Chin. Ann. Math. Ser. B
44(4), 2023, 517–532
DOI: 10.1007/s11401-023-0029-6

Chinese Annals of
Mathematics, Series B

Regular and Maximal Graphs with Prescribed Tripartite Graph as a Star Complement*

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Abstract Let G be a graph of order n and μ be an adjacency eigenvalue of G with multiplicity $k \geq 1$. A star complement H for μ in G is an induced subgraph of G of order $n-k$ with no eigenvalue μ, and the subset $X = V(G-H)$ is called a star set for μ in G. The star complement provides a strong link between graph structure and linear algebra. In this paper, the authors characterize the regular graphs with $K_{2,2,s}$ $(s \geq 2)$ as a star complement for all possible eigenvalues, the maximal graphs with $K_{2,2,s}$ as a star complement for the eigenvalue $\mu = 1$, and propose some questions for further research.

Keywords Adjacency eigenvalue, Star set, Star complement, Regular graph, Maximal graph
2000 MR Subject Classification 05C50

1 Introduction

Let G be a simple graph with vertex set $V(G) = \{1, 2, \cdots, n\} = [n]$ and edge set $E(G)$. The adjacency matrix of G is an $n \times n$ matrix $A(G) = (a_{ij})$, where $a_{ij} = 1$ if vertex i is adjacency to vertex j, and 0 otherwise. We use the notation $i \sim j$ to indicate that i, j are adjacent and the notation $d_G(i)$ (simply, $d(i)$) to indicate the degree of vertex i in G. The adjacency eigenvalues of G are just the eigenvalues of $A(G)$. For more details on graph spectra, see [5]. The join of two graphs G and H, denoted by $G \triangledown H$, is the graph obtained from G and H by connecting each vertex of G to all vertices of H.

Let μ be an eigenvalue of G with multiplicity k. A star set for μ in G is a subset X of $V(G)$ such that $|X| = k$ and μ is not an eigenvalue of $G - X$, where $G - X$ is the subgraph of G induced by $\overline{X} = V(G) \setminus X$. In this situation $H = G - X$ is called a star complement corresponding to μ. Star sets and star complements exist for any eigenvalue of a graph, and they need not to be unique. The basic properties of star sets are established in [6, Chapter 7].

There is another equivalent geometric definition for star sets and star complements. Let G be a graph with vertex set $V(G) = \{1, \cdots, n\}$ and adjacency matrix $A = A(G)$. Let $\{e_1, \cdots, e_n\}$ be the standard orthonormal basis of $\mathbb{R}^n$ and P be the matrix which represents the orthogonal projection of $\mathbb{R}^n$ onto the eigenspace $\mathcal{E}(\mu) = \{x \in \mathbb{R}^n : Ax = \mu x\}$ of A with respect to $\{e_1, \cdots, e_n\}$. Since $\mathcal{E}(\mu)$ is spanned by the vectors Pe_j $(j = 1, \cdots, n)$, there exists

Manuscript received December 14, 2021. Revised October 11, 2022.
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*This work was supported by the National Natural Science Foundation of China (No. 11971180, 12271337) and the Guangdong Provincial Natural Science Foundation (No. 2019A1515012052).

$X \subseteq V(G)$ such that the vectors Pe_j ($j \in X$) form a basis for $\mathcal{E}(\mu)$. Such a subset X of $V(G)$ is called a star set for μ in G. In this situation $H = G - X$ is called a star complement for μ.

For any graph G of order n with distinct eigenvalues $\lambda_1, \cdots, \lambda_m$, there exists a partition $V(G) = V_1 \cup \cdots \cup V_m$ such that V_i is a star set for eigenvalue λ_i ($i = 1, \cdots, m$). Such a partition is called a star partition of G. For any graph G, there exists at least one star partition (see [8]). Each star partition determines a basis for $\mathbb{R}^n$ consisting of eigenvectors of an adjacency matrix. It provides a strong link between graph structure and linear algebra.

In [8], it was proved that if $Y \subset X$ then $X \setminus Y$ is a star set for μ in $G - Y$. Thus the induced subgraph $G - Y$ also has $H = G - X$ as a star complement for μ. If G has H as a star complement for μ, and G is not a proper induced subgraph of some other graph with star complement H for μ, then G is a maximal graph with star complement H for μ, or it is an H-maximal graph for μ. In general, there will be various different maximal graphs, possibly of different orders, but sometimes there is a unique maximal graph.

Let G be a simple graph. The complement and the line graph of G are denoted by $\overline{G}$ and $L(G)$. Let P_n, C_n, S_n, K_n, $K_{m,n}$, $K_{m,n,r}$ be a path, cycle, star, complete graph, complete bipartite graph and complete tripartite graph, respectively (see [4] for more detailed definitions), $S_{m,n}$ be the double star obtained from stars S_m and S_n by joining their centers, R_t and Q_t be defined in [7].

There are a lot of literatures about using star complements to construct and characterize certain graphs. Maximal graphs with a prescribed graph such as S_m, K_m, $S_{m,n}$, $K_{2,5}$, $K_{2,s}$, $K_{1,1,t}$, C_t, P_t, $\overline{L(R_t)}$, $L(R_t)$, $\overline{L(Q_t)}$, $L(Q_t)$, $\overline{K_{1,s}}$, $\overline{K_{1,1,s}}$ or unicyclic graph as a star complement for given eigenvalues (for example, $\mu = 1, -2$) were well studied in the literatures (see [2–3, 7, 10, 15, 17–18, 23–24] and so on). Regular graphs with a prescribed graph such as $K_{2,5}$, $K_{t,s}$, $K_1 \triangledown hK_q$, $K_{1,1,t}$, $K_{1,1,1,t}$, $\overline{sK_1 \cup K_t}$, $P_t(\mu = 1)$, $K_{r,r,r}(\mu = 1)$ or $K_{r,s} \cup tK_1(\mu = 1)$ as a star complement were well studied in the literatures (see [1, 11, 13–16, 19, 22–24] and so on). In this paper, we introduce the fundamental properties of the theory of star complements in Section 2, characterize the regular graphs with the tripartite graph $K_{2,2,s}(s \geq 2)$ as a star complement for all possible eigenvalues in Section 3, the maximal graphs with $K_{2,2,s}$ as a star complement for $\mu = 1$ in Section 4, and propose some questions for further research in Section 5.

2 Preliminaries

In this section, we introduce some results of star sets and star complements that will be required in the sequel. The following fundamental result combines Reconstruction Theorem (see [6, Theorem 7.4.1]) with its converse (see [6, Theorem 7.4.4]).

Theorem 2.1 (see [6]) *Let X be a set of vertices in the graph G. Suppose that G has adjacency matrix*

$$\begin{pmatrix} A_X & B^{\mathrm{T}} \\ B & C \end{pmatrix},$$

where A_X is the adjacency matrix of the subgraph induced by X. Then X is a star set for μ in G if and only if μ is not an eigenvalue of C and

$$\mu I - A_X = B^{\mathrm{T}}(\mu I - C)^{-1}B. \tag{2.1}$$

Note that if X is a star set for μ, then the corresponding star complement $H(= G - X)$ has adjacency matrix C, and (2.1) tells us that G is determined by μ, H and the H-neighbourhood of vertices in X, where the H-neighbourhood of vertex $u \in X$, denoted by $N_H(u)$, is defined as $N_H(u) = \{v \mid v \sim u, v \in V(H)\}$.

It is usually convenient to apply (2.1) in the form

$$m(\mu)(\mu I - A_X) = B^{\mathrm{T}} m(\mu)(\mu I - C)^{-1} B,$$

where $m(x)$ is the minimal polynomial of C. This is because $m(\mu)(\mu I - C)^{-1}$ is given explicitly as follows.

Proposition 2.1 (see [7, Proposition 0.2]) *Let C be a square matrix with minimal polynomial*

$$m(x) = x^{d+1} + c_d x^d + c_{d-1} x^{d-1} + \cdots + c_1 x + c_0.$$

If μ is not an eigenvalue of C, then

$$m(\mu)(\mu I - C)^{-1} = a_d C^d + a_{d-1} C^{d-1} + \cdots + a_1 C + a_0 I,$$

where $a_d = 1$ and for $0 < i \le d$, $a_{d-i} = \mu^i + c_d \mu^{i-1} + c_{d-1} \mu^{i-2} + \cdots + c_{d-i+1}$.

