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Haken 3-Manifolds in Small Covers*

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Abstract The authors prove that a 3-dimensional small cover M is a Haken manifold if and only if M is aspherical or equivalently the underlying simple polytope is a flag polytope. In addition, they find that M being Haken is also equivalent to the existence of a Riemannian metric with non-positive sectional curvature on M.

Keywords Small cover, Haken manifold, Incompressible surface, Flag polytope
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1 Introduction

The notion of small cover is introduced by Davis and Januszkiewicz [5] as an analog of a smooth projective toric variety in the category of closed manifolds with $\mathbb{Z}_2$-torus actions ($\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$). An n-dimensional small cover is a closed n-manifold M with a locally standard $(\mathbb{Z}_2)^n$-action whose orbit space can be identified with a simple convex polytope P in a Euclidean space. A polytope is called simple if every codimension-k face is the intersection of exactly k distinct facets (codimension-one faces) of the polytope. Two polytopes in a Euclidean space are called combinatorially equivalent if there exists a bijection between their posets of faces with respect to the inclusion. All polytopes considered in this paper are convex, so we omit the word "convex" for brevity. Moreover, in most cases, we make no distinction between convex polytopes that are combinatorially equivalent.

The $(\mathbb{Z}_2)^n$-action on the small cover M determines a $(\mathbb{Z}_2)^n$-valued characteristic function λ on the set of facets of P, which encodes all the information of the isotropy groups of the non-free orbits. More specifically, for a facet F of P, the rank-one subgroup $\langle\lambda(F)\rangle \subset (\mathbb{Z}_2)^n$ generated by $\lambda(F)$ is the isotropy group of the codimension-one submanifold $\pi^{-1}(F)$ of M, where $\pi : M \to P$ is the orbit map of the $(\mathbb{Z}_2)^n$-action. The function λ is non-degenerate in the sense that the values of λ on any n facets that are incident to a vertex of P form a linear basis of $(\mathbb{Z}_2)^n$. Conversely, we can recover the manifold M by gluing 2^n copies of P according to the function λ. For any proper face f of P, define:

$$G_f = \text{the subgroup of } (\mathbb{Z}_2)^n \text{ generated by the set } \{\lambda(F) \,|\, f \subset F\}. \tag{1.1}$$

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Moreover define $G_P = \{0\} \subset (\mathbb{Z}_2)^n$. Then M is homeomorphic to the following quotient space

$$P \times (\mathbb{Z}_2)^n / \sim, \tag{1.2}$$

where $(p, g) \sim (p', g')$ if and only if $p = p'$ and $g^{-1}g' \in G_{f(p)}$, and $f(p)$ is the unique face of P that contains p in its relative interior.

It is shown in [5] that many important topological invariants of M can be easily computed in terms of the combinatorial structure of P and the characteristic function λ. For example, we can determine the fundamental group $\pi_1(M)$ of M as follows. Let W_P be a right-angled Coxeter group with one generator s_F and relations $s_F^2 = 1$ for each facet F of P, and $(s_F s_{F'})^2 = 1$ whenever F, F' are adjacent facets of P. Note that if $F \cap F' = \varnothing$, $s_F s_{F'}$ has infinite order in W_P (see [2, Proposition 1.1.1]). According to [5, Lemma 4.4], W_P is isomorphic to the fundamental group of the Borel construction $M_{(\mathbb{Z}_2)^n} = E(\mathbb{Z}_2)^n \times_{(\mathbb{Z}_2)^n} M$ of M. It is shown in [5, Corollary 4.5] that the homotopy exact sequence of the fibration $M \to M_{(\mathbb{Z}_2)^n} \to B(\mathbb{Z}_2)^n$ gives a short exact sequence

$$1 \to \pi_1(M) \xrightarrow{\psi} W_P \xrightarrow{\phi} (\mathbb{Z}_2)^n \to 1, \tag{1.3}$$

where $\phi(s_F) = \lambda(F)$ for any facet F of P, and ψ is induced by the canonical map $M \hookrightarrow M \times E(\mathbb{Z}_2)^n \to M_{(\mathbb{Z}_2)^n}$. So $\pi_1(M)$ is isomorphic to the kernel of ϕ which is a finite index subgroup of W_P. It follows that a small cover M is never simply connected.

Let $\mathcal{F}(P)$ denote the set of facets of P. For any proper face f of P, we have the following definitions.

