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Steady Compressible Euler Equations of Concentration Layers for Hypersonic-limit Flows Passing Three-dimensional Bodies and Generalized Newton-Busemann Pressure Law*

Aifang QU[1] Hairong YUAN[2]

Abstract For stationary hypersonic-limit Euler flows passing a solid body in three-dimensional space, the shock-front coincides with the upwind surface of the body, hence there is an infinite-thin layer of concentrated mass, in which all particles hitting the body move along its upwind surface. By proposing a concept of Radon measure solutions of boundary value problems of the multi-dimensional compressible Euler equations, which incorporates the large-scale of three-dimensional distributions of upcoming hypersonic flows and the small-scale of particles moving on two-dimensional surfaces, the authors derive the compressible Euler equations for flows in concentration layers, which is a stationary pressureless compressible Euler system with source terms and independent variables on curved surface. As a by-product, they obtain a formula for pressure distribution on surfaces of general obstacles in hypersonic flows, which is a generalization of the classical Newton-Busemann law for drag/lift in hypersonic aerodynamics.

Keywords Compressible Euler equations, Hypersonic flow, Concentration layer, Ramp, Cone, Radon measure solution, Newton-Busemann law

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1 Introduction

Mathematical analysis of flow fields of supersonic gas passing bodies in three dimensional space is one of the fundamental problems in aerodynamics. It is a great challenge to both theory and computation of partial differential equations (PDE for short), since the flow contains shocks and behaves very singularly near the boundary of the body. To focus on the mechanical effects, we consider only compressible Euler flows of polytropic gases, thus neglecting viscosity, heat transfer and ionization. Then as Mach number of the upcoming supersonic flow increases, it is observed that shock-fronts approach the upwind part of the body's boundary surface, thus the shock layer, i.e., the region bounded by shock-fronts and the body, becomes narrower and narrower. If the upstream Mach number is infinite, the shock-fronts coincide with the upwind surface and the shock layer is infinitely thin (see [1, Section 3.2, p.58]). However,

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[1]Department of Mathematics, Shanghai Normal University, Shanghai 200234, China.
E-mail: afqu@shnu.edu.cn
[2]School of Mathematical Sciences and Shanghai Key Laboratory of Pure Mathematics and Mathematical Practice, East China Normal University, Shanghai 200241, China. E-mail: hryuan@math.ecnu.edu.cn
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by the hyperbolic nature of supersonic flow, the downstream flow is totally determined by the upstream, hence none gas particles across shock-fronts can rebound from the shock layer. Therefore, concentration of mass occur unavoidably on the upwind surface of the body. Thus the flow field contains two different scales: One for distribution of upstream gas in three dimensional space, the other for flow in concentration layer which is essentially two-dimensional. It is necessary to derive the PDE governing the flow in the mass-concentration layer, to obtain a precise description of hypersonic flows passing bodies.

This paper is devoted to solving this problem by studying Radon measure solutions of boundary value problems of multidimensional compressible Euler equations. We propose a definition, and then construct Radon measure solutions with regular parts (absolutely continuous with respect to the Lebesgue measure of $\mathbb{R}^3$) to describe the upcoming hypersonic flow, and singular parts (weighted Dirac measures supported on the upwind surface of the body) to characterize the infinite-thin shock layer. From the weights of the Dirac measures, we obtain mass density and velocity of the flow in a concentration layer, which satisfy a hyperbolic system of balance laws on a curved surface. The system resembles the compressible pressureless Euler equations, but with source terms manifesting addition of mass and momentum to each point of the concentration layer, from the upstream hypersonic flow. The distribution of the pressure on the upwind surface could also be solved from the system, whose integration over the surface provides the lift/drag force exerting to the body by the hypersonic flow. It provides a generalization of the classical Newton's sine-squared pressure law for straight wedge/cone in hypersonic flow, and its modification by Busemann for curved wedge/cone when centrifugal force of particles moving restrictively on a curved surface is considered in [1, Sections 3.2 and 3.4]. This justified the concept of Radon measure solution and the study of Radon measure solutions from the point view of physics and applications.