In order to find all the graphs with a prescribed star complement for μ, we need to find all solution A_X, B for given μ and C. For any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^t$, where $t = |V(H)|$, let

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^{\mathrm{T}} (\mu I - C)^{-1} \mathbf{y}. \tag{2.2}$$

Let $\mathbf{b}_u$ be the column of B for any $u \in X$. By Theorem 2.1, we have the following corollary.

Corollary 2.1 (see [8, Corollary 5.1.8]) *Suppose that μ is not an eigenvalue of the graph H, where $|V(H)| = t$. There exists a graph G with a star set X for μ such that $G - X = H$ if and only if there exist $(0,1)$-vectors $\boldsymbol{b}_u$ $(u \in X)$ in $\mathbb{R}^t$ such that*

(1) *$\langle \boldsymbol{b}_u, \boldsymbol{b}_u \rangle = \mu$ for all $u \in X$, and*

(2) *$\langle \boldsymbol{b}_u, \boldsymbol{b}_v \rangle = \begin{cases} -1, & u \sim v \\ 0, & u \nsim v \end{cases}$ for all pairs u, v in X.*

In view of the two equations in Corollary 2.1, we have the following lemma.

Lemma 2.1 (see [6]) *Let X be a star set for μ in G, and $H = G - X$.*

(1) *If $\mu \neq 0$, then $V(H)$ is a dominating set for G, that is, the H-neighbourhood of any vertex in X is non-empty.*

(2) *If $\mu \notin \{-1, 0\}$, then $V(H)$ is a location-dominating set for G, that is, the H-neighbourhoods of distinct vertices in X are distinct and non-empty.*

It follows from (2) of Lemma 2.1 that there are only finitely maximal graphs with a prescribed star complement for $\mu \notin \{-1, 0\}$. If $\mu = 0$ and X has distinct vertices u and v with the same neighbourhood in G, then u and v are called duplicate vertices. If $\mu = -1$ and X has distinct vertices u and v with the same closed neighbourhood in G, then u and v are called co-duplicate vertices (see [9]).

Recall that if the eigenspace $\mathcal{E}(\mu)$ is orthogonal to the all-1 vector $\mathbf{j}$, then μ is called a non-main eigenvalue, and we have the following results.

Lemma 2.2 (see [7, Proposition 0.3]) *The eigenvalue μ is a non-main eigenvalue if and only if*

$$\langle \boldsymbol{b}_u, \boldsymbol{j} \rangle = -1 \quad \text{for all } u \in X, \tag{2.3}$$

where $\boldsymbol{j}$ is the all-1 vector.

Lemma 2.3 (see [8, Corollary 3.9.12]) *In an r-regular graph, all eigenvalues other than r are non-main.*

3 Regular Graphs with $K_{2,2,s}$ as a Star Complement

In this section, all the regular graphs with $K_{2,2,s}$ as a star complement for all possible eigenvalues are determined.

In the rest of this paper, we let $H \cong K_{2,2,s}$ $(s \geq 2)$, and (U, V, W) be a tripartition of the graph $K_{2,2,s}$ with $U = \{u_1, u_2\}$, $V = \{v_1, v_2\}$, $W = \{w_1, w_2, \cdots, w_s\}$. We say that a vertex $u \in X$ is of type (a, b, c) if it has a neighbours in U, b neighbours in V, c neighbours in W, thus $(a, b, c) \neq (0, 0, 0)$ and $0 \leq a \leq 2$, $0 \leq b \leq 2$, $0 \leq c \leq s$.

Let C be the adjacency matrix of $H \cong K_{2,2,s}$. Then C has the minimal polynomial

$$m(x) = x(x + 2)(x^2 - 2x - 4s).$$

Since μ is not an eigenvalue of C, we have $\mu \notin \{0, -2\}$ and $\mu^2 - 2\mu \neq 4s$. From Proposition 2.1, we have

$$m(\mu)(\mu I - C)^{-1} = C^3 + \mu C^2 + (\mu^2 - 4s - 4)C + (\mu^3 - 4(s + 1)\mu - 8s)I. \tag{3.1}$$

If μ is a non-main eigenvalue of G, then by (2.3) we have

$$\mu(\mu + 2)(2c - 4s - 2\mu + (a + b)(s + \mu) + c\mu + \mu^2) = 0. \tag{3.2}$$

Using (3.1) to compute $\langle \mathbf{b}_u, \mathbf{b}_u \rangle = \mu$, we obtain the following equation

$$\begin{aligned} &- \mu^5 + (4s + 4)\mu^3 + 8s\mu^2 + 2\mu(\mu + 2)(ac + bc) + (4s + \mu(2 + s))(a^2 + b^2) \\ &+ 2ab\mu(\mu + s) + 4c^2(2 + \mu) + (a + b + c)(\mu^3 - 4(s + 1)\mu - 8s) = 0. \end{aligned} \tag{3.3}$$

Let u, v be distinct vertices in X of type (a, b, c), (α, β, γ), respectively, $\rho_{uv} = |N_H(u) \cap N_H(v)|$, and $a_{uv} = 1$ or 0 according as $u \sim v$ or $u \nsim v$. Using (3.1) to compute $\langle \mathbf{b}_u, \mathbf{b}_v \rangle = -a_{uv}$, we have

$$\begin{aligned} &a_{uv}(\mu^4 - (4s + 4)\mu^2 - 8s\mu) + \mu(\mu + 2)(\alpha c + \beta c + \gamma a + \gamma b) + (4s + \mu(2 + s))(\alpha a + \beta b) \\ &+ \mu(\mu + s)(\alpha b + \beta a) + 4\gamma c(2 + \mu) + \rho_{uv}(\mu^3 - 4(s + 1)\mu - 8s) = 0. \end{aligned} \tag{3.4}$$

Lemma 3.1 *If u, v are of the same type (a, b, c), then $\rho_{uv} = a + b + c - \mu^2 - a_{uv}\mu$.*

Proof Let $\alpha = a$, $\beta = b$, $\gamma = c$ in (3.4). Subtracting (3.3) from (3.4), we have

$$(\mu + 2)(\mu^2 - 2\mu - 4s)(\mu^2 + a_{uv}\mu + \rho_{uv} - a - b - c) = 0.$$

Since $(\mu + 2)(\mu^2 - 2\mu - 4s) \neq 0$, we have $\mu^2 + a_{uv}\mu + \rho_{uv} - a - b - c = 0$. Thus $\rho_{uv} = a + b + c - \mu^2 - a_{uv}\mu$.

Lemma 3.2 *If u and v are of different types (a, b, c) and (b, a, c), then $\rho_{uv} = a + b + c - \mu^2 - a_{uv}\mu - \frac{(a-b)^2}{\mu+2}$.*

Proof Let $\alpha = b$, $\beta = a$, $\gamma = c$ in (3.4). Subtracting (3.3) from (3.4), we have

$$(\mu^2 - 2\mu - 4s)((a - b)^2 - (\mu + 2)(a + b + c - \rho_{uv} - \mu^2 - a_{uv}\mu)) = 0.$$

Since $\mu^2 - 2\mu - 4s \neq 0$, we have $(a - b)^2 - (\mu + 2)(a + b + c - \rho_{uv} - \mu^2 - a_{uv}\mu) = 0$. Thus $\rho_{uv} = a + b + c - \mu^2 - a_{uv}\mu - \frac{(a-b)^2}{\mu+2}$.

Lemma 3.3 *Let $X_1 = \{u \in X \mid u$ is of type $(a, b, c)\}$. If there are vertices $u, v \in X_1$, s.t. $u \sim v$ and vertices $u', v' \in X_1$, s.t. $u' \nsim v'$, then $\mu \in \mathbb{Z}$.*

Proof From Lemma 3.1, we have $\rho_{u'v'} = a + b + c - \mu^2 \in \mathbb{Z}$ by $u' \nsim v'$ and $\rho_{uv} = a + b + c - \mu^2 - \mu \in \mathbb{Z}$ by $u \sim v$. Thus $\mu = \rho_{u'v'} - \rho_{uv} \in \mathbb{Z}$.

