Chin. Ann. Math. Ser. B
44(4), 2023, 533–548
DOI: 10.1007/s11401-023-0030-0

Chinese Annals of Mathematics, Series B

Picard-Type Theorem and Curvature Estimate on an Open Riemann Surface with Ramification*

Zhixue LIU[1] Yezhou LI[2]

Abstract Let M be an open Riemann surface and $G : M \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic map. Consider the conformal metric on M which is given by $\mathrm{d}s^2 = \|\widetilde{G}\|^{2m}|\omega|^2$, where $\widetilde{G}$ is a reduced representation of G, ω is a holomorphic 1-form on M and m is a positive integer. Assume that $\mathrm{d}s^2$ is complete and G is k-nondegenerate$(0 \le k \le n)$. If there are q hyperplanes $H_1, H_2, \cdots, H_q \subset \mathbb{P}^n(\mathbb{C})$ located in general position such that G is ramified over H_j with multiplicity at least $\gamma_j (> k)$ for each $j \in \{1, 2, \cdots, q\}$, and it holds that

$$\sum_{j=1}^{q}\Big(1-\frac{k}{\gamma_j}\Big) > (2n-k+1)\Big(\frac{mk}{2}+1\Big),$$

then M is flat, or equivalently, G is a constant map. Moreover, one further give a curvature estimate on M without assuming the completeness of the surface.

Keywords Picard-type theorem, Holomorphic map, Riemann surface, Curvature estimate
2000 MR Subject Classification 32H25, 32A22, 32H02

1 Introduction

In complex analysis, the Little Picard Theorem says that a non-constant meromorphic function on $\mathbb{C}$ cannot omit more than two points in $\mathbb{C} \cup \{\infty\}$.

For a minimal surface $X : M \to \mathbb{R}^3$, if we choose an isothermal coordinate (u, v) and by letting $z = u + \mathrm{i}v$, one can make M into a Riemann surface. The induced metric $\mathrm{d}s^2$ on M through X from the standard inner product on $\mathbb{R}^3$ can be represented as $\mathrm{d}s^2 = (1 + |g|^2)^2|\omega|^2$, where ω is a holomorphic 1-form and $g : M \to \mathbb{C} \cup \{\infty\}$ is the Gauss map of M which is a meromorphic function.

In 1986, Fujimoto (see [7, Corollary 1.3]) proved an analogous result to the Little Picard Theorem in complex analysis: The Gauss map of a complete non-flat minimal surface in $\mathbb{R}^3$ cannot omit more than four points on the unit sphere. As one knows, the Gauss map of a complete minimal surface in Euclidean space carries many similar value distribution properties

Manuscript received November 18, 2021. Revised March 6, 2023.
[1]Corresponding author. School of Science, Beijing University of Posts and Telecommunications, Beijing 100876, China; Key Laboratory of Mathematics and Information Networks (Beijing University of Posts and Telecommunications), Ministry of Education, China. E-mail: zxliumath@bupt.edu.cn
[2]School of Science, Beijing University of Posts and Telecommunications, Beijing 100876, China; Key Laboratory of Mathematics and Information Networks (Beijing University of Posts and Telecommunications), Ministry of Education, China. E-mail: yezhouli@bupt.edu.cn
*This work was supported by the National Natural Science Foundation of China (Nos. 12101068, 12261106, 12171050).

as the meromorphic functions on the complex plane $\mathbb{C}$. In 2013, Kawakami [13] obtained the following result as an extension of Fujimoto's result.

Theorem 1.1 (see [13, Corollary 2.2]) *Let M be an open Riemann surface with the conformal metric*

$$\mathrm{d}s^2 = (1+|g|^2)^m|\omega|^2,$$

where ω is a holomorphic 1-form, g is a meromorphic function on M and $m \in \mathbb{N}$. Let $\alpha_1, \cdots, \alpha_q \in \mathbb{C}\cup\{\infty\}$ be distinct and $\gamma_1, \cdots, \gamma_q \in \mathbb{N}\cup\{\infty\}$. Suppose the metric $\mathrm{d}s^2$ is complete and

$$\sum_{j=1}^{q}\Big(1-\frac{1}{\gamma_j}\Big) > m+2.$$

If all α_j-points of g have multiplicity at least γ_j, then g is a constant.

In above theorem, the geometric interpretation of the 2 in $m+2$ is the Euler characteristic of the Riemann sphere. Indeed, if $m=0$, $\mathrm{d}s^2 = |\omega|^2$ becomes a flat metric. Owing to the completeness of $\mathrm{d}s^2$, the universal cover of M is the whole complex plane $\mathbb{C}$. Let $\pi : \mathbb{C} \to M$ be the universal covering map, and g can be seen as a holomorphic map from $\mathbb{C}$ into $\mathbb{P}^1(\mathbb{C})$ by replacing g with $g\circ\pi$. By setting $\gamma_j = \infty (1 \le j \le q)$, it recovers the well-known Little Picard Theorem. As an application of this theorem, Kawakami [13] also obtained an analogue of a special case of the Ahlfors islands theorem (see [1] for details of this theorem) for the meromorphic function g on with the complete conformal metric $\mathrm{d}s^2$.

In 1983, Nochka [15] introduced the notion of so-called Nochka weights and obtained a result, which solved the longstanding Cartan's conjecture as follows.

Theorem 1.2 (see [15, Theorem 2]) *Let G be a holomorphic map from $\mathbb{C}$ into $\mathbb{P}^n(\mathbb{C})$. Assume that G is k-nondegenerate for some k with $1 \le k \le n$. If there are q hyperplanes $H_1, H_2, \cdots, H_q \subset \mathbb{P}^n(\mathbb{C})$ located in general position such that G is ramified over H_j with multiplicity at least γ_j for each $j \in \{1, 2, \cdots, q\}$, then*

$$\sum_{j=1}^{q}\Big(1-\frac{k}{\gamma_j}\Big) \le 2n-k+1.$$

Fujimoto [8] and Ru [19] proved that Gauss map of a complete non-flat minimal surfaces immersed in $\mathbb{R}^n$ omits at most $\frac{n(n+1)}{2}$ hyperplanes in $\mathbb{P}^{n-1}(\mathbb{C})$ located in general position. Ros [17] gave a simple and unified proof of the curvature estimate for minimal surface in $\mathbb{R}^3$ whose Gauss map image omits five points. Later, Osserman and Ru [16] obtained a version of the curvature estimate for minimal surfaces in higher dimension.

Theorem 1.3 (see [16, Theorem 1.1]) *Let $X : M \to \mathbb{R}^n$ be a minimal surface. Suppose that its Gauss map G omits more than $\frac{n(n+1)}{2}$ hyperplanes in $\mathbb{P}^{n-1}(\mathbb{C})$, located in general position. Then there exists a constant C, depending on the set of omitted hyperplanes, but not the surface, such that*

$$|K(p)|^{\frac{1}{2}} \le \frac{C}{d(p)},$$

where $K(p)$ is the Gauss curvature of the surface at p, and $d(p)$ is the geodesic distance from p to the boundary of M.

After that, the case of ramification in above theorem was also verified by Liu and Pang [14]. Motivated by the study of the Gauss map of minimal surface, value distribution properties of a holomorphic map on Riemann surface M with ramification are investigated. One first shows a Picard-type theorem for holomorphic maps on M by using the Ahlfors' method in Nevanlinna theory. Furthermore, a curvature estimate on M whose metric is induced from a non-constant holomorphic map $G: M \to \mathbb{P}^n(\mathbb{C})$ is given.

2 Main Results

Let M be an open Riemann surface and $G: M \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic map. Take a locally reduced representation $\widetilde{G} = (g_0, g_1, \cdots, g_n)$ of G, and write $\|\widetilde{G}\|^2 = \sum\limits_{j=0}^{n} |g_j|^2$. Let

$$\mathrm{d}s^2 = \|\widetilde{G}\|^{2m}|\omega|^2$$

be the conformal metric on M, where $m \in \mathbb{N}$, and $\omega = \eta \mathrm{d}z$ is a holomorphic 1-form.

We prove the following result.