We briefly review some related previous works. There are now some significant results on supersonic flow passing bodies, see, for example, [3–7, 10] and references therein. These works are all devoted to studying flow fields with shocks. It is somewhat surprising that there is no theoretical studies of hypersonic flows with concentrations from mathematical point of view before the work [18], which considered hypersonic-limit flow passing straight wedges and proved rigorously the Newton's sine-squared pressure law. Later the authors generalize the ideas to study hypersonic limit flow passing curved wedges/cones, and straight cones with arbitrary cross-sections and attacking angles (see [12–13, 16–17]). These works lead us to solve the general case presented in this paper. Mathematically, we are investigating solutions of compressible Euler equations which are measures, rather than functions. It is interesting to notice that, however, there is a rather long history on measure solutions of hyperbolic conservation laws, debut by Korchinski [14] in 1977, and later rediscovered, named as "delta shocks" and thoroughly studied by Chinese scholars led by Tong Zhang, see, for example, [11, 15, 19–20] and references therein. All these works consider the Cauchy problems, rather than the initial-boundary value problem firstly studied in [18]. It seems that measure solution is inevitable for a general theory of hyperbolic systems of conservation laws, since for large initial data or weakly hyperbolic equations resonance of eigenvalues and eigenvectors may occur so the solution will go to infinite and concentration appears, see [2, Remark 3], and [8, Section 9.6, as well as p.339].

In the following Section 2 we formulate the aforementioned problem of hypersonic limit flow passing a three-dimensional body (Definition 2.2), and derive the PDE for concentration layers, namely (2.53)–(2.57). The generalized Newton-Busemann formula is given by (2.61). These are the main contributions of this paper. The last Section 3 presents some comments and problems in this direction for possible studies in the future.

2 The Steady Compressible Euler Equations of Concentration Layers

This section is the main part of this paper. After a mathematical formulation of hypersonic-limit flows passing bodies, which is named Problem (A), we propose a definition of its Radon measure solutions. Then by studying a special Radon measure solution with a layer of concentrated mass on the boundary of the body, we derive the desired PDE governing motions of particles moving along solid upwind surfaces. As a by-product, we obtain the generalized Newton-Busemann law. Some specific examples are given at the end of this section.

2.1 Preliminaries

In the Euclidean space $\mathbb{R}^3$ equipped with the standard Cartesian coordinates $r=(x,y,z)$, let $\mathcal{B}$ be a surface given by the parametrization

$$r=r(x,s)\doteq(x,\ b_1(x,s),\ b_2(x,s))$$

with $(x,s)\in\mathcal{P}\doteq(\mathbb{R}_+\cup\{0\})\times I$. Here I is a set of real numbers. The functions b_1,b_2 are supposed to be C^2, i.e., their all up to second order derivatives are continuous. For $I=\mathbb{R}$, $\mathcal{B}$ represents the boundary surface of a ramp (see Figure 1), and for $I=S^1\doteq\mathbb{R}/2\pi\mathbb{Z}$ (the unit circle), $\mathcal{B}$ is a conical surface (see Figure 2). The solid obstacle bounded by $\mathcal{B}$ is denoted by $\mathcal{O}$, which is an open set in $\mathbb{R}^3$. For the case of a ramp, it is given by $\mathcal{O}=\{(x,y,z):\ x\geq 0,\ y<b_1(x,z),\ z\in\mathbb{R}\}$. While for the conical body, since $b_1(x,s),\ b_2(x,s)$ are periodic with respect to s, for fixed $x>0$, we suppose the simple Jordan curve $\{y=b_1(x,s),\ z=b_2(x,s)\}$ encloses a bounded domain Σ'_x on the (y,z)-plane. Then $\mathcal{O}=\bigcup\limits_{x\geq 0}(\{x\}\times\Sigma'_x)$. For each case, we choose the orientation following the "right-hand rule", so that the positive direction of the plane $\{x=x_0\}$ is the same as the x-axis, and that as s increases, for any fixed $x_0>0$, the section of the solid body $\Sigma_{x_0}\doteq\mathcal{O}\cap\{x=x_0\}$ always lies at the left-hand side of the curve $\{y=b_1(x_0,s),\ z=b_2(x_0,s)\}$. For simplicity of writing, we assume that $\partial I=\emptyset$. These enable us to apply the divergence theorem later correctly and in a simple way. The space occupied by gas is then the closed set $\mathcal{G}\doteq\{x\geq 0\}\setminus\mathcal{O}$. We also write $\mathcal{I}\doteq\mathcal{G}\cap\{x=0\}$ as the initial boundary, where the upcoming uniform supersonic flow is given. Hence the boundary of $\mathcal{G}$ is $\partial\mathcal{G}=\mathcal{I}\cup\mathcal{B}$.