Lemma 3.4 *Let $X_1 = \{u \in X \mid u$ is of type $(a, b, c)\}$, $X_2 = \{u \in X \mid u$ is of type $(b, a, c)\}$. If there are vertices $u \in X_1$, $v \in X_2$, s.t. $u \sim v$ and vertices $u' \in X_1$, $v' \in X_2$, s.t. $u' \nsim v'$, then $\mu \in \mathbb{Z}$.*

Proof From Lemma 3.2, we have $\rho_{u'v'} = a + b + c - \mu^2 - \frac{(a-b)^2}{\mu+2} \in \mathbb{Z}$ by $u' \nsim v'$ and $\rho_{uv} = a + b + c - \mu^2 - \mu - \frac{(a-b)^2}{\mu+2} \in \mathbb{Z}$ by $u \sim v$. Thus $\mu = \rho_{u'v'} - \rho_{uv} \in \mathbb{Z}$.

Let $H \cong K_{2,2,s}$ $(s \geq 2)$, and (U, V, W) be a tripartition of the graph $K_{2,2,s}$ as above. Let U_i be the set of vertices of type $(1, 2, s)$ in X adjacent to $u_i \in U$, V_i be the set of vertices of type $(2, 1, s)$ in X adjacent to $v_i \in V$, and W_i be the set of vertices of type $(2, 2, 1)$ in X adjacent to $w_i \in W$. We obtain an r-regular graph $G(r)$ with $V(G(r)) = X \cup V(H)$, $X = U_1 \cup U_2 \cup V_1 \cup V_2 \cup W_1 \cup \cdots \cup W_s$, where $|U_i| = |V_j| = \frac{(r+1)(s-1)}{4s-3} - 1$, $|W_i| = \frac{r+1}{4s-3} - 1$, $U_i(V_j, W_k)$ induces a clique for $1 \leq i, j \leq 2$, $1 \leq k \leq s$ and for any i, j, k, each vertex in U_i is adjacent to all vertices in V_j and W_k, each vertex in V_j is adjacent to all vertices in W_k.

For subsets V', V'' of $V(G)$, we write $E(V', V'')$ for the set of edges between V' and V''. The greatest common divisor of a and b is denoted by (a, b). For $\mu = -1$, we have the following theorem.

Theorem 3.1 *If G is an r-regular graph with $H \cong K_{2,2,s}$ $(s \geq 2)$ as a star complement for the eigenvalue $\mu = -1$, then $r \equiv -1 \pmod{4s-3}$ and $G \cong G(r)$.*

Proof Since $K_{2,2,s}$ is connected and $V(K_{2,2,s})$ is a dominating set (see Lemma 2.1), we know G is connected. Let (U, V, W) be a tripartition of the graph $K_{2,2,s}$ defined as above, $u \in X$ be a vertex of type (a, b, c), thus $(a, b, c) \neq (0, 0, 0)$ and $0 \leq a \leq 2$, $0 \leq b \leq 2$, $0 \leq c \leq s$. By Lemma 2.3, we know that $\mu = -1$ is a non-main eigenvalue of G, thus from (3.2), we have

$$4s - c - (a + b)(s - 1) - 3 = 0. \tag{3.5}$$

Let $\mu = -1$ in (3.3). We have

$$4s - (4s - 3)(a + b + c) - 2ac - 2bc + (3s - 2)(a^2 + b^2) + 4c^2 - 2ab(s - 1) - 3 = 0. \tag{3.6}$$

Since $0 \leq a \leq 2$, $0 \leq b \leq 2$, we can consider the following 9 cases, say, $(a, b) \in \{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 1), (2, 2)\}$. For example, if $a = b = 0$, combining (3.5) and (3.6), we have $\begin{cases} c = 0 \\ s = \frac{3}{4} \end{cases}$ or $\begin{cases} c = -\frac{1}{3} \\ s = \frac{2}{3} \end{cases}$, it is a contradiction with the fact that $c \in \mathbb{N}$ and $s \in \mathbb{N}$. Finally, we find that the possible types of vertices in X are $(2, 2, 1)$, $(1, 2, s)$, $(2, 1, s)$, and the feasible solution of (3.4) are shown in Table 1.

We observe that when u, v are of different types, they must be adjacent; when u, v are of the same type, $u \sim v$ if and only if they have the same H-neighbourhood. Thus u, v are co-duplicate vertices. We can add arbitrarily many co-duplicate vertices when constructing graphs with a prescribed star complement for -1.

Now we partition the vertices in X. Let U_i be the set of vertices of type $(1, 2, s)$ in X adjacent to $u_i \in U$, V_i be the set of vertices of type $(2, 1, s)$ in X adjacent to $v_i \in V$, and W_i

(a,b,c)	(α,β,γ)	a_{uv}	ρ_{uv}
$(2,2,1)$	$(2,2,1)$	0	4
$(2,2,1)$	$(2,2,1)$	1	5
$(1,2,s)$	$(1,2,s)$	0	$s+2$
$(1,2,s)$	$(1,2,s)$	1	$s+3$
$(2,1,s)$	$(2,1,s)$	0	$s+2$
$(2,1,s)$	$(2,1,s)$	1	$s+3$
$(2,2,1)$	$(1,2,s)$	1	4
$(2,2,1)$	$(2,1,s)$	1	4
$(1,2,s)$	$(2,1,s)$	1	$s+2$

Table 1 The feasible solution of (3.4).

be the set of vertices of type $(2,2,1)$ in X adjacent to $w_i \in W$. It is clear that any two vertices in U_i (V_i or W_i) are co-duplicate vertices. We do not exclude the possibility that some of the sets U_i, V_i, W_i are empty. Then for any $u_i \in U$, we have

$$d_G(u_i) = 2 + s + |U_i| + \sum_{i=1}^{s} |W_i| + \sum_{i=1}^{2} |V_i|,$$

and for any $v_i \in V$, we have

$$d_G(v_i) = 2 + s + |V_i| + \sum_{i=1}^{s} |W_i| + \sum_{i=1}^{2} |U_i|.$$

Since G is r-regular, we have $|E(X,U)| = |E(X,V)|$ and then $\sum\limits_{i=1}^{2} |V_i| = \sum\limits_{i=1}^{2} |U_i|$. Thus $|U_1| = |U_2| = |V_1| = |V_2|$ by $d_G(u_1) = d_G(u_2) = d_G(v_1) = d_G(v_2)$.

Similarly, for any $w_i \in W$, we have

$$d_G(w_i) = 4 + |W_i| + \sum_{i=1}^{2} |V_i| + \sum_{i=1}^{2} |U_i|.$$

Thus $|W_1| = |W_2| = \cdots = |W_s|$ by $d_G(w_1) = d_G(w_2) = \cdots = d_G(w_s)$. Then we have

$$r = d_G(w_1) = 4 + |W_1| + 4 \cdot |U_1| \quad \text{and} \quad r = d_G(u_1) = 2 + s + s \cdot |W_1| + 3 \cdot |U_1|.$$

It turns out that

$$|W_1| = \frac{r+1}{4s-3} - 1, \quad |U_1| = \frac{(s-1)(r+1)}{4s-3} - 1.$$

Since $|W_1| \in \mathbb{N}$, $|U_1| \in \mathbb{N}$ and

$$(s-1, 4s-3) = (s-1, 4s-3-3(s-1)) = (s-1, s) = 1,$$

we have $r \equiv -1 \pmod{4s-3}$. Consequently, we obtain an r-regular graph $G(r)$.

In the following, we consider the case $\mu \notin \{0,-1\}$. The following lemma lists all possible types of vertices in X.

Lemma 3.5 *Let G be a graph with $H \cong K_{2,2,s}$ $(s \geq 2)$ as a star complement for μ. If μ is a non-main eigenvalue of G and $\mu \notin \{0,-1\}$, then the following statements hold.*

Class	(a,b,c)	s
I	$\left(0,0,\frac{\mu^3+2\mu^2}{\mu-2}\right)$	$\frac{\mu^4+5\mu^3+4\mu}{4\mu-8}$
II	$\left(1,1,\frac{\mu^3+2\mu^2-2}{\mu}\right)$	$\frac{\mu^4+5\mu^3+4\mu^2-2\mu-4}{2\mu}$
III	$\left(0,1,\frac{\mu^3+2\mu^2-1}{\mu-1}\right)$, $\left(1,0,\frac{\mu^3+2\mu^2-1}{\mu-1}\right)$	$\frac{\mu^4+5\mu^3+2\mu^2-2}{3\mu-3}$
IV	$(0,2,\mu^2+2\mu)$, $(2,0,\mu^2+2\mu)$	$\frac{\mu^3}{2}+\frac{5\mu^2}{2}+2\mu$
V	$(1,2,\mu^2+\mu-1)$, $(2,1,\mu^2+\mu-1)$	$\mu^3+4\mu^2+2\mu-2$

Table 2 The possible types of vertices in X.

