in \mathcal{N}_x$ and $V' \in \mathcal{N}_y$ such that for each $k \in \{1, 2, \cdots, n\}$, if $g_k U' \cap V' \neq \emptyset$, then $g_k U' \subseteq O$. Without loss of generality, suppose $U' \subseteq U$, $V' \subseteq V$. Let $H_1 =\{g \in G : gU' \cap V' \neq \emptyset$ and $gU' \nsubseteq O\}$, then $H_1 \neq \emptyset$ if note $(x, y) \notin EqP(X, G)$. Thus, there exists $g_{n'} \in H_1 \subset H_{U,V}$ such that $g_{n'} \neq g_i$ for each $i \in \{1, 2, \cdots, n\}$. This is contrary to $H_{U,V} = \{g \in G : gU \cap V \neq \emptyset$ and $gU \nsubseteq O\}$. This contradiction gives the result.

Lemma 4.2 *Let $x, y, z \in X$ and $z \in \overline{Gx}$. If $(x, y) \notin EqP(X, G)$ and O is a splitting neighborhood of y with regard to x, then $(z, y) \notin EqP(X, G)$ and O is also a splitting neighborhood of y with regard to z.*

Proof Let $W \in \mathcal{N}_z$, $V \in \mathcal{N}_y$ and $g \in G$ such that $x \in U = g^{-1}W$ by $z \in \overline{Gx}$. As $(x, y) \notin EqP(X, G)$ and O is a splitting neighborhood of y with regard to x, there exists $g_1 \in G - \{g^{-1}\}$ such that $g_1 U \cap V \neq \emptyset$ and $g_1 U \nsubseteq O$ by Lemma 4.1. Then $g_1 g^{-1} W \cap V \neq \emptyset$ and $g_1 g^{-1} W \nsubseteq O$. Thus, the result holds.

Remark 4.2 The contrapositive of Lemma 4.2 is: If $(x, y) \in EqP(X, G)$ and $x \in \overline{Gz}$, then

$(z, y) \in EqP(X, G)$.

Corollary 4.3 *Let $x, y, z \in X$. If (y, z) is a trivial (resp. nontrivial) equicontinuity pair of (X, G) and $y \in \overline{Gx}$, then (x, z) is a trivial (resp. nontrivial) equicontinuity pair of (X, G). In particular, if (x, y) and (y, z) are nontrivial equicontinuity pairs of (X, G), then so is (x, z).*

Proof Suppose that (y, z) is a trivial equicontinuity pair of (X, G), then $z \notin \Omega_G(y)$ and $(x, z) \in EqP(X, G)$ by Remark 4.2. Let $V \in \mathcal{N}_y$, $W \in \mathcal{N}_z$ and $F \in \mathrm{Fin}(G)$ such that for all $g \in G - F$, $gV \cap W = \emptyset$. As $y \in \overline{Gx}$, let $g_1 \in G$ be such that $x \in U = g_1^{-1}V$. Then for all $g' \in G - Fg_1$, $g'U \cap W = \emptyset$, which implies $z \notin \Omega_G(x)$. Thus, (x, z) is a trivial equicontinuity pair of (X, G).

Suppose that (y, z) is a nontrivial equicontinuity pair of (X, G), then $(y, z) \in EqP(X, G)$ and $z \in \omega_G(y)$. If $y \in \omega_G(x)$, then $z \in \omega_G(x)$ since $\omega_G(x)$ is G-invariant. If $y \in Gx$, then $\omega_G(x) = \omega_G(y)$ by $\omega_G(x) = \omega_G(gx)$ for each $g \in G$ and $z \in \omega_G(x)$. Therefore, $z \in \omega_G(x)$. But $(x, z) \in EqP(X, G)$ by Remark 4.2, then (x, z) is a nontrivial equicontinuity pair of (X, G) by Remark 4.1.

Suppose that (x, y) and (y, z) are nontrivial equicontinuity pairs of (X, G), then $y \in \omega_G(x)$ by Remark 4.1. From the above discussion, we obtain that (x, z) is a nontrivial equicontinuity pair of (X, G).

Corollary 4.4 *Let (X, G) be a minimal G-system, then for all $x, y \in X$,*

$$(x, y) \in EqP(X, G) \Rightarrow (z, y) \in EqP(X, G) \text{ for all } z \in X.$$

$$(x, y) \notin EqP(X, G) \Rightarrow (z, y) \notin EqP(X, G) \text{ for all } z \in X.$$

Proof As (X, G) is minimal, then $\overline{Gz} = X$ for every $z \in X$. If $(x, y) \in EqP(X, G)$, by Corollary 4.3, then for all $z \in X$, $(z, y) \in EqP(X, G)$ if note $x \in \overline{Gz}$. If $(x, y) \notin EqP(X, G)$, by Lemma 4.2, for all $z \in X$, $(z, y) \notin EqP(X, G)$ as $z \in \overline{Gx}$.