- Define $\mathcal{F}(f^\perp) = \{F \in \mathcal{F}(P) \mid \dim(f \cap F) = \dim(f) - 1\}$. In other words, $\mathcal{F}(f^\perp)$ consists of those facets of P that intersect f transversely.
- We call $M_f = \pi^{-1}(f)$ the facial submanifold of M corresponding to f. It is easy to see that M_f is a small cover over the simple polytope f, whose characteristic function, denoted by λ_f, is determined by λ as follows. Let

$$\rho_f : (\mathbb{Z}_2)^n \to (\mathbb{Z}_2)^n / G_f \cong (\mathbb{Z}_2)^{\dim(f)}$$

be the quotient homomorphism. Then we have

$$\lambda_f(f \cap F) = \rho_f(\lambda(F)), \quad \forall\, F \in \mathcal{F}(f^\perp). \tag{1.4}$$

A submanifold Σ of a manifold M is called π_1-injective if the inclusion of Σ into M induces a monomorphism in the fundamental group.

Theorem 1.1 (see [20, Corollary 1.4]) *Let M be a small cover over a simple polytope P. For a facet F of P, the facial submanifold M_F is π_1-injective in M if and only if F is not contained in any 3-belt on P.*

For any $k \geq 3$, a k-belt on a simple polytope P is a cyclic sequence $(F_1, \cdots, F_k)$ of k different facets of P in which any two consecutive facets have nonempty intersection and no three facets in $(F_1, \cdots, F_k)$ can share a common face.

A simple polytope P is called a flag polytope if a collection of facets of P have common intersection whenever they pairwise intersect. In particular, a flag simple polytope cannot have a 3-belt. Note that if $\dim(P) = 3$, P is a flag polytope if and only if P is not the 3-simplex Δ^3 and has no 3-belts (so in particular P has no 3-gon facets).

By [6, Theorem 2.2.5], a small cover M over a simple polytope P is aspherical if and only if P is a flag polytope. The following theorem gives another description of a small cover being aspherical.

Theorem 1.2 (see [20, Proposition 3.6]) *A small cover M is aspherical if and only if all the facial submanifolds of M are π_1-injective in M.*

In this paper, we focus our study on the geometry and topology of 3-dimensional small covers. The following is the main theorem of the paper.

Theorem 1.3 (see Theorem 2.1) *Let M be a small cover over a 3-dimensional simple polytope P. The following statements are all equivalent.*

(i) *P is a flag polytope.*

(ii) *M is aspherical.*

(iii) *M is $\mathbb{P}^2$-irreducible and $M \ncong \mathbb{R}P^3$.*

(iv) *M is Haken.*

(v) *M admits a Riemannian metric with non-positive sectional curvature.*

Remark 1.1 If P is a flag simple polytope, [6, Theorem 2.2.3] says that every small cover M over P admits a piecewise Euclidean metric which is non-positively curved (in the sense of Aleksandrov) as a metric space. But the above theorem tells us that when $\dim(P) = 3$, M can actually admit a non-positively curved (smooth) Riemannian metric. It is interesting to see whether this is true in higher dimensions.

Remark 1.2 There is a notion of Haken n-manifolds, $n > 3$, defined and studied by Foozwell and Rubinstein [8–10], in analogy with the classical Haken manifolds in dimension 3. According to the discussion in [9, Section 5.2], any small cover over an n-dimensional flag simple polytope is a Haken n-manifold in that sense.

The construction of small covers in (1.2) can be generalized in the following way. A $(\mathbb{Z}_2)^r$-coloring on a simple polytope P is a map $\mu : \mathcal{F}(P) \to (\mathbb{Z}_2)^r$ where the image of μ contains a linear basis of $(\mathbb{Z}_2)^r$. We call μ non-degenerate if at any vertex v of P, the elements $\{\mu(F) \,|\, v \in F \in \mathcal{F}(P)\}$ are linearly independent in $(\mathbb{Z}_2)^r$. We can construct a closed manifold $M(P, \mu)$ from P and μ by a similar rule as in (1.2). When $r = \dim(P)$, $M(P, \mu)$ is just a small cover over P.

Let $\widetilde{\mu}(F_i) = e_i$, $1 \leq i \leq m$ where $\mathcal{F}(P) = \{F_1, \cdots, F_m\}$ and $\{e_1, \cdots, e_m\}$ is a basis of $(\mathbb{Z}_2)^m$, $M(P, \widetilde{\mu})$ is called the real moment-angle manifold of P, denoted by $\mathbb{R}\mathcal{Z}_P$. It is easy to see that for every nondegenerate $(\mathbb{Z}_2)^r$-coloring on P, $\mathbb{R}\mathcal{Z}_P$ is a regular $(\mathbb{Z}_2)^{m-r}$-covering of $M(P, \mu)$.

The paper is organized as follows. In Section 2, we determine what kind of small covers and more generally what kind of $M(P, \mu)$ in dimension 3 are Haken manifolds (see Theorems

2.1–2.2). In Section 3, we prove a few more results related to the fundamental groups of small covers that can be easily derived from other people's work.