(1) *The possible types of vertices in X are shown in Table* 2.

(2) *If G is regular, then all vertices in X are of the same class of the five classes shown in Table* 2.

Proof Let $u \in X$ be a vertex of type (a,b,c), thus $(a,b,c) \neq (0,0,0)$ and $0 \leq a \leq 2$, $0 \leq b \leq 2$, $0 \leq c \leq s$. Since μ is not an eigenvalue of H, we have $\mu \neq 0,-2$ and $\mu^2-2\mu \neq 4s$. Now we apply (3.2)–(3.3). When $a=b=2$, since $\mu \neq 0,-1,-2$, we have $s=\frac{\mu^2}{4}-\frac{\mu}{2}$, which is a contradiction to $\mu^2-2\mu \neq 4s$.

When $a=b=0$, since $\mu \neq 0,-2$, $s(\geq 2) \in \mathbb{N}$ and $c \in \mathbb{N}$, we have $\begin{cases} c=\frac{\mu^3+2\mu^2}{\mu-2}, \\ s=\frac{\mu^4+5\mu^3+4\mu}{4\mu-8}. \end{cases}$

Similarly, we can prove other cases in Table 2 and thus (1) holds.

Now we show (2) holds. If X contains both vertices of Class I and vertices of Class II in Table 2, then we have

$$s=\frac{\mu^4+5\mu^3+4\mu}{4\mu-8}=\frac{\mu^4+5\mu^3+4\mu^2-2\mu-4}{2\mu}. \tag{3.7}$$

Thus $\frac{(\mu^3-3\mu^2-4\mu+4)(\mu+2)^2}{4\mu(\mu-2)}=0$. Since $\mu \neq -2$, we have $\mu^3-3\mu^2-4\mu+4=0$. We substitute the solution of equation $\mu^3-3\mu^2-4\mu+4=0$ into (3.7), and find s is not an integer, it is a contradiction. Thus X cannot contain both vertices of Class I and vertices of Class II.

Similarly, we can prove that any other two classes of vertices in Table 2 cannot exist in X at the same time, except for Class IV and Class V. In the following we show if the vertices in X are of Class IV and Class V, then G is not regular.

From Table 2, we have

$$s=\frac{\mu^3}{2}+\frac{5\mu^2}{2}+2\mu=\mu^3+4\mu^2+2\mu-2.$$

Thus $\frac{(\mu-1)(\mu+2)^2}{2}=0$. Since $\mu \neq -2$, we have $\mu=1$ and $s=5$. Thus the vertices in X are of type $(0,2,3)$, $(2,0,3)$, $(1,2,1)$, $(2,1,1)$. Now we consider the regular graph with $K_{2,2,5}$ as a star complement for $\mu=1$.

From (3.4), we have Table 3. Since $\rho_{uv} \in \mathbb{N}$, the vertices of type (a,b,c) and (α,β,γ) shown in Table 3 cannot exist in X at the same time. If the types of vertices in X are $\{(0,2,3)\}$, $\{(1,2,1)\}$ or $\{(0,2,3),(1,2,1)\}$, then $d_G(u_1) < d_G(v_1)$; if the types of vertices in X are $\{(2,0,3)\}$, $\{(2,1,1)\}$ or $\{(2,0,3),(2,1,1)\}$, then $d_G(u_1) > d_G(v_1)$. Clearly, the above cases imply a contradiction with the regularity of graph G. Thus there is no regular graph with $K_{2,2,5}$ as a star complement for $\mu=1$.

(a,b,c)	(α,β,γ)	ρ_{uv}
$(0,2,3)$	$(2,1,1)$	$\frac{7}{3}-a_{uv}$
$(2,0,3)$	$(1,2,1)$	$\frac{7}{3}-a_{uv}$
$(0,2,3)$	$(2,0,3)$	$\frac{8}{3}-a_{uv}$
$(1,2,1)$	$(2,1,1)$	$\frac{8}{3}-a_{uv}$

Table 3 The infeasible solution of (3.4).

Therefore, if G is a regular graph with $H\cong K_{2,2,s}$ $(s\geq 2)$ as a star complement for μ, then the vertices in X are of only one class.

Now we characterize the regular graphs with the tripartite graph $K_{2,2,s}$ $(s\geq 2)$ as a star complement.

Theorem 3.2 *Let $s\geq 2$. If the r-regular graph G has $H\cong K_{2,2,s}$ as a star complement for an eigenvalue μ of multiplicity k, then one of the following holds*:

(1) $\mu=-1$, $r\equiv -1 \pmod{(4s-3)}$ *and* $G\cong G(r)$;

(2) $\mu=4$, $s=74$ *and G is a* 76*-regular graph of order* 189;

(3) $\mu=1$, $r=6$, $s=2$ *and* $G\cong L(K_5)$;

(4) $\mu=-3$, $r=6$, $s=3$ *and* $G\cong K_{3,3,3}$.

Proof Since μ is not an eigenvalue of $H\cong K_{2,2,s}$ $(s\geq 2)$, we have $\mu\neq 0,-2$ and $\mu^2-2\mu\neq 4s$. If $\mu\neq -1$, then $\mu\notin\{-1,0\}$, and by Lemma 2.1, $V(K_{2,2,s})$ is a location-dominating set, thus G is connected by the fact that $H\cong K_{2,2,s}$ is connected.

Case 1 $\mu=-1$. By Theorem 3.1, (1) holds.

Case 2 $\mu=r$. Since r is an eigenvalue with multiplicity 1, we have $|X|=1$. Since G is regular, we have $d_G(u_1)=d_G(u_2)=d_G(v_1)=d_G(v_2)$. Let $X=\{u\}$. Then either $u\sim u_1$, $u\sim u_2$, $u\sim v_1$, $u\sim v_2$, or $u\nsim u_1$, $u\nsim u_2$, $u\nsim v_1$, $u\nsim v_2$.

If $u\sim u_1$, $u\sim u_2$, $u\sim v_1$, $u\sim v_2$, then

$$d_G(u_1)=d_G(u_2)=d_G(v_1)=d_G(v_2)=s+3,$$

which implies that $d_G(u)=s+3$. It follows that the vertex u is adjacent to $s-1$ vertices of W, and thus vertices $w_1,w_2,\cdots,w_s$ can not have the same degree. It is a contradiction.

If $u\nsim u_1$, $u\nsim u_2$, $u\nsim v_1$, $u\nsim v_2$, then

$$d_G(u)=d_G(u_1)=d_G(u_2)=d_G(v_1)=d_G(v_2)=s+2,$$

which means the vertex u is adjacent to $s+2$ vertices of W. It is a contradiction by the fact that $|W|=s$. Thus there is no regular graph G with $K_{2,2,s}$ as a star complement for the eigenvalue r.

Case 3 $\mu\notin\{-1,r\}$.

By Lemma 2.3, μ is non-main. From Lemma 2.1, we know the H-neighbourhoods of distinct vertices in X are distinct and non-empty. By Lemma 3.5, X contains vertices of only one class. Now we consider the following five subcases.

Subcase 3.1 The vertices in X are of Class I.

Let $u \in X$. If X induces an independent set, then we have $r = c = 2+s$ by $d_G(u) = d_G(u_1)$. Since $c = \frac{\mu^3+2\mu^2}{\mu-2}$, $s = \frac{\mu^4+5\mu^3+4\mu}{4\mu-8}$, we have $2 + s - c = \frac{(\mu+4)(\mu^2-\mu+2)}{4} = 0$. Thus $\mu = -4$ by $\mu \in \mathbb{R}$, and then $s = \frac{10}{3}$. It is a contradiction.