The tangent vectors on $\mathcal{B}$ are given by

$$r_x=(1,\partial_x b_1,\partial_x b_2), \tag{2.1}$$

$$r_s=(0,\partial_s b_1,\partial_s b_2). \tag{2.2}$$

Hence a normal vector is

$$n=(n_1,n_2,n_3)\doteq r_x\times r_s=(\partial_x b_1\partial_s b_2-\partial_x b_2\partial_s b_1,-\partial_s b_2,\ \partial_s b_1). \tag{2.3}$$

In the following, for some computations, we assume that $n_1>0$. A surface element with this property is called upwind. Note that $\frac{n}{|n|}$ is the unit normal on $\mathcal{B}$ pointing into $\mathcal{O}$. Here by $|n|$ we mean the Euclidean norm of the vector n.

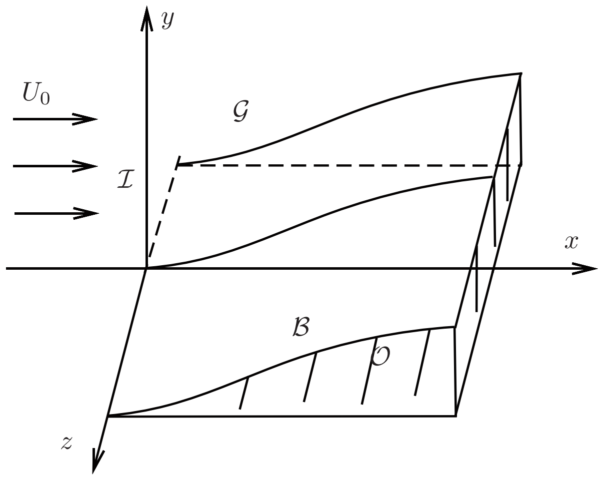

Figure 1 A ramp.

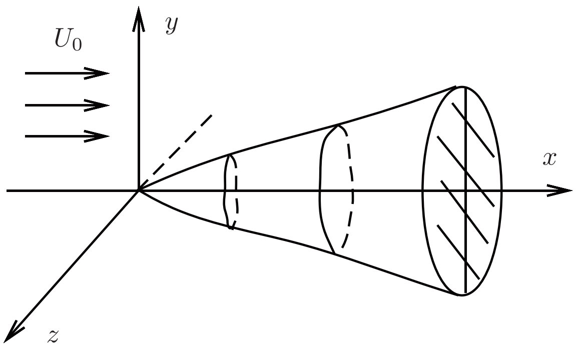

Figure 2 A conical body.