The following theorem was firstly proved by Akin et al. in [1] (see [1, Theorem 2.4]). Now we prove that it is also true for G-systems considered in this paper.

Theorem 4.2 *If a G-system (X, G) is transitive and X has no isolated points and there exists a topological equicontinuity point of (X, G). Then the set of transitive points of (X, G) coincides with the set of topological equicontinuity points of (X, G). In particular, if (X, G) is minimal and (X, G) admits a topological equicontinuity point, then (X, G) is topologically equicontinuous.*

Proof Suppose that $x \in X$ is a topological equicontinuity point of (X, G), then $\omega_G(x) = \Omega_G(x)$ by Proposition 4.5. Since (X, G) is transitive, $\Omega_G(x) = X$. Hence, $\overline{Gx} = X$, i.e., x is a transitive point of (X, G).

Suppose that $y \in X$ is a transitive point of (X, G), then $\overline{Gy} = X$. Since x is a topological equicontinuity point of (X, G), for all $z \in X$, (x, z) is a nontrivial equicontinuity pair of (X, G) by Theorem 4.1. Note that $x \in \overline{Gy}$. Therefore, for every $z \in X$, (y, z) is a nontrivial equicontinuity pair of (X, G) by Corollary 4.3, i.e., y is a topological equicontinuity point of (X, G).

If (X, G) is minimal and there is a topological equicontinuity point of (X, G), by the above discussion, each $w \in X$ is a topological equicontinuity point of (X, G), and therefore (X, G) is topologically equicontinuous.

Theorem 4.3 *Suppose that X is a T_3-space and $x \in X$ is a transitive point of (X, G). If there exists $y \in X$ such that $(x, y) \notin EqP(X, G)$, then (X, G) is Hausdorff sensitive.*

Proof Let $x, y \in X$ and $(x, y) \notin EqP(X, G)$, then there exists $O \in \mathcal{N}_y$ satisfying: For all $U \in \mathcal{N}_x$ and $V \in \mathcal{N}_y$, there exists $g \in G$ such that $gU \cap V \neq \emptyset$, but $gU \nsubseteq O$. Let V_1 and V_2 be open neighborhoods of y with $\overline{V_1} \subseteq O$ and $\overline{V_2} \subseteq V_1$ as X is regular. Then $\mathcal{U} = \{V_1, X - \overline{V_2}\}$ is a finite open cover of X. Let W be an arbitrary nonempty open subset of X, pick $g \in G$ such that $x \in U = g^{-1}W$. Take $g_1 \in G - \{g\}$ such that $g_1U \cap V_2 \neq \emptyset$ and $g_1U \nsubseteq O$ (such a g_1 exists by Lemma 4.1). Then $g_1g^{-1}W \cap V_2 \neq \emptyset$ and $g_1g^{-1}W \nsubseteq O$, i.e., there are $a, b \in W$ such that $g_1g^{-1}a \in V_2$ and $g_1g^{-1}b \notin O$. Then $\{g_1g^{-1}a, g_1g^{-1}b\} \cap V_1 = \{g_1g^{-1}a\}$ and $\{g_1g^{-1}a, g_1g^{-1}b\} \cap \{X - \overline{V_2}\} = \{g_1g^{-1}b\}$. Hence, (X, G) is Hausdorff sensitive.

The next result is the dichotomy theorem between Hausdorff sensitivity and topological equicontinuity, which can be regarded as a generalization of the classical dichotomy theorem given by Auslander and Yorke [4] and [10, Corollary 2.25].

Theorem 4.4 (Revisited Auslander-Yorke Dichotomy II) *If a G-system (X, G) is minimal and X is a T_3-space. Then (X, G) is either topologically equicontinuous or Hausdorff sensitive.*

Proof If (X, G) is not topologically equicontinuous, then there exist $x, y \in X$ such that $(x, y) \notin EqP(X, G)$, but x is a transitive point of (X, G) if note that (X, G) is minimal. Then (X, G) is Hausdorff sensitive by Theorem 4.3.

4.2 Even continuity of G-systems

Even continuity was firstly introduced by Kelley in [16] for a family of mappings from a topological space to another topological space, which is weaker than topological equicontinuity (see Remark 4.4). In this section, we introduce the concept of even continuity for G-systems and we obtain some basic properties of even continuity.