If X induces a clique, then we have $r = c + k - 1 = 2 + s = 4 + \frac{kc}{s}$ by $d_G(u) = d_G(u_1) = d_G(w_1) = d_G(w_2) = \cdots = d_G(w_s)$, where $k = |X|$. Thus $k = 3 + s - c = \frac{(s-2)s}{c}$. From Table 2, we have

$$c(3+s-c) - (s-2)s = -\frac{\mu(\mu+2)(\mu^6+4\mu^5+5\mu^4+14\mu^3-12\mu^2+72\mu+32)}{16(\mu-2)^2} = 0.$$

Since $\mu \neq 0, -2$, we have $\mu^6+4\mu^5+5\mu^4+14\mu^3-12\mu^2+72\mu+32 = 0$, then the value of s obtained by substituting the solution of the equation is not an integer. It is a contradiction.

Otherwise, from Lemma 3.3, we have $\mu \in \mathbb{Z}$. Since $c = \mu^2 + 4\mu + 8 + \frac{16}{\mu-2}$, we have $\frac{16}{\mu-2} \in \mathbb{Z}$. Thus $\mu - 2 \in \{\pm 1, \pm 2, \pm 4, \pm 8, \pm 16\}$. Since $\mu \neq 0, -2$ and $c, s \in \mathbb{Z}^+$, by Table 2, the possible values of s and c are shown in Table 4. Considering the regularity of graph G, we have

μ	s	c
3	57	45
18	2097	405
4	74	48
10	470	150
6	150	72

Table 4 The possible values of s and c.

$r = 2 + s = 4 + \frac{kc}{s}$ by $d_G(u_1) = d_G(w_1) = d_G(w_2) = \cdots = d_G(w_s)$. Thus $k = \frac{(s-2)s}{c} \in \mathbb{Z}^+$. Only when $\mu = 4$, $s = 74$ and $c = 48$, k is an integer. In this case, $k = 111$, $r = 76$, $n = 189$ and $\rho_{uv} = \begin{cases} 32, & u \nsim v, \\ 28, & u \sim v. \end{cases}$

Subcase 3.2 The vertices in X are of Class II.

Let $u \in X$. If X induces an independent set, then we have $r = 2 + c = 2 + s + \frac{k}{2}$ by $d_G(u) = d_G(u_1) = d_G(u_2)$. Thus $k = 2(c - s) \leq 0$. It is a contradiction.

If X induces a clique, then $r = 2 + c + k - 1 = 2 + s + \frac{k}{2} = 4 + \frac{kc}{s}$ by $d_G(u) = d_G(u_1) = d_G(u_2) = d_G(w_1) = d_G(w_2) = \cdots = d_G(w_s)$, and thus $k = 2(s - c + 1) = \frac{c-3}{(\frac{c}{s}-1)}$. From Table 2, we have

$$2(s-c+1)(c-s) - s(c-3) = -\frac{(\mu-1)(\mu+2)^2(\mu^5+4\mu^4+4\mu^3+\mu^2-4\mu-2)}{2\mu^2} = 0.$$

Since $\mu \neq -2$, we have $\mu = 1$ or $\mu^5+4\mu^4+4\mu^3+\mu^2-4\mu-2 = 0$. If $\mu^5+4\mu^4+4\mu^3+\mu^2-4\mu-2 = 0$, then the value of s obtained by substituting the solution of the equation is not an integer, it implies $\mu = 1$, and thus $c = 1$, $s = 2$, $k = 4$. By Lemma 3.1, we have

$$\rho_{uv} = \begin{cases} 2, & u \nsim v, \\ 1, & u \sim v. \end{cases} \tag{3.8}$$

Since $G[X] = K_4$, only $\{N_H(u) \mid u \in X\} = \{\{u_1, v_1, w_1\}, \{u_2, v_1, w_2\}, \{u_1, v_2, w_2\}, \{u_2, v_2, w_1\}\}$ satisfies (3.8). Thus $G \cong L(K_5)$.

Otherwise, we have $\mu \in \mathbb{Z}$ by Lemma 3.3. Notice that $c = \mu^2 + 2\mu - \frac{2}{\mu} \in \mathbb{Z}$, then $\frac{2}{\mu} \in \mathbb{Z}$. Since $\mu \neq -1, -2$, we have $\mu = 1$, $c = 1$, $s = 2$ or $\mu = 2$, $c = 7$, $s = 16$ by Table 2. Considering the regularity of the graph G, we have $r = 2 + s + \frac{k}{2} = 4 + \frac{kc}{s}$ by $d_G(u_1) = d_G(u_2) = d_G(w_1) =$

$d_G(w_2) = \cdots = d_G(w_s)$. Thus $k = \frac{s-2}{(\frac{c}{s}-\frac{1}{2})} \in \mathbb{Z}^+$. If $\mu = 2$, then $k = -224$, which implies $\mu = 1$ and $s = 2$ by $k \in \mathbb{Z}^+$. Counting the edges between X and $V(H)$, we have $3k = 6(r-4)$. Thus $r = \frac{k}{2} + 4 \in \mathbb{Z}^+$ which means k is an even number. Since $\rho_{uv} \neq 0$ by (3.8) and for each vertex u of type $(1,1,1)$, there exists another vertex v of type $(1,1,1)$ with $\rho_{uv} = 0$, we have $k \leq \frac{\binom{2}{1}\binom{2}{1}\binom{2}{1}}{2} = 4$. Thus $k = 2$ or 4. When $k = 2$, we have $r = 5$ and $G[X]$ is a 2-regular graph, which contradicts with $k = |X| = 2$. When $k = 4$, we have $r = 6$ and $G[X] = K_4$. Thus $G \cong L(K_5)$.

Subcase 3.3 The vertices in X are of Class III.

Let

$$X_1 = \Big\{u \in X \mid u \text{ is of type } \Big(0, 1, \frac{\mu^3+2\mu^2-1}{\mu-1}\Big)\Big\},$$
$$X_2 = \Big\{u \in X \mid u \text{ is of type } \Big(1, 0, \frac{\mu^3+2\mu^2-1}{\mu-1}\Big)\Big\}.$$

Considering the regularity of the graph G, we have $|X_1| = |X_2| = \frac{k}{2}$. If one of the following three conditions holds: (1) $G[X_1] \ncong K_{\frac{k}{2}}$ and $G[X_1] \ncong \frac{k}{2}K_1$; (2) $G[X_2] \ncong K_{\frac{k}{2}}$ and $G[X_2] \ncong \frac{k}{2}K_1$; (3) $G[E(X_1,X_2)] \ncong K_{\frac{k}{2},\frac{k}{2}}$ and $E(X_1,X_2) \neq \varnothing$, then from Lemmas 3.3–3.4, we have $\mu \in \mathbb{Z}$. From Lemma 3.2, we have $\frac{1}{\mu+2} \in \mathbb{Z}$ by $\rho_{uv} \in \mathbb{Z}$. Since $\mu \neq -1$, we have $\mu = -3$, then $s = \frac{19}{6}$. It is a contradiction.

Otherwise, we have both (4), (5) and (6) hold: (4) $G[X_1] \cong K_{\frac{k}{2}}$ or $G[X_1] \cong \frac{k}{2}K_1$; (5) $G[X_2] \cong K_{\frac{k}{2}}$ or $G[X_2] \cong \frac{k}{2}K_1$; (6) $G[E(X_1,X_2)] \cong K_{\frac{k}{2},\frac{k}{2}}$ or $E(X_1,X_2) = \varnothing$.

Since $G[X]$ is regular, there are four cases: (a) $G[X] = kK_1$; (b) $G[X] = K_k$; (c) $G[X_1] = K_{\frac{k}{2}}$, $G[X_2] = K_{\frac{k}{2}}$ and $E(X_1,X_2) = \varnothing$; (d) $G[X_1] = \frac{k}{2}K_1$, $G[X_2] = \frac{k}{2}K_1$ and $G[E(X_1,X_2)] = K_{\frac{k}{2},\frac{k}{2}}$. For any of these four cases, there is no regular graph G with $K_{2,2,s}$ as a star complement. Since the proof is similar, we only prove case (a), omitting the proof of the other three cases.