The first fundamental form of the surface $\mathcal{B}$ is

$$\mathrm{d}l^2 = \mathrm{d}r \cdot \mathrm{d}r = E\mathrm{d}x^2 + 2F\mathrm{d}x\mathrm{d}s + G\mathrm{d}s^2, \tag{2.4}$$

where

$$\begin{aligned} E &\doteq r_x \cdot r_x = 1 + (\partial_x b_1)^2 + (\partial_x b_2)^2, \\ F &\doteq r_x \cdot r_s = \partial_x b_1 \partial_s b_1 + \partial_x b_2 \partial_s b_2, \\ G &\doteq r_s \cdot r_s = (\partial_s b_1)^2 + (\partial_s b_2)^2. \end{aligned} \tag{2.5}$$

It is easy to check that

$$EG - F^2 = |n|^2, \tag{2.6}$$

hence the area element of $\mathcal{B}$ is

$$\mathrm{d}\mathcal{H}^2 = |n|\mathrm{d}x\mathrm{d}s, \tag{2.7}$$

where $\mathcal{H}^2$ is the two-dimensional Hausdorff measure on $\mathcal{B}$.

Let $U = (u, v, w)$ be a vector field in $\mathbb{R}^3$. We now consider its decomposition along $\mathcal{B}$, with respect to the moving frames $\{r_x, r_s, n\}$. Suppose that

$$U = \mu r_x + \nu r_s + \omega n. \tag{2.8}$$

Then as $r_x \perp n,\ r_s \perp n$, one has

$$\omega = \frac{U \cdot n}{|n|^2}. \tag{2.9}$$

If U is a tangent vector on $\mathcal{B}$, namely $\omega = 0$, from (2.8) we have

$$u = \mu, \quad v = (\partial_x b_1)\mu + (\partial_s b_1)\nu, \quad w = (\partial_x b_2)\mu + (\partial_s b_2)\nu \tag{2.10}$$

or

$$\mu = \frac{1}{n_1}(v\partial_s b_2 - w\partial_s b_1), \quad \nu = \frac{1}{n_1}(-v\partial_x b_2 + w\partial_x b_1). \tag{2.11}$$

Let $\phi(x, y, z) \in C_c^1(\mathbb{R}^3)$ be a compactly supported continuously differentiable function. Set

$$\widetilde{\phi} = \widetilde{\phi}(x, s) \doteq \phi(x,\ b_1(x, s),\ b_2(x, s)). \tag{2.12}$$

Direct differentiation shows that

$$(\partial_x b_1)\phi_y + (\partial_x b_2)\phi_z = \widetilde{\phi}_x - \phi_x, \quad (\partial_s b_1)\phi_y + (\partial_s b_2)\phi_z = \widetilde{\phi}_s.$$

It follows that

$$\phi_y = \frac{1}{n_1}(\widetilde{\phi}_x \partial_s b_2 - \widetilde{\phi}_s \partial_x b_2 - \phi_x \partial_s b_2), \tag{2.13}$$

$$\phi_z = \frac{1}{n_1}(\widetilde{\phi}_s \partial_x b_1 - \widetilde{\phi}_x \partial_s b_1 + \phi_x \partial_s b_1). \tag{2.14}$$

Then we obtain easily that

$$U \cdot \nabla\phi = u\phi_x + v\phi_y + w\phi_z = \frac{1}{n_1} U \cdot n\phi_x + \mu\widetilde{\phi}_x + \nu\widetilde{\phi}_s, \tag{2.15}$$

where μ and ν are given by (2.11). Notice that this identity holds no matter $U \cdot n = 0$ or not.