In this section, we always assume that (X, G) is a G-system and X is a Hausdorff space.

Definition 4.3 *Let $x, y \in X$. If for each $O \in \mathcal{N}_y$, there exist $U \in \mathcal{N}_x$ and $V \in \mathcal{N}_y$ such that $gU \subseteq O$ for each $g \in N(x, V)$, then (x, y) is called an even continuity pair of (X, G) and y is called an even continuity partner of x.*

Remark 4.3 If X is a compact Hausdorff space, then the concepts of equicontinuity, topological equicontinuity and even continuity are mutually equivalent since equicontinuity and even continuity are equivalent by [16, Theorems 7.22–7.23], and it was proved in [27] that topological equicontinuity and even continuity are equivalent (see [27, Problems 3b and 5c of Chapter 14]).

Denote the set of even continuity pairs of (X,G) by $EvP(X,G)$. If for all $y \in X$, $(x,y) \in EvP(X,G)$, then we say that (X,G) is evenly continuous at $x \in X$. We say that (X,G) is evenly continuous if $EvP(X,G) = X \times X$.

If $(x,y) \notin EvP(X,G)$, by the definition of even continuity pair, there exists $O \in \mathcal{N}_y$ such that for each $U \in \mathcal{N}_x$ and $V \in \mathcal{N}_y$, $gU \nsubseteq O$ for some $g \in N(x,V)$. Meanwhile, we say that O is an even-splitting neighborhood of y with regard to x.

Remark 4.4 Clearly, every equicontinuity pair of (X,G) is an even continuity pair of (X,G). But the converse is not true, see such an example given in [7, Remark 6.5(2)].

By Remark 4.4, if $(x,y) \notin EvP(X,G)$, then every even-splitting neighborhood of y with regard to x is also a splitting neighborhood of y with regard to x.

Proposition 4.6 *If $(x,y) \in EvP(X,G)$, then $(x,gy) \in EvP(X,G)$ for each $g \in G$.*

Proof Let $g \in G$, $O \in \mathcal{N}_{gy}$, then $g^{-1}O \in \mathcal{N}_y$. Since $(x,y) \in EvP(X,G)$, there exist $U \in \mathcal{N}_x$ and $V \in \mathcal{N}_y$ such that $g'U \subseteq g^{-1}O$ for each $g' \in N(x,V)$. Then $gV \in \mathcal{N}_{gy}$. For every $g_1 \in G$, if $g_1x \in gV$ then $g^{-1}g_1x \in V$. Then $g^{-1}g_1U \subseteq g^{-1}O$ which means that $g_1U \subseteq O$.

Proposition 4.7 *Let $x,y \in X$, where X is a Hausdorff space. If $y \notin \omega_G(x)$, then $(x,y) \in EvP(X,G)$.*

Proof Let $O \in \mathcal{N}_y$. Since $y \notin \omega_G(x)$, there exist $V \in \mathcal{N}_y$ and $F \in \mathrm{Fin}(G)$ such that for each $g \in G - F$, $gx \notin V$. By Corollary 4.1, there exist $U' \in \mathcal{N}_x$ and $V' \in \mathcal{N}_y$ such that for each $g \in F$, if $gU' \cap V' \neq \emptyset$, then $gU' \subseteq O$. In particular, for each $g \in F$, if $gx \in V'$, then $gU' \subseteq O$. We suppose $V' \subseteq V \cap O$, then for every $g \in G - F$, $gx \notin V'$. Therefore, $(x,y) \in EvP(X,G)$.

If $(x,y) \notin EvP(X,G)$, then $y \in \omega_G(x)$ by Proposition 4.7. Therefore, we can introduce naturally the following definition of nontrivial even continuity pair.

Definition 4.4 *Let $x,y \in X$ and $(x,y) \in EvP(X,G)$. $(x,y) \in X \times X$ is called a trivial even continuity pair of (X,G) if $y \notin \omega_G(x)$. Otherwise, (x,y) is called a nontrivial even continuity pair of (X,G).*

Remark 4.5 If (x,y) is a nontrivial equicontinuity pair of (X,G), then (x,y) is a nontrivial even continuity pair of (X,G) by Remarks 4.1 and 4.4. And each $(x,y) \in EvP(X,G)$ is either a trivial even continuity pair of (X,G) or $y \in \omega_G(x)$.