2 3-Dimensional Small Covers and Haken Manifolds

First of all, let us recall some basic concepts in the theory of 3-manifolds. A connected 3-manifold is called prime if it cannot be obtained as a connected sum of two 3-manifolds neither of which is the 3-sphere. A connected 3-manifold is called irreducible if every embedded 2-sphere bounds a 3-ball. A prime 3-manifold is irreducible except it is a S^2-bundle over S^1. A 3-manifold is $\mathbb{P}^2$-irreducible if it is irreducible and contains no 2-sided $\mathbb{R}P^2$. It is clear that an orientable manifold is $\mathbb{P}^2$-irreducible if and only if it is irreducible. The following are some well known facts in the theory of 3-manifolds.

Fact-1 A closed connected orientable 3-manifold M is aspherical if and only if M is irreducible and if $\pi_1(M)$ is infinite (this is a consequence of the Sphere theorem and the Poincaré conjecture).

Fact-2 Any aspherical 3-manifold is $\mathbb{P}^2$-irreducible (see [12, Proposition 3.10]).

Fact-3 Any double cover of a $\mathbb{P}^2$-irreducible 3-manifold is still $\mathbb{P}^2$-irreducible (see [13, Lemma 10.4]).

Fact-4 If a closed connected 3-manifold M satisfies $\pi_1(M) \cong G_1 * G_2$, then M is the connected sum of two manifolds M_1 and M_2 where $\pi_1(M_i) \cong G_i$, $i = 1, 2$ (see [13, Chapter 7] on Kneser's conjecture).

Fact-5 Any 2-sided $\mathbb{R}P^2$ in a compact 3-manifold M is always nontrivial in the second homotopy group $\pi_2(M)$ of M (see [12, Proposition 3.10]).

Notice that Fact-1, Fact-2 and Fact-3 together imply that a closed connected 3-manifold M is aspherical if and only if M is $\mathbb{P}^2$-irreducible and if $\pi_1(M)$ is infinite.

A compact connected surface Σ properly embedded in a 3-manifold M is called incompressible in M if Σ is not homeomorphic to S^2 or D^2 and for any embedded 2-disk D in M with $D \cap \Sigma = \partial D$, ∂D also bounds a disk in Σ.

A Haken manifold is a compact, $\mathbb{P}^2$-irreducible 3-manifold that contains a properly embedded two-sided incompressible surface. In particular, a compact orientable 3-manifold is Haken if it is irreducible that contains an orientable, incompressible surface. Note that $\mathbb{R}P^3$ is $\mathbb{P}^2$-irreducible but not Haken.

Let Σ be a properly embedded compact connected surface in M that is not homeomorphic to S^2 or D^2. If Σ is π_1-injective, then Σ is incompressible. But conversely an incompressible surface in a 3-manifold is not necessarily π_1-injective. However, if we assume that Σ is two-sided in M, the loop theorem implies that Σ is incompressible if and only if Σ is π_1-injective (see [13, p. 59]).

Let $\pi : M \to P$ be a small cover over a 3-dimensional simple polytope P whose characteristic function is λ. We have the following classification.

- If the fundamental group of M is finite, then P has to be 2-neighborly (see [15, Corollary 2.6]). This implies that P is a 3-simplex and $M = \mathbb{R}P^3$.
- If the fundamental group of M is infinite:

– If P is a flag polytope, then M is aspherical, hence $\mathbb{P}^2$-irreducible (by Fact-2).

– If $P \neq \Delta^3$ is not a flag polytope, then P must have a 3-belt. Up to a change of basis of $(\mathbb{Z}_2)^3$, the value of the characteristic function λ on the 3-belt must be one of the patterns shown in Figures 1–3, where D is a triangle in P that intersects the 3-belt of P transversely.

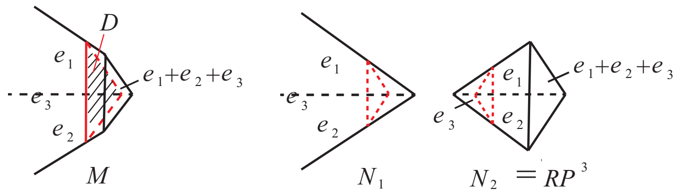

Figure 1 Characteristic function around a 3-belt.

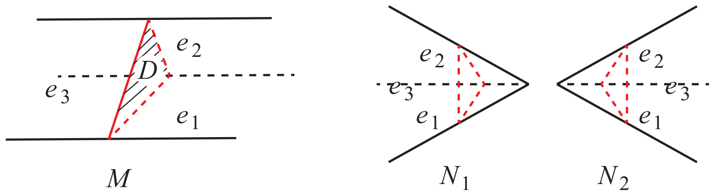

Figure 2 Characteristic function around a 3-belt.