Theorem 2.1 *Let M be an open Riemann surface and $G: M \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic map. Consider the conformal metric on M which is given by $\mathrm{d}s^2 = \|\widetilde{G}\|^{2m}|\omega|^2$, where $\widetilde{G}$ is a reduced representation of G, ω is a holomorphic 1-form on M and m is a nonnegative integer. Assume that $\mathrm{d}s^2$ is complete and G is k-nondegenerate$(0 \le k \le n)$. If there are q hyperplanes $H_1, H_2, \cdots, H_q \subset \mathbb{P}^n(\mathbb{C})$ located in general position such that G is ramified over H_j with multiplicity at least $\gamma_j(> k)$ for each $j \in \{1, 2, \cdots, q\}$, and it holds that*

$$\sum_{j=1}^{q}\Big(1 - \frac{k}{\gamma_j}\Big) > (2n - k + 1)\Big(\frac{mk}{2} + 1\Big),$$

then M is flat, or equivalently, G is a constant map.

Remark 2.1 As discussed in Introduction of this paper, the universal cover of complete Riemann surface M is the whole complex plane $\mathbb{C}$ in the case of $m = 0$. We thus get that k-nondegenerate holomorphic map G of $\mathbb{C}$ omits at most $2n - k + 1$ hyperplanes in $\mathbb{P}^n(\mathbb{C})$ located in general position. So, Theorem 2.1 recovers the Nochka's result (i.e., Theorem 1.2).

From Theorem 2.1, one gets immediately the following corollary which is an extension of [5, Corollary 1].

Corollary 2.1 *Let M be an open Riemann surface and $G: M \to \mathbb{P}^n(\mathbb{C})$ be a nonconstant holomorphic map. Take a reduced representation $\widetilde{G}$ of G, and let*

$$\mathrm{d}s^2 = \|\widetilde{G}\|^{2m}|\omega|^2$$

be the conformal metric defined on M, where ω is a holomorphic 1-form and m is a positive integer. Assume that $\mathrm{d}s^2$ is complete. If there are q hyperplanes $H_1, H_2, \cdots, H_q \subset \mathbb{P}^n(\mathbb{C})$ located in general position such that G is ramified over H_j with multiplicity at least $\gamma_j(> n)$ for each $j \in \{1, 2, \cdots, q\}$, then

$$\sum_{j=1}^{q}\Big(1 - \frac{n}{\gamma_j}\Big) \le \frac{n+1}{2}(mn + 2).$$

Proof To see how Theorem 2.1 implies the above Corollary 2.1, one knows that if G is not constant, then G is always k-nondegenerate with some $1 \leq k \leq n$. From Theorem 2.1, G is ramified over a set of hyperplanes $\{H_j\}_{j=1}^q$ with multiplicity at least $\gamma_j(> n)$ for each $j \in \{1, 2, \cdots, q\}$, and

$$\sum_{j=1}^{q}\Big(1-\frac{k}{\gamma_j}\Big) \leq (2n-k+1)\Big(\frac{mk}{2}+1\Big). \tag{2.1}$$

Set

$$\begin{aligned}Q(k) &= (2n-k+1)\Big(\frac{mk}{2}+1\Big)\\ &= -\frac{m}{2}\Big(k^2-\Big(2n+1-\frac{2}{m}\Big)k\Big)+2n+1.\end{aligned}$$

Obviously, $\max\limits_{1\leq k\leq n, k\in\mathbb{N}} Q(k) = \max\{Q(n-1), Q(n)\}$. Note that $m \in \mathbb{N}^+$ and

$$\begin{aligned}Q(n-1) &= \frac{n+1}{2}(mn+2)+1-m,\\ Q(n) &= \frac{n+1}{2}(mn+2).\end{aligned}$$

Hence, $Q(k) = (2n-k+1)\big(\frac{mk}{2}+1\big) \leq \frac{n+1}{2}(mn+2)$ holds for all $1 \leq k \leq n$. Together with (2.1), we thus prove Corollary 2.1.

Let M be an open Riemann surface with a conformal metric $\mathrm{d}s^2 = \mu^2|\mathrm{d}z|^2$, where μ is a smooth positive function in terms of a holomorphic local coordinate. Define the Gauss curvature $K(p)$ of the metric $\mathrm{d}s^2$ of M at p by

$$K(p) := -\frac{\Delta\log\mu}{\mu^2}.$$

A curve $\Gamma(t)(0 \leq t < 1)$ in Riemann surface M is said to be divergent if for every compact subset K, there exists $t_0 < 1$ such that $\Gamma(t) \notin K$ for any $t > t_0$ (see [6]). We define the distance $d(p)(\leq \infty)$ from a point $p \in M$ to the boundary of M as the greatest lower bound of the lengths of all continuous curves which are divergent in M.

Motivated by the results of [9, 12, 16], we give a curvature estimate for the surface M with the metric $\mathrm{d}s^2 = \|\widetilde{G}\|^{2m}|\omega|^2$ which is not necessary complete.

Theorem 2.2 (Curvature estimate) *Let M be an open Riemann surface and $G : M \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic map. Take a reduced representation $\widetilde{G}$ of G, and let*

$$\mathrm{d}s^2 = \|\widetilde{G}\|^{2m}|\omega|^2$$

be the conformal metric on M, where m is a positive integer and ω is a holomorphic 1-form. If there are q hyperplanes $H_1, H_2, \cdots, H_q \subset \mathbb{P}^n(\mathbb{C})$ located in general position such that G is ramified over H_j with multiplicity at least γ_j for each $j \in \{1, 2, \cdots, q\}$, and

$$\sum_{j=1}^{q}\Big(1-\frac{n}{\gamma_j}\Big) > \frac{n+1}{2}(mn+2),$$

then there exists a constant C, depending on the set of hyperplanes, such that

$$|K(p)|^{\frac{1}{2}} \leq \frac{C}{d(p)}, \tag{2.2}$$

where $K(p)$ is the Gauss curvature of M at p with respect to the conformal metric $\mathrm{d}s^2$, and $d(p)$ is the geodesic distance from p to the boundary of M.

Remark 2.2 If $\mathrm{d}s^2$ in Theorem 2.2 is complete, then $d(p) \equiv \infty$ for any $p \in M$. So (2.2) is a trivial result.

3 Basic Notions and Auxiliary Results

Prior to proving our main results, we introduce some preliminary definitions and auxiliary results.

Let $H = \{[z_0 : z_1 : \cdots : z_k] \mid a_0 z_0 + \cdots + a_k z_k = 0\}$ be a hyperplane in $\mathbb{P}^k(\mathbb{C})$, here $\mathbf{a} = (a_0, \cdots, a_k) \in \mathbb{C}^{k+1} \setminus \{\mathbf{0}\}$ is called the normal vector associated to H. Hyperplanes $H_1, \cdots, H_q$ are said to be in n-subgeneral position (with $n \geq k$) if and only if for every injective map $\mu : \{0, 1, \cdots, n\} \to \{1, \cdots, q\}$, the linear span of those corresponding normal vectors $\mathbf{a}_{\mu(0)}$, $\cdots$, $\mathbf{a}_{\mu(n)}$ is $\mathbb{C}^{k+1}$. When $k = n$, then we just say the $H_1, \cdots, H_q$ are in general position in $\mathbb{P}^k(\mathbb{C})$. It is clearly that if hyperplanes $H_1, \cdots, H_q$ in $\mathbb{P}^n(\mathbb{C})$ are in general position, regarding $\mathbb{P}^k(\mathbb{C}) \subset \mathbb{P}^n(\mathbb{C}) (k \leq n)$, the restricted hyperplanes $H_1 \cap \mathbb{P}^k(\mathbb{C}), \cdots, H_q \cap \mathbb{P}^k(\mathbb{C})$ are located in n-subgeneral position.

Lemma 3.1 (see [4, 15]) *Let $\{H_j\}_{j=1}^q$ be a set of hyperplanes in $\mathbb{P}^k(\mathbb{C})$ in n-subgeneral position. Then there exist some functions $\varpi(j)$ and a number $\theta > 0$ such that*:
- $0 < \varpi(j)\theta \leq 1$ *for all* $1 \leq j \leq q$.
- $q - 2n + k - 1 = \theta\big(\sum\limits_{j=1}^{q} \varpi(j) - k - 1\big)$.
- $1 \leq \frac{n+1}{k+1} \leq \theta \leq \frac{2n-k+1}{k+1}$.