Lemma 4.3 *Let $x,y \in X$. Suppose $(x,y) \notin EvP(X,G)$ and O is an even-splitting neighborhood of y with regard to x. Then for each $U \in \mathcal{N}_x$ and $V \in \mathcal{N}_y$, the set $H_{U,V} = \{g \in G : gx \in V$ and $gU \nsubseteq O\}$ is infinite.*

Proof Let $U \in \mathcal{N}_x$, $V \in \mathcal{N}_y$. Then $H_{U,V} \neq \emptyset$ if note $(x,y) \notin EvP(X,G)$. If $H_{U,V}$ is finite, say $H_{U,V} = \{g_1, g_2, \cdots, g_n\}$. By Corollary 4.1, there exist $U' \in \mathcal{N}_x$ and $V' \in \mathcal{N}_y$ such that for each $k \in \{1, 2, \cdots, n\}$, if $g_kU' \cap V' \neq \emptyset$, then $g_kU' \subseteq O$ which means that if $g_kx \in V'$ then $g_kU' \subseteq O$. Without loss of generality, suppose $U' \subseteq U$, $V' \subseteq V$. Let $H_1 =\{g \in G : gx \in V'$

and $gU' \nsubseteq O\}$, then $H_1 \neq \emptyset$ if note $(x, y) \notin EvP(X, G)$. Thus, there exists $g_{n'} \in H_1 \subset H_{U,V}$ such that $g_{n'} \neq g_i$ for each $i \in \{1, 2, \cdots, n\}$. This is contrary to $H_{U,V} = \{g \in G : gx \in V$ and $gU \nsubseteq O\}$. This contradiction gives the result.

Proposition 4.8 *Let $x, y \in X$, $(x, y) \notin EvP(X, G)$ and O be an even-splitting neighborhood of y with regard to x. Then for every $g \in G$, $(gx, y) \notin EvP(X, G)$ and O is also an even-splitting neighborhood of y with regard to gx.*

Proof Let $W \in \mathcal{N}_{gx}$ and $V \in \mathcal{N}_y$, then $x \in U = g^{-1}W$. Let $H =\{g_1 \in G : g_1x \in V$ and $g_1U \nsubseteq O\}$, then H is infinite by Lemma 4.3. Take $g_1 \in G - \{g^{-1}\}$ such that $g_1 \in H$, then $g_1g^{-1}gx \in V$, $g_1g^{-1}W \nsubseteq O$. Then $(gx, y) \notin EvP(X, G)$ by Definition 4.3.

Remark 4.6 Let $x, y, z \in X$. Suppose $(x, y) \in EvP(X, G)$ and $x \in Gz$, then by Proposition 4.8, $(z, y) \in EvP(X, G)$.

Theorem 4.5 *If $x \in X$ has no even continuity partners, then x is a transitive point of (X, G) and (X, G) admits no equicontinuity pairs.*

Proof Suppose $x \in X$ has no even continuity partners, then for all $y \in X$, $(x, y) \notin EvP(X, G)$, and thus $y \in \omega_G(x)$ by Definition 4.4, which implies $\omega_G(x) = X$, i.e., x is a transitive point of (X, G).

Pick arbitrarily $y, z \in X$ and assume that O is an even-splitting neighborhood of z with regard to x. Let $V \in \mathcal{N}_y$, $W \in \mathcal{N}_z$, then there exists $g \in G$ such that $gx \in V$ since x is a transitive point of (X, G). By Proposition 4.8, $(gx, z) \notin EvP(X, G)$ and O is also an even-splitting neighborhood of z with regard to gx. Hence, there exists $g' \in G$ such that $g'gx \in W$ and $g'V \nsubseteq O$ (i.e., $g'V \cap W \neq \emptyset$ and $g'V \nsubseteq O$) which implies $(y, z) \notin EqP(X, G)$.

Acknowledgement The authors greatly thank the anonymous referees for the careful reading and many helpful comments.

References

[1] Akin, E., Auslander, J. and Berg, K., When is a transitive map chaotic? *Convergence in Ergodic Theory and Probability,* **5**(3), 1996, 25–40.

[2] Auslander, J., Minimal Flows and Their Extensions, Elsevier, Amsterdam, 1988.

[3] Auslander, J., Greschonig, G. and Nagar, A., Reflections on equicontinuity, *Proc. Amer. Math. Soc.,* **142**(9), 2014, 3129–3137.

[4] Auslander, J. and Yorke, J., Interval maps, factors of maps, and chaos, *Tohoku Math. J.,* **32**(2), 1980, 177–188.

[5] Cairns, G., Kolganova, A. and Nielsen, A., Topological transitivity and mixing notions for group actions, *Rocky Mountain J. Math.,* **37**(2), 2007, 371–397.