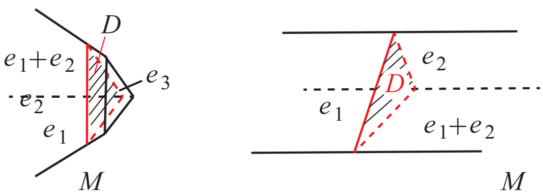

Figure 3 Characteristic function around a 3-belt.

• If λ has a local pattern shown in Figure 1 or Figure 2, then M is the connected sum of two 3-manifolds along the 2-sphere $\pi^{-1}(D)$. So M is not irreducible.

• If λ has a local pattern shown in Figure 3, then $\pi^{-1}(D)$ is a disjoint union of two copies of two-sided $\mathbb{R}P^2$'s in M. So M is not $\mathbb{P}^2$-irreducible.

By definition, a 3-manifold is not irreducible implies that it is not $\mathbb{P}^2$-irreducible. So the above classification tells us that $M \not\cong \mathbb{R}P^3$ is $\mathbb{P}^2$-irreducible if and only if P is a flag polytope.

Theorem 2.1 *Let M be a small cover over a 3-dimensional simple polytope P. The following statements are all equivalent.*

(i) *P is a flag polytope.*

(ii) *M is aspherical.*

(iii) *M is $\mathbb{P}^2$-irreducible and $M \neq \mathbb{R}P^3$.*

(iv) *M is Haken.*

(v) *M admits a Riemannian metric with non-positive sectional curvature.*

Proof The equivalence of (i) and (ii) is given by [6, Theorem 2.2.5]. The equivalence of (i) and (iii) is proved in the above discussion.

(iv)⇒(iii). It follows from the definition of Haken manifold and the fact that $\mathbb{R}P^3$ is not Haken.

(i)⇒(iv). If P is a flag polytope, then M is aspherical hence $\mathbb{P}^2$-irreducible (by Fact-2). Choose an arbitrary facet F of P. Then F is an n-gon with $n \geq 4$ since P is flag. By Theorem 1.2, the facial submanifold M_F is an embedded π_1-injective surface in M. Then since M_F is not simply connected, M_F is an incompressible surface of M. Note that the Euler characteristic $\chi(M_F) = 4 - n \leq 0$.

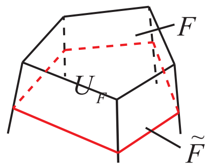

Figure 4 A small neighborhood of a facet F in P.

Let $\widetilde{F}$ be a section of P parallel to F in a small neighborhood U_F of F in P (see Figure 4). Let $\pi : M \to P$ be the orbit map. By the definition of small covers, it is easy to see that $\pi^{-1}(U_F)$ is a tubular neighborhood of M_F in M. By abuse of notation, we let

$$M_{\widetilde{F}} = \pi^{-1}(\widetilde{F}) = \text{the boundary of } \pi^{-1}(U_F).$$

Clearly $M_{\widetilde{F}}$ is a double cover of M_F. Let $\eta : M_{\widetilde{F}} \to M_F$ denote the double covering map.

- If η is a trivial double cover, then M_F is two-sided. So M_F is a two-sided incompressible surface in M.

- If η is a non-trivial double cover, it is easy to see that $M_{\widetilde{F}}$ is an embedded two-sided connected surface in M. Moreover, we can show that $M_{\widetilde{F}}$ is an incompressible surface of M. Indeed, let $i_1 : M_F \hookrightarrow M$ and $i_2 : M_{\widetilde{F}} \hookrightarrow M$ be the inclusion maps. Since η is a covering map, the homomorphism $\eta_* : \pi_1(M_{\widetilde{F}}) \to \pi_1(M_F)$ is injective. Then since M_F is π_1-injective, the composition $(i_1)_* \circ \eta_* : \pi_1(M_{\widetilde{F}}) \to \pi_1(M)$ is also injective. On the other hand, the map i_2 is homotopic to $i_1 \circ \eta$ via the deformation retraction of $\pi^{-1}(U_F)$ onto M_F. Then $(i_2)_* : \pi_1(M_{\widetilde{F}}) \to \pi_1(M)$ is also injective. So $M_{\widetilde{F}}$ is π_1-injective. In addition, since F is not a 3-gon, M_F is not $\mathbb{R}P^2$. So $M_{\widetilde{F}}$ is not S^2. Therefore, $M_{\widetilde{F}}$ is an incompressible surface of M.

By the above argument, we can always find a two-sided incompressible surface (with non-positive Euler characteristic) in M. So M is a Haken manifold.