Here $\varpi(j)$ are called the Nochka weights associated to the hyperplanes $H_j (1 \leq j \leq q)$.

Let $F : \Delta_R \to \mathbb{P}^k(\mathbb{C})$ be a linearly non-degenerate holomorphic map, where $\Delta_R := \{z \mid |z| < R\} (0 < R \leq \infty)$. Take a reduced representation $\widetilde{F} = (f_0, f_1, \cdots, f_k)$ of F, i.e., $\widetilde{F} : \Delta_R \to \mathbb{C}^{k+1} \setminus \{\mathbf{0}\}$ and let $\|\widetilde{F}\|^2 = \big(\sum\limits_{j=0}^{k} |f_j|^2\big)$. Define

$$\widetilde{F}_s = \widetilde{F}^{(0)} \wedge \widetilde{F}^{(1)} \wedge \cdots \wedge \widetilde{F}^{(s)} : \Delta_R \to \bigwedge^{s+1} \mathbb{C}^{k+1},$$

where $\widetilde{F}^{(s)} = (f_0^{(s)}, f_1^{(s)}, \cdots, f_k^{(s)})$ is the s-th derivative of $\widetilde{F}$ for each $0 \leq s \leq k$. Obviously, $\widetilde{F}_{k+1} \equiv 0$. Let $\mathbb{P}$ be the natural projection, and $F_s = \mathbb{P}(\widetilde{F}_s)$. We call the map F_s the s-th associated map of F.

For holomorphic functions $f_0, f_1, \cdots, f_k$, one says that

$$W(f_0, f_1, \cdots, f_k) := \begin{vmatrix} f_0, & f_1, & \cdots, & f_k \\ f_0', & f_1', & \cdots, & f_k' \\ \vdots & \vdots & \vdots & \vdots \\ f_0^{(k)}, & f_1^{(k)}, & \cdots, & f_k^{(k)} \end{vmatrix}$$

is the Wronskian of $f_0, f_1, \cdots, f_k$. Let $\{\varepsilon_i\}_{i=0}^{k}$ be the standard basis of $\mathbb{C}^{k+1}$. For $0 \le s \le k$, one can write

$$\widetilde{F}_s = \sum_{0 \le i_0 < \cdots < i_s \le k} W(f_{i_0}, f_{i_1}, \cdots, f_{i_s})\varepsilon_{i_0} \wedge \cdots \wedge \varepsilon_{i_s}.$$

For a hyperplane H_j in $\mathbb{P}^k(\mathbb{C})$ with the normal vector $\mathbf{a}_j = (a_{j0}, \cdots, a_{jk})$, we define for $0 \le s \le k, 1 \le j \le q$,

$$\|(\widetilde{F}_s, H_j)\|^2 = \sum_{0 \le i_1 < \cdots < i_s \le k} \Big| \sum_{t \ne i_1, \cdots, i_s} a_{jt} W(f_t, f_{i_1}, \cdots, f_{i_s}) \Big|^2. \tag{3.1}$$

From above, we see that $\|(\widetilde{F}_s, H_j)\| \equiv 0$ if and only if

$$\sum_{t \ne i_1, \cdots, i_s} a_{jt} W(f_t, f_{i_1}, \cdots, f_{i_s}) \equiv 0$$

for all $i_1, \cdots, i_s$. Then if F is linearly non-degenerate, $\|(\widetilde{F}_s, H_j)\| \not\equiv 0$ for all $0 \le s \le k$ and $1 \le j \le q$. Indeed, if $(\widetilde{F}_s, H_j) \equiv 0$ for some s and j, then

$$W\Big(\sum_{t \ne i_1, \cdots, i_s} a_{jt} f_t, f_{i_1}, \cdots, f_{i_s} \Big) = \sum_{t \ne i_1, \cdots, i_s} a_{jt} W(f_t, f_{i_1}, \cdots, f_{i_s}) \equiv 0,$$

i.e.,

$$W((\widetilde{F}, H_j), f_{i_1}, \cdots, f_{i_s}) \equiv 0$$

for all $i_1, \cdots, i_s$. This implies that $(\widetilde{F}, H_j), f_{i_1}, \cdots, f_{i_s}$ are linearly dependent, which contradicts the linearly non-degeneracy of F.

From (3.1), when $s = 0$ or k, one gets the following:

$$\|(\widetilde{F}, H_j)\| = \|(\widetilde{F}_0, H_j)\| = |a_{j0} f_0 + a_{j1} f_1 + \cdots + a_{jk} f_k|$$

and

$$\|(\widetilde{F}_k, H_j)\| = \|\widetilde{F}_k\| = |W(f_0, f_1, \cdots, f_k)|.$$

Note that for every $z \in \mathbb{C}$, $(\widetilde{F}_s, H_j)(z)$ denote some complex vectors for $1 \le s \le k-1$ while $(\widetilde{F}_s, H_j)(z)$ denote some complex numbers when $s = 0$ or k. In addition, F is ramified over H with multiplicity at least γ if all zeros of $(\widetilde{F}, H_j)$ have orders at least γ. If $\gamma = \infty$, one says that the map F omits the hyperplane H.

The following result was obtained by Ru, which plays an important role in the proof of Theorem 2.1.

Lemma 3.2 (see [20, Main Lemma]) *Let $F = [f_0 : \cdots : f_k] : \Delta_R \to \mathbb{P}^k(\mathbb{C})$ be a non-degenerate holomorphic map, $H_1, H_2, \cdots, H_q$ be hyperplanes in $\mathbb{P}^k(\mathbb{C})$ in n-subgeneral position, and $\varpi(j)$ be their Nochka weights. Take a reduced representation $\widetilde{F} = (f_0, f_1, \cdots, f_k)$ of F. If F is ramified over H_j with multiplicity at least γ_j for each $j \in \{1, 2, \cdots, q\}$ and*

$$\sum_{j=1}^{q} \Big(1 - \frac{k}{\gamma_j}\Big) > 2n - k + 1,$$

$$N > \frac{2q(k^2 + 2k)}{\sum\limits_{j=1}^{q} \varpi(j)\big(1 - \frac{k}{\gamma_j}\big) - (k+1)},$$

then there exists a positive constant C such that

$$\|\widetilde{F}\|^{\sum\limits_{j=1}^{q}\varpi(j)(1-\frac{k}{\gamma_j})-(k+1)-\frac{2q(k^2+2k-1)}{N}} \frac{\prod\limits_{s=0}^{k-1}\prod\limits_{j=1}^{q}\|(\widetilde{F}_s,H_j)\|^{\frac{4}{N}}\|\widetilde{F}_k\|^{1+\frac{2q}{N}}}{\prod\limits_{j=1}^{q}|(\widetilde{F},H_j)|^{\varpi(j)(1-\frac{k}{\gamma_j})}}$$
$$\leq C\Big(\frac{2R}{R^2-|z|^2}\Big)^{\frac{1}{2}k(k+1)+\frac{2q}{N}\sum\limits_{s=0}^{k}s^2}.$$

To prove Theorem 2.2, one needs some results on the geometric orbifolds introduced by Campana in [2]. In this paper, we use some notations and results of geometric orbifold as shown in [3, 18]. An orbifold consists of a compact irreducible complex space together with a Weil $\mathbb{Q}$-divisor. Let (X,D) be an orbifold with $D:=\sum\limits_{j\in I}\big(1-\frac{1}{\gamma_j}\big)H_j$, where $\gamma_j\in\mathbb{N}\cup\{\infty\}$ are multiplicities and H_j are distinct hyperplanes. One also say D is an orbifold structure on X. Orbifold can be regarded as a complex space endowed with an additional structure in the form of a certain Weil $\mathbb{Q}$-divisor. A holomorphic map f from the unit disk $\Delta=\{z\in\mathbb{C}:|z|<1\}$ to an orbifold (X,D) is an orbifold morphism if $f(\Delta)\not\subset\operatorname{supp}(D)$ and $\operatorname{mult}_z(f^*H_j)\geq\gamma_j(1\leq j\leq q)$ for $z\in\Delta$ with $f(z)\in\operatorname{supp}(H_j)$.