[6] Corbacho, E., Tarieladze, V. and Vidal, R., Equicontinuity and quasi-uniformities, *Georgian Math. J.,* **11**(4), 2004, 681–690.

[7] Corbacho, E., Tarieladze, V. and Vidal, R., Even continuity and topological equicontinuity in topologized semigroups, *Topology Appl.,* **156**(7), 2009, 1289–1297.

[8] Devaney, R. L., An Introduction to Chaotic Dynamical Systems, 2nd ed., Studies in Nonlinearity, Westview Press, Boulder, CO, 2003.

[9] Engelking, R., General Topology, 2nd ed., Sigma Series in Pure Mathematics, **6**, Heldermann Verlag, Berlin, 1989.

[10] Good, C., Leek, R. and Mitchell, J., Equicontinuity, transitivity and sensitivity, the Auslander-Yorke dichotomy revisied, *Discrete Contin. Dyn. Syst.,* **40**(4), 2020, 2441–2474.

[11] Good, C. and Macías, S., What is topological about topological dynamics? *Discrete Contin. Dyn. Syst.,* **38**(3), 2018, 1007–1031.

[12] Huang, W., Khilko, D., Kolyada, S. and Zhang, G., Dynamical compactness and sensitivity, *J. Differential Equations,* **260**(9), 2016, 6800–6827.

[13] Huang, W., Khilko, D., et al., Finite intersection property and dynamical compactness, *J. Dynam. Differential Equations,* **30**(3), 2018, 1221–1245.

[14] Huang, W., Kolyada, S. and Zhang, G. H., Analogues of Auslander-Yorke theorems for multi-sensitivity, *Ergodic Theory Dyna. Systems,* **38**(2), 2016, 651–665.

[15] James, I., Topologies and Uniformities, Springer-Verlag, London, 1999.

[16] Kelley, J. L., General Topology, Graduate Texts in Mathematics, Springer-Verlag, New York, Heidelberg, Berlin, 1955.

[17] Li, J. and Tu, S. M., Density-equicontinuity and density-sensitivity, *Acta Math Sin.* (*Engl. Ser.*)*,* **37**(2), 2021, 345–361.

[18] Li, J., Tu, S. M. and Ye, X. D., Mean equicontinuity and mean sensitivity, *Ergodic Theory and Dyna. Systems,* **35**(8), 2015, 2587–2612.

[19] Li, J. and Ye, X. D., Recent development of chaos theory in topological dynamics, *Acta Math. Sin.* (*Engl. Ser.*)*,* **32**(1), 2016, 83–114.

[20] Li, J., Ye, X. D. and Yu, T., Equicontinuity and sensitivity in mean forms, *J. Dynam. Differential Equations,* **34**(1), 2022, 133–154.

[21] Liu, K. and Zhou, X. M., Auslander-Yorke type dichotomy theorems for stronger versions of r-sensitivity, *Proc. Amer. Math. Soc.,* **147**(6), 2019, 2609–2617.

[22] Moothathu, T., Stronger forms of sensitivity for dynamical systems, *Nonlinearity,* **20**(9), 2007, 2115–2126.

[23] Noble, N., Ascoli theorems and the exponential map, *Trans. Amer. Math. Soc.,* **143**, 1969, 393–411.

[24] Oprocha, P. and Zhang, G. H., Topological Aspects of Dynamics of Pairs, Tuples and Sets., Recent Progress in General Topology III, Atlantis Press, Paris, 2014, 665–709.

[25] Polo, F., Sensitive dependence on initial conditions and chaotic group actions, *Proc. Amer. Math. Soc.,* **138**(8), 2010, 2815–2826.

[26] Poppe, H., Convergence of evenly continuous nets in general function spaces, *Real Anal. Exchange,* **18**(2), 1992, 459–464.

[27] Royden, H. L., Real Analysis, 3nd ed., Prentice-Hall, Inc., 2006.

[28] Wang, H. Y., Equicontinuity, shadowing and distality in general topological spaces, *Czechoslovak Math. J.,* **70**(3), 2020, 711–726.

[29] Wang, H. Y. and Zhong, Y. H., A note on sensitivity in uniform spaces, *J. Dyn. Control Syst.,* **24**(4), 2018, 625–633.

[30] Willard, S., General Topology, Reading, Mass., Addison-Wesley, 1970.

[31] Zhang, G. H., Relativization of complexity and sensitivity, *Ergodic Theory and Dyna. Systems,* **27**(4), 2007, 1349–1371.