(v)⇒(ii). If M admits a non-positive sectional curvature, the Cartan-Hadamard theorem implies that its universal covering space is diffeomorphic to an Euclidean space, which implies that M is aspherical.

(ii)⇒(v). By [11], every finitely generated Coxeter group W is virtually special (i.e., there exists a finite index subgroup of W that is isomorphic to a subgroup of a finitely generated right-angled Artin group). Then since $\pi_1(M)$ is isomorphic to a subgroup of the right-angled

Coxeter group W_P (see (1.3)), $\pi_1(M)$ is also virtually special. Moreover, by [18, Corollary 1.4] a compact aspherical 3-manifold has virtually special fundamental group if and only if it admits a Riemannian metric of non-positive sectional curvature (also see [16]). This finishes the proof.

For a non-degenerate $(\mathbb{Z}_2)^r$-coloring μ on a 3-dimensional simple polytope P, we may ask when $M(P,\mu)$ is Haken as well. To answer this question, we first prove a lemma.

Lemma 2.1 *Suppose $\pi : M_2 \to M_1$ is a double covering where M_1 and M_2 are both closed connected 3-manifolds. Then the following statements are equivalent.*

(i) *M_1 is $\mathbb{P}^2$-irreducible and $M_1 \neq \mathbb{R}P^3$ or S^3.*

(ii) *M_2 is $\mathbb{P}^2$-irreducible and $M_2 \neq S^3$.*

Proof (i)$\Rightarrow$(ii). This follows from the Fact-3.

(ii)$\Rightarrow$(i). Since $M_2 \neq S^3$, M_1 cannot be $\mathbb{R}P^3$. And since M_2 is connected, M_1 cannot be S^3 either. For any embedded 2-sphere S in M_1, it is clear that $\pi^{-1}(S)$ consists of two disjoint 2-spheres $\widetilde{S}_1$ and $\widetilde{S}_2$ in M_2. Then since M_2 is irreducible, $\widetilde{S}_1$ and $\widetilde{S}_2$ both bound some 3-balls in M_2. This implies that S is the zero element in $\pi_2(M_1)$. Then by [12, Proposition 3.10], S bounds a 3-ball in M_1. So M_1 is irreducible.

If M_1 contains an embedded surface Σ which is a 2-sided $\mathbb{R}P^2$, then $\pi^{-1}(\Sigma)$ is either an embedded 2-sphere or a disjoint union of two 2-sided $\mathbb{R}P^2$s in M_2. But since M_2 is $\mathbb{P}^2$-irreducible, $\pi^{-1}(\Sigma)$ has to be an embedded 2-sphere and so bounds a 3-ball in M_2. This implies that Σ represents the zero element in $\pi_2(M_1)$. But this contradicts the Fact-5. So M_1 cannot contain any 2-sided $\mathbb{R}P^2$. Then M_1 is $\mathbb{P}^2$-irreducible. The lemma is proved.

Theorem 2.2 *Let P be a 3-dimensional simple polytope and μ be a non-degenerate $(\mathbb{Z}_2)^r$-coloring on P. The following statements are all equivalent.*

(i) *P is a flag polytope.*

(ii) *$M(P,\mu)$ is aspherical.*

(iii) *$M(P,\mu)$ is $\mathbb{P}^2$-irreducible and $M(P,\mu) \neq \mathbb{R}P^3$ or S^3.*

(iv) *$M(P,\mu)$ is Haken.*

(v) *$M(P,\mu)$ admits a Riemannian metric with non-positive sectional curvature.*

Proof First of all, we choose an arbitrary small cover M over P (the existence of M is guaranteed by the Four Color theorem).

(i)$\Leftrightarrow$(ii). By [6, Theorem 2.2.5], P is a flag polytope if and only if M is aspherical. Then since $\mathbb{R}\mathcal{Z}_P$ is a covering space of both M and $M(P,\mu)$, M is aspherical if and only if $M(P,\mu)$ is aspherical.

(ii)$\Rightarrow$(iii). This follows from Fact-2.

(iii)$\Rightarrow$(i). Since $\mathbb{R}\mathcal{Z}_P$ is a regular $(\mathbb{Z}_2)^{m-r}$-covering of $M(P,\mu)$, we can think of $\mathbb{R}\mathcal{Z}_P$ as the total space of an $(m-r)$-stage iterated double covering of $M(P,\mu)$. Then by Lemma 2.1, the conditions on $M(P,\mu)$ given in (iii) implies that $\mathbb{R}\mathcal{Z}_P$ is $\mathbb{P}^2$-irreducible and $\mathbb{R}\mathcal{Z}_P \neq S^3$ (since P cannot be the 3-simplex). Moreover, since $\mathbb{R}\mathcal{Z}_P$ is a regular $(\mathbb{Z}_2)^{m-3}$-covering of the small cover M, we can think of $\mathbb{R}\mathcal{Z}_P$ as the total space of an $(m-3)$-stage iterated double covering of M. Then using Lemma 2.1 again we can deduce that M is $\mathbb{P}^2$-irreducible and $M \neq \mathbb{R}P^3$. So by Theorem 2.1, P is a flag polytope.