Lemma 3.3 (see [18, Theorem 5.3]) *Let $H_1,H_2,\cdots,H_q$ be q hyperplanes in general position in $\mathbb{P}^n(\mathbb{C})$ with $q>2n$. Let $D:=\sum\limits_{1\leq j\leq q}\big(1-\frac{1}{\gamma_j}\big)H_j$ with $\deg(D)=\sum\limits_{1\leq j\leq q}\big(1-\frac{1}{\gamma_j}\big)>q-\frac{q}{n}+1+\frac{1}{n}$. Then every orbifold morphism $f:\mathbb{C}\to(\mathbb{P}^n(\mathbb{C}),D)$ is constant.*

Lemma 3.4 (see [18, Theorem 5.1]) *Let $H_1,H_2,\cdots,H_q$ be q hyperplanes in general position in $\mathbb{P}^n(\mathbb{C})$ with $q>2n$. Let $D:=\sum\limits_{1\leq j\leq q}\big(1-\frac{1}{\gamma_j}\big)H_j$ with $\deg(D)=\sum\limits_{1\leq j\leq q}\big(1-\frac{1}{\gamma_j}\big)>q-\frac{q}{n}+1+\frac{1}{n}$. Then $(\mathbb{P}^n(\mathbb{C}),D)$ is hyperbolic and hyperbolically imbedded in $\mathbb{P}^n(\mathbb{C})$.*

Lemma 3.5 (see [11, Proposition 10]) *Let $\mathrm{d}s^2$ be a Hermitian metric on X compact. Assume that the orbifold (X,D) is hyperbolic and hyperbolically imbedded in X, then the set of all orbifold morphisms $f:\Delta\to(X,D)$ is relatively compact in $\operatorname{Hol}(\Delta,X)$, where $\operatorname{Hol}(\Delta,X)$ denotes the set of all holomorphic maps of Δ into X.*

Lemma 3.6 (see [3, Proposition 7]) *Let $f_n:(X,\Delta)\to(X',\Delta')$ be a sequence of orbifold morphisms. Assume that $\{f_n\}$, regarded as a sequence of holomorphic maps from X to X', converges locally uniformly to a holomorphic map $f:X\to X'$. Then either $f(X)\subset\operatorname{Supp}(\Delta')$ or f is an orbifold morphism from (X,Δ) to (X',Δ').*

4 The Proof of Theorem 2.1

The following lemma is needed for the proof of Theorem 2.1.

Lemma 4.1 (see [10, Lemma 1.6.7]) *Let $\mathrm{d}\sigma^2$ be a conformal flat metric on an open Riemann surface M. Then for each point $p\in M$, there exists a local diffeomorphism Φ of a disk $\Delta_R=\{w\in\mathbb{C}\mid|w|<R\}(0<R\leq\infty)$ onto an open neighborhood of p with $\Phi(0)=p$ such that Φ is local isometry (i.e., the pullback $\Phi^*(\mathrm{d}\sigma^2)$ is equal to the standard Euclidean metric $\mathrm{d}s_E^2$ on Δ_R), and there exists a point a_0 with $|a_0|=1$, the Φ-image Γ_{a_0} of the line $L_{a_0}=\{w=a_0t:0<t<R\}$ is divergent in M.*

Based on the similar method as shown in [5, Theorem 1] (also see the arguments in [16, 19–20]), we prove Theorem 2.1 and show the details as follows.

By taking the universal cover of M if necessary, one can assume that M is simply connected. It follows from the uniformization theorem that M is conformally equivalent to unit disc Δ or $\mathbb{C}$. For the case of $m=0$, $\mathrm{d}s^2=|\omega|^2$ becomes a flat metric. Owing to the completeness of $\mathrm{d}s^2$, the universal cover of M is the whole complex plane $\mathbb{C}$. Assume $\pi:\mathbb{C}\to M$ is the universal covering map. G can be regarded as the holomorphic map from $\mathbb{C}$ into $\mathbb{P}^n(\mathbb{C})$ by replacing G with $G\circ\pi$, one thus knows G is a constant map by Theorem 1.2.

For the case of $m\in\mathbb{N}^+$, one has that G is a constant map by using Theorem 1.2 again if M is conformally equivalent to $\mathbb{C}$. So it suffices to consider the case that M is conformally equivalent to unit disc Δ. If G is nonconstant, then there exists $k(1\le k\le n)$ such that the image of G is contained in $\mathbb{P}^k(\mathbb{C})\subset\mathbb{P}^n(\mathbb{C})$, but not in any subspace whose dimension is lower than k. In other words, G can be regarded as a linearly non-degenerate map from Δ into $\mathbb{P}^k(\mathbb{C})$. Take a reduced representation $\widetilde{G}=(g_0,g_1,\cdots,g_k)$ of G and let $\widetilde{H}_j:=H_j\cap\mathbb{P}^k(\mathbb{C})$, $1\le j\le q$. Obviously, hyperplanes $\widetilde{H}_1,\cdots,\widetilde{H}_j,\cdots,\widetilde{H}_q$ are in n-subgeneral position in $\mathbb{P}^k(\mathbb{C})$. Furthermore, one may assume that each $\widetilde{H}_j$ is given by

$$\widetilde{H}_j: a_{j0}z_0+a_{j1}z_1+\cdots+a_{jk}z_k=0,\quad 1\le j\le q.$$

For each $j(1\le j\le q)$, $\widetilde{\varpi}(j)$ is the Nochka weight associated to the hyperplane $\widetilde{H}_j$. By Lemma 3.1, one has

$$0<\widetilde{\varpi}(j)\theta\le 1$$

and

$$q-2n+k-1=\theta\Big(\sum_{j=1}^{q}\widetilde{\varpi}(j)-k-1\Big).$$

Hence

$$\begin{aligned}\frac{2\Big(\sum\limits_{j=1}^{q}\widetilde{\varpi}(j)\Big(1-\frac{k}{\gamma_j}\Big)-k-1\Big)}{mk(k+1)}&=\frac{2\theta\Big(\sum\limits_{j=1}^{q}\widetilde{\varpi}(j)-k-1-\sum\limits_{j=1}^{q}\widetilde{\varpi}(j)\frac{k}{\gamma_j}\Big)}{\theta mk(k+1)}\\&\ge\frac{2\Big(q-2n+k-1-\sum\limits_{j=1}^{q}\frac{k}{\gamma_j}\Big)}{\theta mk(k+1)}\\&=\frac{2\Big(\sum\limits_{j=1}^{q}\Big(1-\frac{k}{\gamma_j}\Big)-2n+k-1\Big)}{\theta mk(k+1)}.\end{aligned}$$

Together with $\theta\le\frac{2n-k+1}{k+1}$,

$$\frac{2\Big(\sum\limits_{j=1}^{q}\widetilde{\varpi}(j)\Big(1-\frac{k}{\gamma_j}\Big)-k-1\Big)}{mk(k+1)}\ge\frac{2\Big(\sum\limits_{j=1}^{q}\Big(1-\frac{k}{\gamma_j}\Big)-2n+k-1\Big)}{mk(2n-k+1)}.$$

The condition $\sum\limits_{j=1}^{q}\big(1-\frac{k}{\gamma_j}\big)>(2n-k+1)\big(\frac{mk}{2}+1\big)$ implies

$$\frac{2\Big(\sum\limits_{j=1}^{q}\widetilde{\varpi}(j)\Big(1-\frac{k}{\gamma_j}\Big)-k-1\Big)}{mk(k+1)}>1,$$

which is equivalent to

$$\sum_{j=1}^{q}\widetilde{\varpi}(j)\Big(1-\frac{k}{\gamma_j}\Big)-k-1-\frac{mk}{2}(k+1)>0.$$

We thus choose some N such that

$$\frac{\sum\limits_{j=1}^{q}\widetilde{\varpi}(j)\big(1-\frac{k}{\gamma_j}\big)-k-1-\frac{mk}{2}(k+1)}{\frac{2m}{q}+k^2+2k-1+m\sum\limits_{s=0}^{k}s^2}<\frac{2q}{N}<\frac{\sum\limits_{j=1}^{q}\widetilde{\varpi}(j)\big(1-\frac{k}{\gamma_j}\big)-k-1-\frac{mk}{2}(k+1)}{k^2+2k-1+m\sum\limits_{s=0}^{k}s^2}.$$