Next we utilize Green's theorem to calculate

$$\begin{aligned}
&\int_{\mathcal{B}} (\mu\widetilde{\phi}_x + \nu\widetilde{\phi}_s)\,\mathrm{d}\mathcal{H}^2 = \int_{\mathcal{P}} (\mu\widetilde{\phi}_x + \nu\widetilde{\phi}_s)|n|\,\mathrm{d}x\mathrm{d}s \\
&= \int_{\mathcal{P}} [\partial_x(\widetilde{\phi}\mu|n|) + \partial_s(\widetilde{\phi}\nu|n|)]\,\mathrm{d}x\mathrm{d}s - \int_{\mathcal{P}} \widetilde{\phi}[\partial_x(\mu|n|) + \partial_s(\nu|n|)]\,\mathrm{d}x\mathrm{d}s \\
&= \int_{\partial\mathcal{P}} \widetilde{\phi}\mu|n|\,\mathrm{d}s - \widetilde{\phi}\nu|n|\,\mathrm{d}x - \int_{\mathcal{P}} \widetilde{\phi}[\partial_x(\mu|n|) + \partial_s(\nu|n|)]\,\mathrm{d}x\mathrm{d}s \\
&= -\int_I (\widetilde{\phi}\mu|n|)(0, s)\,\mathrm{d}s - \int_{\mathcal{B}} \widetilde{\phi}\frac{1}{|n|}[\partial_x(\mu|n|) + \partial_s(\nu|n|)]\,\mathrm{d}\mathcal{H}^2 \\
&= -\int_I (\widetilde{\phi}\mu|n|)(0, s)\,\mathrm{d}s - \int_{\mathcal{B}} \widetilde{\phi}\,\mathrm{div}\,\varpi\,\mathrm{d}\mathcal{H}^2,
\end{aligned} \tag{2.16}$$

where in the last line we have set

$$\varpi \doteq \mu r_x + \nu r_s = \mu\partial_x + \nu\partial_s, \tag{2.17}$$

and the forth equality utilized the assumption $\partial I = \emptyset$. Recall the definition of divergence operator in a Riemannian manifold with the metric $\mathrm{d}l^2$ (see (2.4)):

$$\operatorname{div}\varpi \doteq \frac{1}{|n|}[\partial_x(\mu|n|) + \partial_s(\nu|n|)].$$

Notice that by considering a vector field as an operator on scalar functions, (2.10) can also be written as [1]

$$u = \varpi(x), \quad v = \varpi(b_1), \quad w = \varpi(b_2). \tag{2.18}$$

We then could calculate the integral $\int_{\mathcal{B}} U\cdot\nabla\phi\,\mathrm{d}\mathcal{H}^2$, where $U:\mathcal{B}\to\mathbb{R}^3$ is a C^1 vector field, and $\phi\in C_c^1(\mathbb{R}^3)$. By (2.15), we have

$$\begin{aligned}\int_{\mathcal{B}} U\cdot\nabla\phi\,\mathrm{d}\mathcal{H}^2 &= \int_{\mathcal{B}}\frac{1}{n_1}U\cdot n\phi_x\,\mathrm{d}\mathcal{H}^2 + \int_{\mathcal{B}}(\mu\widetilde{\phi}_x + \nu\widetilde{\phi}_s)\,\mathrm{d}\mathcal{H}^2\\ &= \int_{\mathcal{B}}\frac{1}{n_1}U\cdot n\phi_x\,\mathrm{d}\mathcal{H}^2 - \int_{\mathcal{B}}\operatorname{div}(\varpi)\widetilde{\phi}\,\mathrm{d}\mathcal{H}^2 - \int_I(\widetilde{\phi}\mu|n|)(0,s)\,\mathrm{d}s.\end{aligned} \tag{2.19}$$

2.2 Radon measure solutions of three-dimensional compressible Euler equations

The three-dimensional steady compressible Euler equations consist of the following conservation of mass, momentum and energy (see [8, (3.3.29) in p.62]):

$$\operatorname{Div}(\rho U) = 0, \tag{2.20}$$

$$\operatorname{Div}(\rho U\otimes U) + \nabla p = 0, \tag{2.21}$$

$$\operatorname{Div}(\rho U H) = 0, \tag{2.22}$$

where ρ, p and H are the density, the pressure and total enthalpy of the gas, respectively, and $U = (u, v, w)$ is the velocity. We use Div and ∇ to denote respectively the divergence and gradient operator in $\mathbb{R}^3$, and the tensor product $U\otimes U$ is given by the matrix $U^\top U$ in Cartesian coordinates. For a polytropic gas, the state function is

$$p = \frac{\gamma-1}{\gamma}\rho\cdot\Big(H - \frac{1}{2}|U|^2\Big), \tag{2.23}$$