(iv)$\Rightarrow$(iii). This follows from the definition of Haken manifolds.

(i)$\Rightarrow$(iv). If P is a flag polytope, then $M(P,\mu)$ is aspherical hence $\mathbb{P}^2$-irreducible (by Fact-2). It remains to prove that $M(P,\mu)$ has a two-sided incompressible surface. Let $\pi : M \to P$ and $\pi_\mu : M(P,\mu) \to P$ be the projections. In addition, let $\xi : \mathbb{R}\mathcal{Z}_P \to M$ and $\xi_\mu : \mathbb{R}\mathcal{Z}_P \to M(P,\mu)$ be the regular $(\mathbb{Z}_2)^{m-3}$-covering map and regular $(\mathbb{Z}_2)^{m-r}$-covering map, respectively. Clearly $\pi \circ \xi = \pi_\mu \circ \xi_\mu : \mathbb{R}\mathcal{Z}_P \to P$ is the natural projection. So we have the following diagram.

$$\begin{tikzcd} & \mathbb{R}\mathcal{Z}_P \arrow[dl, "\xi"'] \arrow[dr, "\xi_\mu"] & \\ M \arrow[dr, "\pi"'] & & M(P,\mu) \arrow[dl, "\pi_\mu"] \\ & P & \end{tikzcd}$$

Let F be an arbitrary facet of P. By the proof of Theorem 2.1, $M_F = \pi^{-1}(F)$ is a π_1-injective embedded surface of M with non-positive Euler characteristic.

- Let N_F be a connected component of $\pi_\mu^{-1}(F)$ in $M(P,\mu)$.
- Let Σ_F be a connected component of $\xi^{-1}(M_F)$ in $\mathbb{R}\mathcal{Z}_P$.

It is easy to see that $\xi|_{\Sigma_F} : \Sigma_F \to M_F$ and $\xi_\mu|_{\Sigma_F} : \Sigma_F \to N_F$ are both regular coverings whose deck transformation groups are $\mathbb{Z}_2$-tori.

Claim-1 Σ_F is a π_1-injective embedded surface of $\mathbb{R}\mathcal{Z}_P$.

Let $i : M_F \to M$ and $\widetilde{i} : \Sigma_F \to \mathbb{R}\mathcal{Z}_P$ be the inclusion maps. Obviously we have $\xi \circ \widetilde{i} = i \circ \xi|_{\Sigma_F} : \Sigma_F \to M$.

$$\begin{tikzcd} \Sigma_F \arrow[r, "\widetilde{i}"] \arrow[d, "\xi|_{\Sigma_F}"'] & \mathbb{R}\mathcal{Z}_P \arrow[d, "\xi"] \\ M_F \arrow[r, "i"] & M \end{tikzcd}$$

So the induced homomorphisms on the fundamental groups satisfy

$$\xi_* \circ \widetilde{i}_* = i_* \circ (\xi|_{\Sigma_F})_* : \pi_1(\Sigma_F) \to \pi_1(M).$$

By our construction, $\xi_* : \pi_1(\mathbb{R}\mathcal{Z}_P) \to \pi_1(M)$, $(\xi|_{\Sigma_F})_* : \pi_1(\Sigma_F) \to \pi_1(M_F)$ and $i_* : \pi_1(M_F) \to \pi_1(M)$ are all injective. So $\widetilde{i}_* : \pi_1(\Sigma_F) \to \pi_1(\mathbb{R}\mathcal{Z}_P)$ must also be injective. Claim-1 is proved.

Claim-2 N_F is a π_1-injective embedded surface of $M(P,\mu)$.

Let $i_\mu : N_F \to M(P,\mu)$ be the inclusion map.

$$\begin{tikzcd} \Sigma_F \arrow[r, "\widetilde{i}"] \arrow[d, "\xi_\mu|_{\Sigma_F}"'] & \mathbb{R}\mathcal{Z}_P \arrow[d, "\xi_\mu"] \\ N_F \arrow[r, "i_\mu"] & M(P,\mu) \end{tikzcd}$$

Then since $\xi_\mu \circ \widetilde{i} = i_\mu \circ \xi_\mu|_{\Sigma_F} : \Sigma_F \to M(P,\mu)$, we have

$$(\xi_\mu)_* \circ \widetilde{i}_* = (i_\mu)_* \circ (\xi_\mu|_{\Sigma_F})_* : \pi_1(\Sigma_F) \to \pi_1(M(P,\mu)).$$