Now we show if $G[X] = kK_1$, there is no regular graph G with $K_{2,2,s}$ as a star complement. In fact, if $G[X] = kK_1$, we have $r = 1 + c = 2 + s + \frac{k}{4} = 4 + \frac{kc}{s}$ by $d_G(u) = d_G(u_1) = d_G(u_2) = d_G(w_1) = d_G(w_2) = \cdots = d_G(w_s)$. Thus $k = 4(c-s-1) = \frac{s(c-3)}{c}$. From Table 2, we have

$$4c(c-s-1) - s(c-3) = -\frac{5\mu^7 + 23\mu^6 + 9\mu^5 - 33\mu^4 + 10\mu^3 - 6\mu + 4}{3(\mu-1)^2} = 0.$$

Thus $5\mu^7 + 23\mu^6 + 9\mu^5 - 33\mu^4 + 10\mu^3 - 6\mu + 4 = 0$. However, the value of s obtained by substituting the solution of the equation is not an integer. It is a contradiction.

Subcase 3.4 The vertices in X are of Class IV.

Let $X_1 = \{u \in X \mid u \text{ is of type } (0,2,\mu^2+2\mu)\}$, $X_2 = \{u \in X \mid u \text{ is of type } (2,0,\mu^2+2\mu)\}$. Considering the regularity of graph G, we have $|X_1| = |X_2| = \frac{k}{2}$.

If $c = 2$, then $\mu = \pm\sqrt{3} - 1$ and $s = 3$. Since $r = \frac{k}{2} + 2 + 3 = 4 + \frac{2k}{3}$ by $d_G(u_1) = d_G(u_2) = d_G(w_1) = d_G(w_2) = d_G(w_3)$, we have $k = 6$. However, from Lemma 3.1, we have $\rho_{uv} = \pm\sqrt{3}(a_{uv} - 2) + a_{uv} \notin \mathbb{Z}$. Hence, the star set X can only contain one vertex of type $(0,2,2)$ and one vertex of type $(2,0,2)$, which contradicts with $k = 6$.

If $c \neq 2$, we have $\mu = \frac{2s-3c}{c-2}$ by Table 2, then $\mu \in \mathbb{Q}$ by the fact that $s, c \in \mathbb{Z}$. Notice that μ is an algebraic integer, then $\mu \in \mathbb{Z}$. Since $\rho_{uv} \in \mathbb{Z}$ for $u \in X_1$ and $v \in X_2$, from Lemma 3.2, we have $\frac{4}{\mu+2} \in \mathbb{Z}$. Since $\mu \neq 0, -1$, we have $\mu \in \{-6, -4, -3, 2\}$.

Now we show $\mu \neq -4, -6, 2$. If $\mu = -4$, then $s = 0 \notin \mathbb{Z}^+$, it is a contradiction. If $\mu = -6$, then $s = -30 \notin \mathbb{Z}^+$. If $\mu = 2$, then $c = 8$ and $s = 18$. Considering the regularity of graph G, we have $r = \frac{k}{2} + 2 + 18 = 4 + \frac{8k}{18}$ by $d_G(u_1) = d_G(u_2) = d_G(w_1) = d_G(w_2) = \cdots = d_G(w_{18})$. Hence $k = -288 \notin \mathbb{Z}^+$, it is a contradiction.

Therefore, $\mu = -3$, and thus $c = s = 3$. Considering the regularity of graph G, we have $r = \frac{k}{2} + 2 + 3 = 4 + k$ by $d_G(u_1) = d_G(u_2) = d_G(w_1) = d_G(w_2) = d_G(w_3)$. So $k = 2$, $r = 6$. Since $\rho_{uv} = \begin{cases} 0, & u \nsim v \\ 3, & u \sim v \end{cases}$, we have $G[X] = K_2$ and

$$\{N_H(u) \mid u \in X\} = \{\{u_1, u_2, w_1, w_2, w_3\}, \{v_1, v_2, w_1, w_2, w_3\}\}.$$

Thus $G \cong K_{3,3,3}$.

Subcase 3.5 The vertices in X are of Class V.

Let $X_1 = \{u \in X \mid u \text{ is of type } (1, 2, \mu^2 + \mu - 1)\}$, $X_2 = \{u \in X \mid u \text{ is of type } (2, 1, \mu^2 + \mu - 1)\}$. Considering the regularity of graph G, we have $|X_1| = |X_2| = \frac{k}{2}$. If $c = 0$, then $\mu = \frac{\pm\sqrt{5}-1}{2}$ and $s = 1$, which contradicts with $s \geq 2$. If $c \neq 0$, we have $\mu = \frac{s-1}{c} - 3$ by Table 2, then $\mu \in \mathbb{Q}$. Notice that μ is an algebraic integer, then $\mu \in \mathbb{Z}$. Since $\rho_{uv} \in \mathbb{Z}$ for $u \in X_1$ and $v \in X_2$, from Lemma 3.2, we have $\frac{1}{\mu+2} \in \mathbb{Z}$. Since $\mu \neq -1$, we have $\mu = -3$, and then $s = 1$, $c = 5 > s$, which implies a contradiction by $0 \leq c \leq s$.

Combining the above arguments, we complete the proof.

Remark 3.1 The graph $L(K_5)$ is strongly regular with parameters $(10, 6, 3, 4)$, and the spectrum is $[-2^5, 1^4, 6]$; the graph $K_{3,3,3}$ is strongly regular with parameters $(9, 6, 3, 6)$, and the spectrum is $[-3^2, 0^6, 6]$.

4 Maximal Graphs with $K_{2,2,s}$ as a Star Complement for $\mu = 1$

In this section, we characterize the maximal graphs with $K_{2,2,s}$ as a star complement for the eigenvalue $\mu = 1$.

Lemma 4.1 *Let $s \geq 2$. Then $K_{2,2,2}$, $K_{2,2,5}$, $K_{2,2,6}$ and $K_{2,2,20}$ are the only graphs among $K_{2,2,s}$ which can be star complements for $\mu = 1$.*

Proof Let G be the maximal graph with $H \cong K_{2,2,s}$ as a star complement for $\mu = 1$. Then G is connected since $K_{2,2,s}$ is connected and $V(K_{2,2,s})$ is a location-dominating set (see Lemma 2.1). Let $u \in X$ be a vertex of type (a, b, c), thus $(a, b, c) \neq (0, 0, 0)$ and $0 \leq a \leq 2$, $0 \leq b \leq 2$, $0 \leq c \leq s$. We distinguish the following six cases.

Case 1 $a = b = 0$.

By (3.3), we have $12s(1 - c) + 12c^2 - 3c + 3 = 0$. Clearly, $c \neq 1$. So

$$\frac{3c + 1}{4c - 4} = s - c \in \mathbb{Z}. \tag{4.1}$$

Notice that $0 < \frac{3c+1}{4c-4} < 1$ whenever $c > 5$, and in this case $s - c$ is not an integer, a contradiction. So $0 \leq c \leq 5$, then we have $c = 5$ and $s = 6$ by $s \in \mathbb{Z}$ and (4.1).

Case 2 $a = b = 1$.

By (3.3), we have $12c^2 + 9c + 3 - 12cs = 0$. Clearly, $c \neq 0$. So

$$\frac{3c + 1}{4c} = s - c \in \mathbb{Z}. \tag{4.2}$$

If $c > 1$, we have $0 < \frac{3c+1}{4c} < 1$, which contradicts with $s - c \in \mathbb{Z}$. So $c = 1$, and thus $s = 2$ by (4.2).

Case 3 $a = b = 2$.

By (3.3), we have $12s(1-c)+12c^2+21c+15=0$. Clearly, $c \neq 1$. So

$$\frac{3c+13}{4c-4} = s-c-2 \in \mathbb{Z}. \tag{4.3}$$

Notice that $0 < \frac{3c+13}{4c-4} < 1$ whenever $c > 17$, and in this case $s-c-2$ is not an integer, a contradiction. For $0 \leq c \leq 17$, we have $c = 17$ and $s = 20$ by $s \in \mathbb{Z}$ and (4.3).

Case 4 $a = 0$, $b = 1$ or $a = 1$, $b = 0$.

By (3.3), we have $s(5-12c)+12c^2+3c+2=0$. So $\frac{4c-7}{12c-5} = c-s+1 \in \mathbb{Z}$.