where $\gamma > 1$ is the adiabatic exponent. Then the sonic speed is $c_{\text{sonic}} \doteq \sqrt{\frac{\gamma p}{\rho}}$, and the Mach number is defined by $M_{\text{ach}} \doteq \frac{|U|}{c_{\text{sonic}}}$. After some unidimensional scalings, we may assume that the uniform upcoming supersonic flow is given by

$$\rho = \rho_0 = 1, \quad U = U_0 = (1,0,0), \quad H = H_0 \quad \text{on } \mathcal{I} \tag{2.24}$$

with $H_0 > \frac{1}{2}$ a constant. The Mach number of this upstream flow is (see [18, (8) in p. 4])

$$M_{\text{ach},0} = \frac{1}{(\gamma-1)\big(H_0-\frac{1}{2}\big)}.$$

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[1] Here $\varpi(x)$ means the vector field $\varpi = \mu\partial_x + \nu\partial_s$ acting on the function x, rather than the value of ϖ at x, which does not make sense in this paper.

Hence the hypersonic-limit, namely $M_{\mathrm{ach},0} \to +\infty$, is the limit $\gamma \to 1$, and for hypersonic-limit flow, we may take $\gamma = 1$ in (2.23). Therefore the limiting hypersonic gas is pressureless. It is known that the resultant pressureless Euler equations (2.20)–(2.22), constituting a system of conservation laws, are hyperbolic along the x-axis, but only weakly hyperbolic, in the sense that all the eigenvalues are real and coincident, and the associated eigenvectors are not complete in the state space $\mathbb{R}^5$.

We now formulate a boundary value problem of the pressureless Euler equations (2.20)–(2.22) in the domain $\mathcal{G}$, subjected with the initial data (2.24), and the slip condition

$$U \cdot n = 0 \quad \text{on } \mathcal{B}. \tag{2.25}$$

For simplicity, in the sequel we call it Problem (A). This is a mathematical model of stationary limiting hypersonic Euler flows passing three-dimensional obstacles $\mathcal{O}$.

The above formulation of Problem (A) is deduced for classical solutions, namely C^1 flow fields without singularities. We now rewrite it for integrable weak solutions, to allow for flow fields with discontinuities, thus motivate our definition of Radon measure solutions later.

We write the Lebesgue measure in $\mathbb{R}^n$ as $\mathcal{L}^n$. Locally integrable functions (ρ, U, H) (with respect to the Lebesgue measure $\mathcal{L}^3$) are called weak solutions to Problem (A), provided that:

- For any $\phi \in C_c^1(\mathbb{R}^3)$, there hold

$$\int_{\mathcal{G}} \rho U \cdot \nabla\phi \,\mathrm{d}\mathcal{L}^3 + \int_{\mathcal{I}} \rho_0 u_0 \phi \,\mathrm{d}\mathcal{L}^2 = 0, \tag{2.26}$$

$$\int_{\mathcal{G}} \rho H U \cdot \nabla\phi \,\mathrm{d}\mathcal{L}^3 + \int_{\mathcal{I}} \rho_0 u_0 H_0 \phi \,\mathrm{d}\mathcal{L}^2 = 0. \tag{2.27}$$

- For any vector field $\varphi \in C_c^1(\mathbb{R}^3; \mathbb{R}^3)$, there holds

$$\begin{aligned}&\int_{\mathcal{G}} \rho U \otimes U : \nabla\varphi \,\mathrm{d}\mathcal{L}^3 + \int_{\mathcal{G}} p \,\mathrm{Div}\,\varphi \,\mathrm{d}\mathcal{L}^3 + \int_{\mathcal{I}} \rho_0 (1,0,0) U_0 \otimes U_0 \varphi \,\mathrm{d}\mathcal{L}^2 \\ &= \int_{\mathcal{B}} p\varphi \cdot \frac{n}{|n|} \,\mathrm{d}\mathcal{H}^2.\end{aligned} \tag{2.28}$$