By our assumption, $\widetilde{i}_* : \pi_1(\Sigma_F) \to \pi_1(\mathbb{R}\mathcal{Z}_P)$, $(\xi_\mu)_* : \pi_1(\mathbb{R}\mathcal{Z}_P) \to \pi_1(M(P,\mu))$ and $(\xi_\mu|_{\Sigma_F})_* : \pi_1(\Sigma_F) \to \pi_1(N_F)$ are all injective. But these conditions do not directly imply the injectivity of $(i_\mu)_* : \pi_1(N_F) \to \pi_1(M(P,\mu))$. Note that the Euler characteristic of N_F is non-positive. So $\pi_1(N_F)$ is torsion-free. Then the kernel of $(i_\mu)_* : \pi_1(N_F) \to \pi_1(M(P,\mu))$ is either trivial or an infinite subgroup of $\pi_1(N_F)$. On the other hand, since Σ_F is a finite-sheeted covering of N_F, the image of $(\xi_\mu|_{\Sigma_F})_*$ is a finite index subgroup of $\pi_1(N_F)$. If $(i_\mu)_*$ is not injective, then $\ker(i_\mu)_*$ is an infinite group. So the intersection of the image of $(\xi_\mu|_{\Sigma_F})_*$ and $\ker(i_\mu)_*$ is a finite index subgroup of $\ker(i_\mu)_*$ hence also an infinite group. But this contradicts the fact that $(i_\mu)_* \circ (\xi_\mu|_{\Sigma_F})_* = (\xi_\mu)_* \circ \widetilde{i}_*$ is injective. Therefore, $(i_\mu)_*$ has to be injective. Claim-2 is proved.

Finally, if N_F is two-sided in $M(P,\mu)$, then N_F is a two-sided π_1-injective embedded connected surface with non-positive Euler characteristic in $M(P,\mu)$. So $M(P,\mu)$ is a Haken manifold. Otherwise if N_F is one-sided in $M(P,\mu)$, let $\widetilde{N}_F$ be the boundary of a tubular neighborhood of N_F in $M(P,\mu)$. Using the similar argument as in the proof of (i)⇒(iv) in Theorem 2.1, we can prove that $\widetilde{N}_F$ is a two-sided π_1-injective embedded connected surface with non-positive Euler characteristic in $M(P,\mu)$. So we finish the proof.

(v)⇒(ii). This follows from the Cartan-Hadamard theorem.

(ii)⇒(v). It is shown in [5] that $\pi_1(\mathbb{R}\mathcal{Z}_P)$ is isomorphic to the commutator subgroup $[W_P, W_P]$ of W_P. Then since W_P is virtually special, so is $\pi_1(\mathbb{R}\mathcal{Z}_P)$. Moreover, since $\mathbb{R}\mathcal{Z}_P$ is a regular $(\mathbb{Z}_2)^{m-r}$-covering of $M(P,\mu)$, $\pi_1(\mathbb{R}\mathcal{Z}_P)$ is a finite index subgroup of $\pi_1(M(P,\mu))$. This implies that $\pi_1(M(P,\mu))$ is also virtually special. So by [18, Corollary 1.4], $M(P,\mu)$ admits a Riemannian metric with non-positive sectional curvature.

Remark 2.1 Recently, more results on the existence of incompressible surfaces in $M(P,\mu)$ are obtained in Erokhovets [7].

3 More Results Related to the Fundamental Groups of Small Covers

3.1 Borel conjecture for small covers

A closed manifold M is said to be topologically rigid if any homotopy equivalence from M to another closed manifold is homotopic to a homeomorphism. In particular, if M is topologically rigid, any closed manifold homotopy equivalent to M is homeomorphic to M.

Borel Conjecture Closed aspherical manifolds are topologically rigid.

Proposition 3.1 *For any $n \neq 4$, the Borel Conjecture holds for n-dimensional aspherical small covers.*

Proof Haken 3-manifolds are known to satisfy the Borel Conjecture (see [19]). So by Theorem 2.1, the Borel Conjecture holds for aspherical small covers in dimension 3. In fact, Thurston's Geometrization Conjecture implies that every closed 3-manifold with torsion-free fundamental group is topologically rigid. So in particular, the Borel Conjecture holds in dimension 3 (see [14, Theorem 0.7]).