If $c \geq 2$, we have $0 < \frac{4c-7}{12c-5} < 1$, which contradicts with $c-s+1 \in \mathbb{Z}$. If $c = 1$ or $c = 0$, we have $s \notin \mathbb{Z}$, which implies a contradiction.

Case 5 $a = 0$, $b = 2$ or $a = 2$, $b = 0$.

By (3.3), we have $s(8-12c)+12c^2+9c+5=0$. So $\frac{5c+13}{12c-8} = s-c-1 \in \mathbb{Z}$.

If $c > 3$, we have $0 < \frac{5c+13}{12c-8} < 1$, which contradicts with $s-c-1 \in \mathbb{Z}$. If $0 \leq c \leq 3$, then we have $c = 3$ and $s = 5$ by $s \in \mathbb{Z}$.

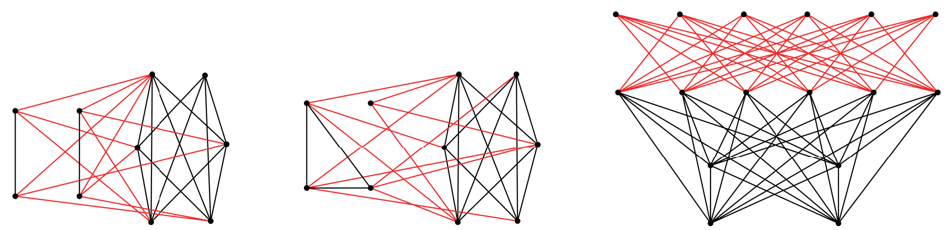
Case 6 $a = 1$, $b = 2$ or $a = 2$, $b = 1$.

By (3.3), we have $s(5-12c)+12c^2+15c+8=0$. So $\frac{8c+13}{12c-5} = s-c-1 \in \mathbb{Z}$.

If $c \geq 5$, we have $0 < \frac{8c+13}{12c-5} < 1$, which contradicts with $s-c-1 \in \mathbb{Z}$. If $0 \leq c \leq 4$, we have $c = 1$ and $s = 5$ by $s \in \mathbb{Z}$.

Combining the above six cases, we complete the proof.

Theorem 4.1 *Let $\mu = 1$. Then the graphs $L(K_5)$, G_1 and G_2 (see Figure 1 are the three non-isomorphic maximal graphs with $K_{2,2,2}$ as a star complement for μ; $K_2 \nabla GQ(2,4)$ is the unique maximal graph with $K_{2,2,5}$ as a star complement for μ; G_3 (see Figure 1) is the unique maximal graph with $K_{2,2,6}$ as a star complement for μ.*



(a) G_1 (b) G_2 (c) G_3

Figure 1 Graphs G_1, G_2 and G_3.

Proof Let $s = 2$. By Case 2 of the proof of Lemma 4.1, we know the vertices in X are of type $(1,1,1)$, thus by Lemma 3.1 we have $\rho_{uv} = \begin{cases} 2, & u \nsim v \\ 1, & u \sim v \end{cases}$. From Lemma 2.1, we know the H-neighbourhoods of distinct vertices in X are distinct and non-empty. Since $\rho_{uv} \neq 0$, $\{N_H(u) \mid u \in X\}$ can only contain at most one element from each of the following four sets: $\{\{u_1,v_1,w_1\},\{u_2,v_2,w_2\}\}$, $\{\{u_1,v_2,w_1\},\{u_2,v_1,w_2\}\}$, $\{\{u_1,v_2,w_2\},\{u_2,v_1,w_1\}\}$, $\{\{u_1,v_1,w_2\},\{u_2,v_2,w_1\}\}$. Thus $k \leq 4$. Considering the symmetry of $K_{2,2,2}$, there are three non-isomorphic maximal families $\{N_H(u) \mid u \in X\}$ which are shown as follows:

$$\{\{u_1,v_1,w_1\},\{u_2,v_1,w_2\},\{u_1,v_2,w_2\},\{u_2,v_2,w_1\}\};$$

$$\{\{u_1, v_1, w_1\}, \{u_1, v_2, w_1\}, \{u_1, v_2, w_2\}, \{u_1, v_1, w_2\}\};$$
$$\{\{u_1, v_1, w_1\}, \{u_1, v_2, w_1\}, \{u_1, v_2, w_2\}, \{u_2, v_2, w_1\}\}.$$

Therefore, the maximal graphs with $K_{2,2,2}$ as a star complement for $\mu = 1$ are $L(K_5)$, G_1 and G_2.

Let $s = 5$. Then the vertices in X are of type $(0, 2, 3)$, $(2, 0, 3)$, $(1, 2, 1)$ or $(2, 1, 1)$. From Table 3, we know the vertices in X can contain vertices of type $(0, 2, 3)$ and $(1, 2, 1)$, or contain vertices of type $(2, 0, 3)$ and $(2, 1, 1)$. Observe the symmetry of a and b in (3.4), we only consider the first case. By (3.4), we have Table 5. There are 10 possibilities for the H-neighbourhood of a vertex of each type. By Table 5, we know all 20 possibilities are compatible. Thus $|X| = 20$ and the maximal graph with $K_{2,2,5}$ as a star complement for $\mu = 1$ is $K_2 \triangledown GQ(2, 4)$, where $GQ(2, 4)$ is an order-$(2, 4)$ generalized quadrangle.

(a, b, c)	(α, β, γ)	a_{uv}	ρ_{uv}
$(0, 2, 3)$	$(1, 2, 1)$	0	3
$(0, 2, 3)$	$(1, 2, 1)$	1	2
$(0, 2, 3)$	$(0, 2, 3)$	0	4
$(0, 2, 3)$	$(0, 2, 3)$	1	3
$(1, 2, 1)$	$(1, 2, 1)$	0	3
$(1, 2, 1)$	$(1, 2, 1)$	1	2

Table 5 The feasible solution of (3.4) when $s = 5$.

Let $s = 6$. Then the vertices in X are of type $(0, 0, 5)$ and $\rho_{uv} = \begin{cases} 4, & u \nsim v, \\ 3, & u \sim v \end{cases}$ by Lemma 3.1. Thus $G[X] = 6K_1$ and $\{N_H(u) \mid u \in X\} = \{\{w_2, w_3, w_4, w_5, w_6\}, \{w_1, w_3, w_4, w_5, w_6\}, \{w_1, w_2, w_4, w_5, w_6\}, \{w_1, w_2, w_3, w_5, w_6\}, \{w_1, w_2, w_3, w_4, w_6\}, \{w_1, w_2, w_3, w_4, w_5\}\}$. Thus $G \cong G_3$.

Combining the above arguments, we complete the proof.

Remark 4.1 The graph G_1 is a graph of order 10 with spectrum $[-2^4, -1.46410, 1^4, 5.46410]$; the graph G_2 is a graph of order 10 with spectrum $[-2^4, -1.60555, 1^4, 5.60555]$; the graph $K_2 \triangledown GQ(2, 4)$ is a graph of order 29 with spectrum $[-5^6, -3.88819, 0, 1^{20}, 13.88819]$, where generalized quadrangle $GQ(2, 4)$ is strongly regular with parameters $(27, 10, 1, 5)$; the graph G_3 is a graph of order 16 with spectrum $[-6.58872, -2, -1^5, 0^2, 1^6, 7.58872]$.

Now we consider the maximal graph with $K_{2,2,20}$ as a star complement for $\mu = 1$. Let (U, V, W) be a tripartition of the graph $K_{2,2,20}$ with $U = \{u_1, u_2\}$, $V = \{v_1, v_2\}$, $W = \{w_1, w_2, \cdots, w_{20}\}$. We know all vertices in X are of type $(2, 2, 17)$ by Case 3 of the proof of Lemma 4.1 and $\rho_{uv} = \begin{cases} 20, & u \nsim v, \\ 19, & u \sim v \end{cases}$ by Lemma 3.1. Let $N_W(u) = \{v \mid v \sim u, v \in W\}$ be the W-neighbourhood of vertex $u \in X$ and $\rho'_{uv} = |N_W(u) \cap N_W(v)|$. Then $\rho'_{uv} = \begin{cases} 16, & u \nsim v, \\ 15, & u \sim v. \end{cases}$ Let $\mathcal{F}_{17}$ be a family of 17-subsets of W, $W^{(3)}$ be the family of all the 3-subsets of W and $\mathcal{F}_3 \subset W^{(3)}$. We say that the family $\mathcal{F}_{17}$ is compatible if $|S_1 \cap S_2| \in \{15, 16\}$ for any distinct sets $S_1, S_2 \in \mathcal{F}_{17}$, equivalently $W \setminus S_1 := \overline{S_1} \nsubseteq S_2$. In the following we give an algorithm for finding the maximal compatible families $\mathcal{F}_{17}$ (see Algorithm 1), and then give some examples.