In (2.28), φ is considered as a column vector, and $\nabla\varphi$ is the Jacobian matrix. Recall that ':' means the standard inner product of matrices A and B, namely, $A : B \doteq \mathrm{tr}(AB)$. For pressureless flow, we shall take p in the second integrand in the left-hand-side of (2.28) to be zero, while to take into account of the effects of solid boundary $\mathcal{B}$, the p in the right-hand-side of (2.28) shall be calculated if we study concentration of mass. This is the essential difference from the treatment of Cauchy problems of steady pressureless Euler equations. To guarantee uniqueness and stability of weak solutions, one also needs certain entropy admissibility conditions, such as the Lax E-condition. However, for our purpose in this paper, we do not need such conditions presently. Hence they were ignored.

There is a Newton theory of infinite-thin shock layers for Problem (A), for which the shock front coincides with the upwind boundary of the obstacle, and mass concentrates on its boundary. This theory is fundamental in hypersonic aerodynamics, see [1, Chapter 3] and [9, Chapter III]. We wish to derive the PDE governing motions of particles in the concentration layer. To this end, we need to introduce Radon measure solutions to Problem (A).

We review some basic facts about Radon measures. Let $\mathcal{F}$ be the Borel σ-algebra of $\mathbb{R}^3$. By Riesz representation theorem, a (signed) Radon measure m on $(\mathbb{R}^3, \mathcal{F})$ is a continuous linear functional on $C_c(\mathbb{R}^3)$, the space of compactly supported continuous functions, expressed by

$$\langle m, \phi\rangle = \int_{\mathbb{R}^3} \phi \,\mathrm{d}m, \quad \forall\, \phi \in C_c(\mathbb{R}^3). \tag{2.29}$$

The restrictions of Lebesgue measure $\mathcal{L}^3$ on a measurable set $\Omega \in \mathcal{F}$, denoted by $\mathcal{L}^3\lfloor(\Omega)$, is a Radon measure. The other important example is the following weighted Dirac measure $W\delta_{\mathcal{S}}$ supported on a lower dimensional sub-manifold $\mathcal{S}$ of $\mathbb{R}^3$. Suppose that the Hausdorff dimension of $\mathcal{S}$ is k, and W is a locally integrable function on $\mathcal{S}$, with respect to the Hausdorff measure $\mathcal{H}^k$. Then we define

$$\langle W\delta_{\mathcal{S}}, \phi\rangle \doteq \int_{\mathcal{S}} W\phi \,\mathrm{d}\mathcal{H}^k. \tag{2.30}$$

Hence the weight W may be considered as the density of mass on $\mathcal{S}$. Particularly, for a subset $\Omega \in \mathcal{F}$, we have

$$W\delta_{\mathcal{S}}(\Omega) = \int_{\mathcal{S}\cap\Omega} W \,\mathrm{d}\mathcal{H}^k,$$

which is the mass contained in Ω. Note that $W\delta_{\mathcal{S}}$ is singular with respect to the Lebesgue measure $\mathcal{L}^3$ if $k \neq 3$.

For a (signed) measure m and a nonnegative Radon measure ρ, if m is absolutely continuous with respect to ρ, we write it as $m \ll \rho$. The Radon-Nikodym derivative is denoted by $\frac{\mathrm{d}m}{\mathrm{d}\rho}$, namely, it is a ρ-measurable function, and there holds

$$\int_\Omega \mathrm{d}m = \int_\Omega \frac{\mathrm{d}m}{\mathrm{d}\rho}\,\mathrm{d}\rho, \quad \forall\, \Omega \in \mathcal{F}.$$

To define Radon measure solutions of the Euler equations, we start from the simple but typical case that m is an incompressible vector field in $\mathbb{R}^3$, namely,

$$\operatorname{Div} m = 0. \tag{2.31}$$

By a weak solution of (2.31) in $\mathcal{G}$, we mean that m is a locally integrable vector-valued function on $(\mathbb{R}^3, \mathcal{F}, \mathcal{L}^3)$ so that the following makes sense:

$$\int_{\mathcal{G}} m\cdot\nabla\phi \,\mathrm{d}\mathcal{L}^3 + \int_{\mathcal{I}} m\cdot(1,0,0)\phi \,\mathrm{d}\mathcal{L}^2 = \int_{\mathcal{B}} \phi m\cdot\frac{n}{|n|}\,\mathrm{d}\mathcal{H}^2. \tag{2.32}$$

Motivated by this formula, we propose the following definition.