Let

$$\beta:=\sum_{j=1}^{q}\widetilde{\varpi}(j)\Big(1-\frac{k}{\gamma_j}\Big)-(k+1)-\frac{2q}{N}(k^2+2k-1)$$

and

$$\tau:=\frac{m}{\beta}\Big(\frac{1}{2}k(k+1)+\frac{2q}{N}\sum_{s=0}^{k}s^2\Big).$$

From how to choose the N, one has

$$0<\tau<1,\quad 0<N\beta(1-\tau)<4m.$$

Since $G:\Delta\to\mathbb{P}^k(\mathbb{C})$ is linearly non-degenerate, none of the $\|(\widetilde{G}_s,\widetilde{H}_j)\|, 0\le s\le k, 1\le j\le q$, vanishes identically. Thus, by (3.1) for each $\|(\widetilde{G}_s,\widetilde{H}_j)\|$, there exist $i_1,i_2,\cdots,i_s$ such that

$$\xi_{js}:=\sum_{t\neq i_1,\cdots,i_s}a_{jt}W(g_t,g_{i_1},\cdots,g_{i_s}) \tag{4.1}$$

does not vanish identically. Here, let $\xi_{j0}=(\widetilde{G},\widetilde{H}_j)$. Note that every ξ_{js} is a holomorphic function and has only isolated zeros.

For the holomorphic 1-form ω of the conformal metric $\mathrm{d}s^2$, one can write it as $\omega=\eta\mathrm{d}z$, where η is a no-where vanishing holomorphic function. We define a new metric

$$\mathrm{d}\sigma^2=\Bigg(\frac{\prod\limits_{j=1}^{q}|(\widetilde{G},\widetilde{H}_j)|^{\widetilde{\varpi}(j)\big(1-\frac{k}{\gamma_j}\big)}}{\|\widetilde{G}_k\|^{1+\frac{2q}{N}}\prod\limits_{j=1}^{q}\big(\prod\limits_{s=0}^{k-1}|\xi_{js}|\big)^{\frac{4}{N}}}\Bigg)^{\frac{2m}{(1-\tau)\beta}}|\eta|^{\frac{2}{1-\tau}}|\mathrm{d}z|^2 \tag{4.2}$$

on the subset $M_0:=\Delta\backslash\big\{p\in\Delta\mid \text{either } \widetilde{G}_k=0 \text{ or } \prod\limits_{j=1}^{q}\prod\limits_{s=0}^{k-1}|\xi_{js}|=0\big\}$.

Notice that

$$\Big\{z:\prod_{j=1}^{q}|(\widetilde{G},\widetilde{H}_j)|(z)=0\Big\}\subseteq\{z:\|\widetilde{G}_k\|(z)=0\}.$$

In fact, one may assume that $(\widetilde{G},\widetilde{H}_j)=\sum\limits_{i=0}^{k}a_{ji}g_i$, here $(a_{j0},a_{j1},\cdots,a_{jk})$ is the normal vector associated to H_j. For any zero point z_0 of $(\widetilde{G},\widetilde{H}_j)$, $(\widetilde{G},\widetilde{H}_j)(z_0)=0$ and $(\widetilde{G},\widetilde{H}_j)^{(s)}(z_0)=0$ for

$1 \le s \le k$ since G is ramified over H_j with multiplicity at least $\gamma_j(> k)$ for each $j \in \{1, 2, \cdots, q\}$. Without loss of generality, we assume that $a_{j0} \neq 0$. Then

$$a_{j0}\|\widetilde{G}_k\| = a_{j0}|W(g_0, g_1, \cdots, g_k)| = \begin{vmatrix} (\widetilde{G}, \widetilde{H}_j), & g_1, & \cdots, & g_k \\ (\widetilde{G}, \widetilde{H}_j)', & g_1', & \cdots, & g_k' \\ \vdots & \vdots & \vdots & \vdots \\ (\widetilde{G}, \widetilde{H}_j)^{(k)}, & g_1^{(k)}, & \cdots, & g_k^{(k)} \end{vmatrix}$$

vanishes at z_0. So, $\mathrm{d}\sigma^2$ is a flat metric on M_0.

Fix a point $p_0 \in M_0$, by Lemma 4.1, there exists a local diffeomorphism Φ of a disk $\Delta_R = \{w \in \mathbb{C} : |w| < R\}(0 < R \le \infty)$ onto an open neighborhood of p_0 with $\Phi(0) = p_0$ such that Φ is local isometry. Furthermore, there exists a point a_0 with $|a_0| = 1$, the Φ-image Γ_{a_0} of the line $L_{a_0} = \{w = a_0 t : 0 < t < R\}$ is divergent in M_0. On the other hand, $G \circ \Phi$ is a holomorphic map from Δ_R into $\mathbb{P}^n(\mathbb{C})$ and R is finite by Theorem 1.2.

Next, we will show Φ-image Γ_{a_0} actually is divergent to the boundary of Δ. To this end, we assume the contrary: The curve Γ_{a_0} is divergent to a point z_0 which either satisfies $\|\widetilde{G}_k\|(z_0) = 0$ or $|\xi_{js}|(z_0) = 0$ for some s with $0 \le s \le k-1$ and j with $1 \le j \le q$. Let $\mathrm{d}\sigma = \mu|\mathrm{d}z|$, one has the following expression from (4.2),

$$\begin{aligned} \mu^{\frac{(1-\tau)\beta}{m}} &= \frac{\prod\limits_{j=1}^{q} |(\widetilde{G}, \widetilde{H}_j)|^{\widetilde{\varpi}(j)\left(1-\frac{k}{\gamma_j}\right)}}{\|\widetilde{G}_k\|^{1+\frac{2q}{N}} \prod\limits_{j=1}^{q} \big(\prod\limits_{s=0}^{k-1} |\xi_{js}|\big)^{\frac{4}{N}}} \cdot |\eta|^{\frac{\beta}{m}} \\ &= \frac{\prod\limits_{j=1}^{q} |(\widetilde{G}, \widetilde{H}_j)|^{\widetilde{\varpi}(j)\left(1-\frac{k}{\gamma_j}\right)}}{\|\widetilde{G}_k\|} \cdot \frac{|\eta|^{\frac{\beta}{m}}}{\|\widetilde{G}_k\|^{\frac{2q}{N}} \prod\limits_{j=1}^{q} \big(\prod\limits_{s=0}^{k-1} |\xi_{js}|\big)^{\frac{4}{N}}}. \end{aligned}$$

By [20, Lemma 3.1], one gets that $\frac{\prod_{j=1}^{q} \|(\widetilde{G}, \widetilde{H}_j)\|^{\widetilde{\varpi}(j)(1-\frac{k}{\gamma_j})}}{\|\widetilde{G}_k\|}$ has no zeros and the multiplicity of poles of μ is at least $\delta_0 = \frac{4m}{N\beta(1-\tau)}(> 1)$. We thus get

$$\begin{aligned} R &= \int_{L_{a_0}} \Phi^* \mathrm{d}\sigma = \int_{\Gamma_{a_0}} \mathrm{d}\sigma \\ &= \int_{\Gamma_{a_0}} \Bigg(\frac{\prod\limits_{j=1}^{q} |(\widetilde{G}, \widetilde{H}_j)|^{\widetilde{\varpi}(j)\left(1-\frac{k}{\gamma_j}\right)}}{\|\widetilde{G}_k\|^{1+\frac{2q}{N}} \prod\limits_{j=1}^{q} \big(\prod\limits_{s=0}^{k-1} |\xi_{js}|\big)^{\frac{4}{N}}} \Bigg)^{\frac{m}{(1-\tau)\beta}} |\eta|^{\frac{1}{1-\tau}} |\mathrm{d}z| \\ &\ge c \int_{\Gamma_{a_0}} \frac{1}{|z - z_0|^{\delta_0}} |\mathrm{d}z| = \infty, \end{aligned}$$

which contradicts the fact $R < \infty$. Therefore $\Gamma_{a_0} = \Phi(L_{a_0})$ is divergent to the boundary of Δ.