Theorem 4.2 *Let G be a graph with $K_{2,2,20}$ as a star complement for $\mu = 1$, (U, V, W) be a tripartition of the graph $K_{2,2,20}$ with $|U| = |V| = 2$, $|W| = 20$. Then G is a maximal*

Algorithm 1 Maximal Family $\mathcal{F}_{17}$

```
F_17 ← ∅, F_3 ← W^(3)
while F_3 ≠ ∅ do
    Select F ∈ F_3
    F_17 ← F_17 ∪ \overline{F}
    F_3 ← F_3 \ (\overline{F}^(3) ∪ {F})
end while
return F_17
```

graph if and only if all vertices in X are of type $(2,2,17)$ and the family of W-neighbourhoods $N_W(u)$ $(u \in X)$ can be obtained from Algorithm 1.

Proof If the family of W-neighbourhoods $N_W(u)$ $(u \in X)$ can be obtained from Algorithm 1, then X will be given. Let $\mathcal{F}_{17}$ be a family obtained from Algorithm 1. Firstly, we prove that $\mathcal{F}_{17}$ is a compatible family. Let $S_i^{(3)}$ be the family of all the 3-subsets of S_i. If there are two sets $S_1, S_2 \in \mathcal{F}_{17}$ such that $|S_1 \cap S_2| = 14$, then $\overline{S_2} \in S_1^{(3)}$. Suppose that in Algorithm 1, S_1 is the first selected into $\mathcal{F}_{17}$, then after S_1 is selected, the elements in $S_1^{(3)}$ are deleted from $\mathcal{F}_3$. Thus $\overline{S_2} \notin \mathcal{F}_3$ and then $S_2 \notin \mathcal{F}_{17}$. It is a contradiction. Secondly, we prove that $\mathcal{F}_{17}$ is maximal. Suppose that $\mathcal{F}'_{17}$ is a compatible family and $\mathcal{F}_{17} \subset \mathcal{F}'_{17}$ with $S_0 \in \mathcal{F}'_{17} \setminus \mathcal{F}_{17}$, then for any $S_1 \in \mathcal{F}'_{17}$, we have $|S_0 \cap S_1| \neq 14$. Since $\mathcal{F}_{17} \subset \mathcal{F}'_{17}$, for any $S_2 \in \mathcal{F}_{17}$, we have $|S_0 \cap S_2| \neq 14$ and $S_0 \neq S_2$. Thus $\overline{S_0} \notin S_2^{(3)} \cup \{\overline{S_2}\}$ and then $\overline{S_0} \in \mathcal{F}_3 \neq \varnothing$ in Algorithm 1. It is a contradiction. So $\mathcal{F}_{17}$ is a maximal compatible family. Therefore, G is maximal.

Conversely, let G be a maximal graph, and $\mathcal{F}_{17} = \{S_1, S_2, \cdots, S_k\}$ be the family of W-neighbourhoods of all vertices $u \in X$. Then $\mathcal{F}_{17}$ is a maximal compatible family. Now we show that $\mathcal{F}_{17}$ can be obtained from Algorithm 1. Let F_i be the i-th selected element in $\mathcal{F}_3$ in Algorithm 1 and $\overline{F_i} = S_i$. We will prove that $\{F_1, F_2, \cdots, F_k\}$ satisfies Algorithm 1. For $1 \le i \le k$, let $\mathcal{F}_{17,i} = \{S_1, S_2, \cdots, S_i\}$ and $\mathcal{F}_{3,i} = W^{(3)} \setminus (S_1^{(3)} \cup S_2^{(3)} \cup \cdots \cup S_i^{(3)} \cup \{\overline{S_1}, \overline{S_2}, \cdots, \overline{S_i}\})$. Since $\mathcal{F}_{17} = \{S_1, S_2, \cdots, S_k\}$ is compatible, for any $j \le i$ where $i < k$, we have $S_{i+1} \neq S_j$ and $|S_j \cap S_{i+1}| \neq 14$. Thus $F_{i+1} \neq \overline{S_j}$, $F_{i+1} = \overline{S_{i+1}} \nsubseteq S_j$ and then $F_{i+1} \notin S_1^{(3)} \cup S_2^{(3)} \cup \cdots \cup S_i^{(3)} \cup \{\overline{S_1}, \overline{S_2}, \cdots, \overline{S_i}\}$. Therefore, $F_{i+1} \in \mathcal{F}_{3,i}$ and we can select F_{i+1} in the $(i+1)$-th step of the algorithm. Next, we prove that after the algorithm proceeds to the k-th step, $\mathcal{F}_{3,k} = W^{(3)} \setminus (S_1^{(3)} \cup S_2^{(3)} \cup \cdots \cup S_k^{(3)} \cup \{\overline{S_1}, \overline{S_2}, \cdots, \overline{S_k}\}) = \varnothing$. If $\mathcal{F}_{3,k} \neq \varnothing$, then there exists $F \in \mathcal{F}_{3,k}$, s.t. $F \notin S_1^{(3)} \cup S_2^{(3)} \cup \cdots \cup S_k^{(3)} \cup \{\overline{S_1}, \overline{S_2}, \cdots, \overline{S_k}\}$. Thus for any $i \in [k]$, $\overline{F} \neq S_i$, $|\overline{F} \cap S_i| \neq 14$ and then $\{S_1, S_2, \cdots, S_k, \overline{F}\}$ is compatible, which contradicts with G is maximal. To sum up, $\mathcal{F}_{17}$ can be obtained from Algorithm 1.

It is easy to verify that the following two examples can be obtained from Algorithm 1.

Example 4.1 Let G_4 be a maximal graph with $H = K_{2,2,20}$ as a star complement for $\mu = 1$ where all vertices in X are of type $(2,2,17)$ and the family of W-neighbourhoods $N_W(u)$ $(u \in X)$ is $\{S \mid S \subset W \setminus \{w_1\}, |S| = 17\}$. Then G_4 is a graph of order 195 and it is easy to show that G_4 is the graph of the maximum order among all maximal graphs with $K_{2,2,20}$ as a star complement for $\mu = 1$ by Erdős-Ko-Rado Theorem (see [12, Theorem 4.1]).

Example 4.2 Let G_5 be a maximal graph with $H = K_{2,2,20}$ as a star complement for $\mu = 1$ where all vertices in X are of type $(2,2,17)$ and the family of W-neighbourhoods $N_W(u)$ $(u \in X)$ is $\overline{\{w_2, w_3, w_4\}} \cup \mathcal{F}$, where $\mathcal{F} = \{\overline{F} \mid F \subset W, |F| = 3, w_1 \in F, F \cap \{w_2, w_3, w_4\} \neq \varnothing\}$. Then G_5 is a graph of order 76, which is the graph with the second largest order among all maximal graphs with $K_{2,2,20}$ as a star complement for $\mu = 1$ by Hilton-Milner Theorem (see [12, Theorem 8.1]).

5 Concluding Remarks

In Theorem 3.2, we characterize the regular graphs with $K_{2,2,s}(s \geq 2)$ as a star complement for an eigenvalue $\mu \in \mathbb{R}$. But the existence and structure of G in (2) of Theorem 3.2 are not clear, which is a question worth investigating. In Section 4, we characterize the maximal graphs with $K_{2,2,s}$ as a star complement for $\mu = 1$. The structure of maximal graphs with $K_{2,2,s}$ as a star complement for other eigenvalues is also an interesting question worth studying. Thus, we propose the following problems.

Question 5.1 Can we give a specific characterization of the structure of G in (2) of Theorem 3.2?

Question 5.2 Let $\mu \neq 1$. What are the maximal graphs with $K_{2,2,s}$ as a star complement for the eigenvalue μ?

Acknowledgement The authors would like to thank the referees for their valuable comments, corrections and suggestions, which lead to an improvement of the original manuscript.

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