For higher dimensions, let M be a small cover over a simple polytope P and assume that M is aspherical. Then P is a flag polytope. Define

$$\mathscr{L}_P = P \times W_P / \sim,$$

where $(p, \omega) \sim (p', \omega')$ if and only if $p = p'$ and $\omega' \omega^{-1}$ belongs to the subgroup of W_P that is generated by $\{s_F \mid F \text{ is a facet of } P \text{ that contains } p\}$. There is a canonical action of W_P on $\mathscr{L}_P$ defined by:

$$\omega' \cdot [(p, \omega)] = [(p, \omega' \omega)], \quad p \in P, \ \omega, \omega' \in W_P, \tag{3.1}$$

where $[(p, \omega)]$ is the equivalence class of (p, ω) in $\mathscr{L}_P$.

By [3, Corollary 10.2], $\mathscr{L}_P$ is a simply connected manifold, which is the universal covering of the small cover M. Moreover, there is a natural piecewise Euclidean cubical metric $d_\square$ on $\mathscr{L}_P$ where the canonical action of W_P on $\mathscr{L}_P$ is isometric. It is shown in [6, Section 1.6] that the metric $d_\square$ is non-positively curved (in the sense of Aleksandrov) if and only if P is a flag polytope (also see [4, Chapter 1]). So in our case $(\mathscr{L}_P, d_\square)$ is a CAT(0) space.

Since $\pi_1(M)$ is a finite index subgroup of W_P (see (1.3)), $\pi_1(M)$ acts isometrically and cocompactly on $(\mathscr{L}_P, d_\square)$ (also see [20, Proposition 2.4]). Then $\pi_1(M)$ belongs to the class of groups $\mathcal{B}$ defined in Bartels-Lück [1, Theorem A], which implies that the Borel Conjecture holds for all aspherical small covers in dimension ≥ 5.

The Borel Conjecture is also very likely to hold for 4-dimensional aspherical small covers. But we do no have any tools to prove this statement.

3.2 Prime decomposition of 3-dimensional small covers

It is well known that every closed, orientable 3-manifold is the connected sum of a unique (up to homeomorphism) finite collection of prime 3-manifolds (see [13]). The following proposition tells us that all the prime factors of a 3-dimensional small cover are still small covers.

Proposition 3.2 *Let M be a small cover over a 3-dimensional simple polytope P. If M is orientable, then all the prime factors in the prime decomposition of M are either aspherical small covers or $\mathbb{R}P^3$.*

Proof Let λ be the characteristic function of M. Since M is orientable, we can assume that the range of λ is in the subset $\{e_1, e_2, e_3, e_1 + e_2 + e_3\}$ of $(\mathbb{Z}_2)^3$, where $\{e_1, e_2, e_3\}$ is a basis of $(\mathbb{Z}_2)^3$ (see [17, Theorem 1.7]). If $P \neq \Delta^3$ has no 3-belts, M is aspherical by [6, Theorem 2.2.5] and hence prime (see Fact-2).

Now suppose there exist some 3-belts in P. Note that up to a change of basis of $(\mathbb{Z}_2)^3$, the value of λ on any 3-belt on P must be equivalent to one of the pictures in Figures 1–2. Then M is the connected sum of two closed 3-manifolds N_1 and N_2 along the embedded 2-sphere $\pi^{-1}(D)$, each of which is an orientable small cover. Clearly the underlying simple polytopes of N_1 and N_2 both have less 3-belts than P. Since P has only finitely many 3-belts, by iterating this argument, we can write M as a connected sum $M_1 \# \cdots \# M_k$, where each M_i is an orientable small cover over a simple polytope P_i that has no 3-belts. If $P_i = \Delta^3$, M_i must be $\mathbb{R}P^3$. Otherwise P_i is a flag polytope and so M_i is aspherical. This proves the proposition.

Corollary 3.1 *If two 3-dimensional small covers M_1 and M_2 are orientable and have isomorphic fundamental groups, then M_1 is homeomorphic to M_2.*

Proof It is well known that the indecomposable factors of any finitely generated group is unique up to order and isomorphism (see [13, Section 8]). Suppose

$$\pi_1(M_1) \cong \pi_1(M_2) \cong G_1 * \cdots * G_k,$$

where each G_i is indecomposable (cannot be further factored via free product). Then by Fact-4, there exist prime decompositions:

$$M_i = N_{i,1} \# \cdots \# N_{i,k}, \quad \text{where } \pi_1(N_{i,j}) \cong G_j,\ 1 \leq j \leq k,\ i = 1, 2.$$

By Proposition 3.2, each $N_{i,j}$ is either an aspherical small cover or $\mathbb{R}P^3$. Then by Proposition 3.1, $N_{1,j} \cong N_{2,j}$ for all $1 \leq j \leq k$. So M_1 is homeomorphic to M_2.

It is well known that every closed, non-orientable 3-manifold is a connected sum of irreducible 3-manifolds and non-trivial S^2 bundles over S^1. But it is not clear to us whether the factors in such a decomposition for a non-orientable 3-dimensional small cover are still small covers or not.

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