By proving the finiteness of the length of Γ_{a_0} with respect to the metric $\mathrm{d}s^2 = \|\widetilde{G}\|^{2m}|\omega|^2$, one gets a contradiction for the completeness of $\mathrm{d}s^2$.

Define some functions on $\{w \mid |w| < R\}$ as follows:

$$f_s(w) := g_s(\Phi(w)), \quad 0 \le s \le k$$

and $F(w) := \widetilde{G} \circ \Phi(w) = (f_0(w), f_1(w), \cdots, f_k(w))$. For $1 \le j \le q,\ 0 \le s \le k$, we define

$$(F, \widetilde{H}_j) := a_{j0} f_0 + \cdots + a_{jk} f_k, \quad F_k := W(f_0, f_1, \cdots f_k)$$

and

$$\zeta_{js} := \sum_{t \ne i_1, \cdots, i_s} a_{jt} W(f_t, f_{i_1}, \cdots, f_{i_s}),$$

where $(i_1, \cdots, i_s)$ is the index in the definition of ξ_{js} in (4.1). Noticing the fact that, for $0 \le s \le k$,

$$F_s(w) = (F \wedge F' \wedge \cdots \wedge F^{(s)})(w) = (\widetilde{G} \wedge \cdots \wedge \widetilde{G}^{(s)})(z) \Big(\frac{\mathrm{d}z}{\mathrm{d}w}\Big)^{\frac{s(s+1)}{2}}.$$

From (4.2) and the selection of τ, one has

$$\begin{aligned}
\Phi^* \mathrm{d}\sigma &= \Phi^* \Bigg(\frac{\prod_{j=1}^{q} |(\widetilde{G}, \widetilde{H}_j)|^{\widetilde{\varpi}(j)\left(1-\frac{k}{\gamma_j}\right)}}{\|\widetilde{G}_k\|^{1+\frac{2q}{N}} \prod_{j=1}^{q} \big(\prod_{s=0}^{k-1} |\xi_{js}| \big)^{\frac{4}{N}}} \Bigg)^{\frac{m}{(1-\tau)\beta}} \cdot |\eta(\Phi(w))|^{\frac{1}{1-\tau}} |\mathrm{d}z| \\
&= \Bigg(\frac{\prod_{j=1}^{q} |(F, \widetilde{H}_j)|^{\widetilde{\varpi}(j)\left(1-\frac{k}{\gamma_j}\right)}}{\|F_k\|^{1+\frac{2q}{N}} \prod_{j=1}^{q} \big(\prod_{s=0}^{k-1} |\zeta_{js}| \big)^{\frac{4}{N}}} \Bigg)^{\frac{m}{(1-\tau)\beta}} \\
&\quad \times \Big| \frac{\mathrm{d}z}{\mathrm{d}w} \Big|^{\frac{\left(1+\frac{2q}{N}\right)\frac{mk(k+1)}{2} + \frac{4mq}{N} \sum_{s=0}^{k-1} \frac{s(s+1)}{2}}{(1-\tau)\beta}} \cdot |\eta(\Phi(w))|^{\frac{1}{1-\tau}} |\mathrm{d}z| \\
&= \Bigg(\frac{\prod_{j=1}^{q} |(F, \widetilde{H}_j)|^{\widetilde{\varpi}(j)\left(1-\frac{k}{\gamma_j}\right)}}{\|F_k\|^{1+\frac{2q}{N}} \prod_{j=1}^{q} \big(\prod_{s=0}^{k-1} |\zeta_{js}| \big)^{\frac{4}{N}}} \Bigg)^{\frac{m}{(1-\tau)\beta}} \Big| \frac{\mathrm{d}z}{\mathrm{d}w} \cdot \eta(\Phi(w)) \Big|^{\frac{1}{1-\tau}} |\mathrm{d}w|.
\end{aligned}$$

Using the isometry property of Φ, i.e., $|\mathrm{d}w| = \Phi^* \mathrm{d}\sigma$, we get

$$\Big| \frac{\mathrm{d}w}{\mathrm{d}z} \Big| = \Bigg(\frac{\prod_{j=1}^{q} |(F, \widetilde{H}_j)|^{\widetilde{\varpi}(j)\left(1-\frac{k}{\gamma_j}\right)}}{\|F_k\|^{1+\frac{2q}{N}} \prod_{j=1}^{q} \big(\prod_{s=0}^{k-1} |\zeta_{js}| \big)^{\frac{4}{N}}} \Bigg)^{\frac{m}{\beta}} |\eta(\Phi(w))|. \tag{4.3}$$

Now, denote by $l(\Gamma_{a_0})$ the length of the curve Γ_{a_0} with respect to the metric $\|\widetilde{G}\|^{2m}|\omega|^2$, then from (4.3),

$$\begin{aligned}
l(\Gamma_{a_0}) &= \int_{\Gamma_{a_0}} \|\widetilde{G}\|^m |\omega| = \int_{L_{a_0}} \|\widetilde{G}(\Phi(w))\|^m |\eta(\Phi(w))| \Big| \frac{\mathrm{d}z}{\mathrm{d}w} \Big| |\mathrm{d}w| \\
&= \int_{L_{a_0}} \|F\|^m \Bigg(\frac{\|F_k\|^{1+\frac{2q}{N}} \prod_{j=1}^{q} \big(\prod_{s=0}^{k-1} |\zeta_{js}| \big)^{\frac{4}{N}}}{\prod_{j=1}^{q} |(F, \widetilde{H}_j)|^{\widetilde{\varpi}(j)\left(1-\frac{k}{\gamma_j}\right)}} \Bigg)^{\frac{m}{\beta}} |\mathrm{d}w|
\end{aligned}$$

$$\leq \int_{L_{a_0}} \Big(\frac{\|F\|^{\beta} \|F_k\|^{1+\frac{2q}{N}} \prod_{j=1}^{q} \big(\prod_{s=0}^{k-1} \|(F_s, \widetilde{H}_j)\| \big)^{\frac{4}{N}}}{\prod_{j=1}^{q} |(F, \widetilde{H}_j)|^{\widetilde{\varpi}(j)\left(1-\frac{k}{\gamma_j}\right)}} \Big)^{\frac{m}{\beta}} |\mathrm{d}w|.$$

In above inequality, we use the fact that $|\zeta_{js}| \leq \|(F_s, \widetilde{H}_j)\|$ for all $0 \leq s \leq k,\ 1 \leq j \leq q$. Noticing that $0 < \tau < 1$, we conclude from Lemma 3.2 that

$$l(\Gamma_{a_0}) \leq C \int_0^R \Big(\frac{2R}{R^2 - |w|^2} \Big)^{\tau} |\mathrm{d}w| < \infty,$$

which contradicts the completeness of the metric $\|\widetilde{G}\|^{2m}|\omega|^2$. We thus complete the proof of Theorem 2.1.

5 The Proof of Theorem 2.2

Lemma 5.1 (see [16, Lemma 2.1]) *Let Δ_r be the disk centered at the origin with radius r, $0 < r < 1$, and let R be the hyperbolic radius of Δ_r in the unit disc. Let $\mathrm{d}s^2 = \mu^2(z)|\mathrm{d}z|^2$ be any conformal metric on Δ_r with the property that geodesic distance from the origin to a point z on $|z| = r$ is greater than or equal to R. If the Gauss curvature K of the metric $\mathrm{d}s^2$ satisfies $-1 \leq K \leq 0$, then the distance of any point to the origin in the metric $\mathrm{d}s^2$ is greater than or equal to the hyperbolic distance.*

Lemma 5.2 (see [16, Lemma 2.2]) *Let $\{\mathrm{d}s_l^2\}$ be a sequence of conformal metrics on the unit disc Δ whose curvatures satisfy $-1 \leq K_l \leq 0$. Suppose that Δ is a geodesic disk of radius R_l with respect to the metric $\mathrm{d}s_l^2$, where $R_l \to \infty$, and that the metric $\{\mathrm{d}s_l^2\}$ converges, uniformly on compact sets, to a metric $\mathrm{d}s^2$. Then all distances to the origin with respect to $\mathrm{d}s^2$ are greater than or equal to the corresponding hyperbolic distances in Δ. In particular, $\mathrm{d}s^2$ is complete.*

The following result was obtained by the author and Chen et al, which is needed for the proof of Theorem 2.2.

Proposition 5.1 (see [5, Proposition 1]) *Let M be an open simply connected Riemann surface and let $G^{(l)} : M \to \mathbb{P}^n(\mathbb{C})$ be a sequence of holomorphic maps. Fix a globally reduced representation $\widetilde{G}^{(l)} = (g_0^{(l)}, g_1^{(l)}, \cdots, g_n^{(l)})$ of $G^{(l)}$ (such representation exists because M is simply connected) and let $\|\widetilde{G}^{(l)}\|^2 = \sum_{j=0}^{n} |g_j^{(l)}|^2$. Define a sequence of the conformal metrics $\mathrm{d}s_l^2$ on M as follows:*

$$\mathrm{d}s_l^2 = \|\widetilde{G}^{(l)}\|^{2m} |\mathrm{d}z|^2,$$

where $m \in \mathbb{N}$. Denote by K_l the Gauss curvature of M with respect to the above metric. Assume that $\{G^{(l)}\}$ converges to a non-constant holomorphic map G uniformly on every compact subset of M and $\{|K_l|\}$ is uniformly bounded. Then one of the following statements must be true.

(i) *There is a subsequence $\{K_{l_i}\}$ of $\{K_l\}$ which converges to zero*;

(ii) *for each $0 \leq j \leq n$, there exists a subsequence $\{g_j^{(l_i)}\}$ of $\{g_j^{(l)}\}$ which converges to a holomorphic function ϕ_j on M. Furthermore, $\phi_0, \cdots, \phi_n$ have no common zeros.*

Proof of Theorem 2.2 The proof of Theorem 2.2 basically follows the argument in [5] (see also the arguments in [16]) by using Proposition 5.1. We include our proof here for the convenience of the reader.

If $\mathrm{d}s^2$ is complete, i.e., $d(p)=\infty$ holds for all $p\in M$. Then by Theorem 2.1, G is a constant and $|K(p)|^{\frac{1}{2}}=0$. Hence (2.2) is a trivial result. We may assume that the metric $\mathrm{d}s^2$ is not complete on M.

If (2.2) does not hold, one can construct a sequence of open Riemann surfaces M_l (one may assume that M_l is simply connected by taking universal cover of M_l if necessary), points $p_l\in M_l$ and a sequence of holomorphic map $G^{(l)}:M_l\to\mathbb{P}^n(\mathbb{C})$ such that $|K_l(p_l)|d_l^2(p_l)\to\infty$, and such that $G^{(l)}$ is ramified over a fixed set of hyperplanes $\{H_j\}_{j=1}^q$ with multiplicity at least γ_j for each $j\in\{1,2,\cdots,q\}$. For each l, $K_l(p_l)$ denotes the Gauss curvature of the surface M_l at p_l with respect to the metric $\mathrm{d}s_l^2=\|\widetilde{G}^{(l)}\|^{2m}|\omega^{(l)}|^2$, $\widetilde{G}^{(l)}=(g_0^{(l)}:g_1^{(l)}:\cdots:g_k^{(l)})$ is a reduced representation of $G^{(l)}$, and $d_l(p_l)$ is the geodesic distance from p_l to the boundary of M_l with respect to the metric $\mathrm{d}s_l^2$. It is worth pointing out that the Gauss curvature K_l is independent of the universal cover of M_l. In fact, for a conformal metric $\mathrm{d}\sigma$ on M, it shows that

$$\mathrm{d}\sigma=\mu(z)|\mathrm{d}z|=\mu(z(w))\Big|\frac{\mathrm{d}z}{\mathrm{d}w}\Big||\mathrm{d}w|$$

and

$$K(\mathrm{d}\sigma^2)=-\frac{\Delta_w\log\big(\mu(z(w))\big|\frac{\mathrm{d}z}{\mathrm{d}w}\big|\big)}{\big(\mu(z(w))\big|\frac{\mathrm{d}z}{\mathrm{d}w}\big|\big)^2}=-\frac{\Delta_z\log\mu}{\mu^2}\circ z(w)=K(\mathrm{d}\sigma^2(z(w))).$$

By using a similar method in [16] (also see [5]), one may assume that the surfaces M_l and points p_l can be chosen such that $K_l(p_l)=-\frac{1}{4}$, $-1\le K_l\le 0$ on M_l for all l, and $d_l(p_l)\to\infty$ when $l\to\infty$. And the uniformization theorem implies that M_l is either conformally equivalent to $\mathbb{C}$ or to the unit disc Δ.

For the case M_l is the complex plane $\mathbb{C}$, $G^{(l)}$ is an orbifold morphism of $\mathbb{C}$ into $(\mathbb{P}^n(\mathbb{C}),D)$, where $D:=\sum\limits_{1\le j\le q}\big(1-\frac{1}{\gamma_j}\big)H_j$ with $\deg(D)=\sum\limits_{1\le j\le q}\big(1-\frac{1}{\gamma_j}\big)$. Then by Lemma 3.3, $G^{(l)}$ is a constant. Indeed, the holomorphic map $G^{(l)}:\mathbb{C}\to\mathbb{P}^n(\mathbb{C})$ is ramified over H_j with multiplicity at least γ_j for each j, and

$$\sum_{j=1}^q\Big(1-\frac{n}{\gamma_j}\Big)>\frac{n+1}{2}(mn+2),$$

thus we obtain

$$\sum_{j=1}^q\Big(1-\frac{1}{\gamma_j}\Big)>q-\frac{q}{n}+\frac{n+1}{n}+\frac{m(n+1)}{2}.$$

Then there exists a no-where vanishing holomorphic function g_l such that $\mathrm{d}s_l^2=((n+1)|g_l|^2)^m\cdot|\mathrm{d}z|^2$, i.e., $K_l\equiv 0$, which contradicts with the fact that $K_l(p_l)=-\frac{1}{4}$, a contradiction.

For the other case M_l is conformally equivalent to the unit disc Δ, as discussed in the same argument above and one thus gets from Lemma 3.4 that $(\mathbb{P}^n(\mathbb{C}),D)$ is hyperbolically imbedded in $\mathbb{P}^n(\mathbb{C})$. Furthermore, Lemma 3.5 implies that $\{G^{(l)}\}$ is normal, i.e., there exists a subsequence of holomorphic maps $\{G^{(l_i)}\}$ of $\{G^{(l)}\}$, still denoted by $\{G^{(l)}\}$, converges to a holomorphic map g uniformly on every compact subset of the unit disc Δ.

If g is a constant map, then g maps Δ into a single point Q. Take a hyperplane H not containing the point Q, and let U, V be two disjoint neighborhoods of H, Q, respectively. So, g omits a neighborhood of H in $\mathbb{P}^2(\mathbb{C})$. Since $G^{(l)}$ converges to a holomorphic map g uniformly on $\Delta_r(r<1)$. So, $G^{(l)}$ also omits a neighborhood of H in $\mathbb{P}^n(\mathbb{C})$ for l large enough. Then by [5, Theorem 3], there exists a constant C such that

$$|K_l(p_l)|^{\frac{1}{2}} d_l(r) \leq C, \quad p_l \in \Delta_r,$$

where $K_l(p_l)$ is the Gauss curvature of the surface Δ_r at point p_l, and $d_l(r)$ is the geodesic distance from p_l to the boundary of Δ_r. Using the condition that $K_l(p_l) = -\frac{1}{4}$, we get, for l large enough,

$$d_l(r) \leq 2C. \tag{5.1}$$

On the other hand, one may choose a suitable $r<1$ such that the hyperbolic distance R from $z=0$ to $|z|=r$ satisfies

$$R > 2C. \tag{5.2}$$

Now, we will use Lemma 5.1 to derive a lower bound for $d_l(r)$. The surface M_l is a geodesic disk of radius $R_l(<+\infty)$ and the fact $d_l(p_l) \to \infty$ when $l \to \infty$ implies that $R_l \to \infty$. So, some $r_l(<1)$ can be selected such that $\{w : |w| < r_l\}$ has a hyperbolic radius R_l. One thus knows $r_l \to 1$ as $l \to \infty$. Furthermore, we re-parameterize it by letting $w = r_l z$ and thus the circle $|z| = 1$ corresponds to $|w| = r_l$. By the condition that $-1 \leq K_l(z) \leq 0$ for $z \in \Delta$, one knows $-1 \leq K_l(z(w)) \leq 0$ for all $w \in \{w : |w| < r_l\}$. For these disks $\{w : |w| < r_l\}$, by Lemma 5.1, we get for $r<1$ that the distance with the metric from the origin to any points on the circle $|w| = r_l r$, or equivalently, $|z| = r$, is not less than the hyperbolic distance from the origin to any points on $|w| = r_l r$. By the choice of R in (5.2), $d_l(r) \geq R$ for l large enough and one further gets $d_l(r) > 2C$ which yields a contradiction for (5.1). Hence, g is not a constant.

Let $\widetilde{G}^{(l)} = (g_0^{(l)}, \cdots, g_n^{(l)})$ be a reduced representation of $G^{(l)}$ and $\omega^{(l)} = \eta_l \mathrm{d}z$ for each l, where η_l is a no-where vanishing holomorphic function. Hence, the metric $\mathrm{d}s_l^2$ can be written as the form of

$$\mathrm{d}s_l^2 = (|g_0^{(l)}\eta_l|^2 + \cdots + |g_n^{(l)}\eta_l|^2)^m |\mathrm{d}z|^2.$$

By Proposition 5.1, there is a subsequence of $\{g_j^{(l)}\eta_l\}$, say itself, which converges to ϕ_j uniformly on every compact subset of the unit disc Δ for each j with $0 \leq j \leq n$. Furthermore, $\phi_0, \cdots, \phi_n$ have no common zeros. So we get a holomorphic map $[\phi_0 : \cdots : \phi_n] : M \to \mathbb{P}^n(\mathbb{C})$. Obviously, $g = [\phi_0 : \cdots : \phi_n]$. Note that $d_l(p_l) \to \infty$ when $l \to \infty$, by Lemma 5.2, the metric $\mathrm{d}s^2 := \sum_{j=0}^{n} |\phi_j|^2 |\mathrm{d}z|^2$ is complete on the unit disc Δ. It follows from Lemma 3.6 that g is an orbifold morphism of Δ into $(\mathbb{P}^n(\mathbb{C}), D)$ or $g(\Delta) \subset \mathrm{supp}(D)$.

If g is ramified over hyperplanes H_j with multiplicities at least γ_j for all $j = 1, 2, \cdots, q$, then Corollary 2.1 implies that g is a constant, a contradiction. So, there exists a set of hyperplanes $\{H_j\}_{j \in J}$, $J \subset \{1, \cdots, q\}$ such that $g(\Delta) \subseteq \bigcap_{j \in J} H_j$. And g is ramified over hyperplanes H_j with multiplicities at least γ_j for all $j \in \{1, 2, \cdots, q\} \setminus J$. Without loss of generality, one may assume that $J = \{1, 2, \cdots, k\} (1 \leq k \leq n)$ and $g(\Delta) \subseteq \bigcap_{j=1}^{k} H_j = \mathbb{P}(V)$, where V is a subspace of $\mathbb{C}^{n+1}$ of dimension $n+1-k$. Obviously, $\{H_j \cap (\bigcap_{j=1}^{k} H_j)\}_{j=k+1}^{q}$ is a set of hyperplanes in $\mathbb{P}(V)$ located in general position. On the other hand, g can be regarded as a holomorphic map from Δ into $\mathbb{P}(V)$, and g is ramified over hyperplanes H_j with multiplicities at least γ_j for each $k+1 \leq j \leq q$. Furthermore, one has the following inequality:

$$\begin{aligned}\sum_{j=k+1}^{q}\Big(1-\frac{n-k}{\gamma_j}\Big) &\geq \sum_{j=1}^{q}\Big(1-\frac{n}{\gamma_j}\Big)-\sum_{j=1}^{k}\Big(1-\frac{n}{\gamma_j}\Big)\\ &> \frac{n+1}{2}(mn+2)-k\\ &> \frac{n-k+1}{2}(m(n-k)+2).\end{aligned}$$

Hence, g is a constant by Corollary 2.1, this is a contradiction. We thus complete the proof of Theorem 2.2.

Acknowledgements The authors are grateful to Professor Min Ru for his constant encouragement and guidance. The authors also would like to thank the anonymous reviewers for their careful reading and valuable comments.

References

[1] Bergweiler, W., The role of the Ahlfors five islands theorem in complex dynamics, *Conform. Geom. Dyn.*, **4**, 2000, 22–34.

[2] Campana, F., Orbifolds, special varieties and classification theory, *Ann. Inst. Fourier*, **54**(3), 2004, 499–630.

[3] Campana, F. and Winkelmann, J., A Brody theorem for orbifolds, *Manuscripta Math.*, **128**(2), 2009, 195–212.

[4] Chen, W., Cartan Conjecture: Defect Relation for Merommorphic Maps from Parabolic Manifold to Projective Space, Thesis, University of Notre Dame, 1987.

[5] Chen, X. D., Li, Y. Z., Liu, Z. X. and Ru, M., Curvature estimate on an open Riemann surface with the induced metric, *Math. Z.*, **298**, 2021, 451–467.

[6] Chern, S. S. and Osserman, R., Complete minimal surfaces in euclidean n-space, *J. Anal. Math.*, **19**, 1967, 15–34.

[7] Fujimoto, H., On the number of exceptional values of the Gauss maps of minimal surfaces, *J. Math. Soc. Japan*, **40**(2), 1988, 235–247.

[8] Fujimoto, H., Modified defect relations for the Gauss map of minimal surfaces. II, *J. Differential Geom.*, **31**(2), 1990, 365–385.

[9] Fujimoto, H., On the Gauss curvature of minimal surfaces, *J. Math. Soc. Japan*, **44**(3), 1992, 427–439.

[10] Fujimoto, H., Value distribution theory of the Gauss map of minimal surface in $\mathbb{R}^m$, Aspects of Mathematics, **E21**, Friedr. Vieweg and Sohn, Braunschweig, 1993.

[11] Ha, P. H., An estimate for the Gaussian curvature of minimal surfaces in $\mathbb{R}^m$ whose Gauss map is ramified over a set of hyperplanes, *Differential Geom. Appl.*, **32**, 2014, 130–138.

[12] Kawakami, Y., On the maximal number of exceptional values of Gauss maps for various classes of surfaces, *Math. Z.*, **274**(3–4), 2013, 1249–1260.

[13] Kawakami, Y., Function-theoretic properties for the Gauss maps of various classes of surfaces, *Canad. J. Math.*, **67**(6), 2015, 1411–1434.

[14] Liu, X. J. and Pang, X. C., Normal family theory and Gauss curvature estimate of minimal surfaces in $\mathbb{R}^m$, *J. Differential Geom.*, **103**(2), 2016, 297–318.

[15] Nochka, E. I., On the theory of meromorphic functions, *Dokl. Akad. Nauk SSSR*, **269**(3), 1983, 547–552.

[16] Osserman, R. and Ru, M., An estimate for the Gauss curvature of minimal surfaces in $\mathbb{R}^m$ whose Gauss map omits a set of hyperplanes, *J. Differential Geom.*, **45**, 1997, 578–593.

[17] Ros, A., The Gauss map of minimal surfaces, *Differential Geom.*, *Valencia* 2001, 2002, 235–252.

[18] Rousseau, E., Hyperbolicity of geometric orbifolds, *Trans. Amer. Math. Soc.*, **362**(7), 2010, 3799–3826.

[19] Ru, M., On the Gauss map of minimal surfaces immersed in $\mathbb{R}^n$, *J. Differential Geom.*, **34**(2), 1991, 411–423.

[20] Ru, M., Gauss map of minimal surfaces with ramification, *Trans. Amer. Math. Soc.*, **339**(2), 1993, 751–764.