Definition 2.1 *A Radon measure m defined on $\overline{\mathcal{G}}$ is a solution to* (2.31)*, subjected to the boundary conditions*

$$m\cdot(1,0,0)\lfloor(\mathcal{I}) = m_{1,0}\mathrm{d}\mathcal{L}^2, \tag{2.33}$$

$$m\cdot\frac{n}{|n|}\lfloor(\mathcal{B}) = W\delta_{\mathcal{B}} \tag{2.34}$$

(*where $m_{1,0}$ and W are functions*)*, if there holds*

$$\langle m, \nabla\phi\rangle + \int_{\mathcal{I}} m_{1,0}\phi \,\mathrm{d}\mathcal{L}^2 = \langle W\delta_{\mathcal{B}}, \phi\rangle. \tag{2.35}$$

Now suppose that m has the following regular-singular decomposition

$$m = m_r \mathcal{L}^3 \lfloor (\mathcal{G}) + m_s \delta_{\mathcal{B}}, \tag{2.36}$$

where m_r is a continuous vector field on $\mathcal{G}$ and continuously differentiable in $\mathcal{G} \setminus \mathcal{B}$, such that

$$\mathrm{Div}\, m_r = 0 \quad \text{in } \ \mathcal{G} \setminus \mathcal{B}, \tag{2.37}$$

$$m_r \cdot (1,0,0)|_{\mathcal{I}} = m_{1,0}, \tag{2.38}$$

and $m_s : \mathcal{B} \to \mathbb{R}^3$ is a C^1 vector field, defined only on the surface $\mathcal{B}$. Substituting (2.36) into the left-hand side of (2.35), it follows from (2.32), (2.37) and (2.19) that

$$\begin{aligned}
&\langle m, \nabla\phi\rangle + \int_{\mathcal{I}} m_{1,0}\phi \,\mathrm{d}\mathcal{L}^2 \\
&= \int_{\mathcal{G}} m_r \cdot \nabla\phi \,\mathrm{d}\mathcal{L}^3 + \int_{\mathcal{B}} m_s \cdot \nabla\phi \,\mathrm{d}\mathcal{H}^2 + \int_{\mathcal{I}} m_{1,0}\phi \,\mathrm{d}\mathcal{L}^2 \\
&= \int_{\mathcal{B}} \phi m_r \cdot \frac{n}{|n|} \,\mathrm{d}\mathcal{H}^2 + \int_{\mathcal{B}} m_s \cdot \nabla\phi \,\mathrm{d}\mathcal{H}^2 \\
&= \int_{\mathcal{B}} \widetilde{\phi}\Big[m_r \cdot \frac{n}{|n|} - \mathrm{div}\,(\varpi)\Big] \,\mathrm{d}\mathcal{H}^2 + \int_{\mathcal{B}} \frac{1}{n_1} m_s \cdot n\phi_x \,\mathrm{d}\mathcal{H}^2 - \int_I (\widetilde{\phi}\mu|n|)(0,s)\,\mathrm{d}s.
\end{aligned}$$

Here we used (2.8) and (2.17), where $\varpi = \mu r_x + \nu r_s$, $\widetilde{\phi} = \phi|_{\mathcal{B}}$ and

$$m_s = \varpi + \frac{m_s \cdot n}{|n|^2} n.$$

Then from (2.35) we get

$$\begin{aligned}
&\int_{\mathcal{B}} \widetilde{\phi}\Big[m_r \cdot \frac{n}{|n|} - \mathrm{div}\,(\varpi) - W\Big] \,\mathrm{d}\mathcal{H}^2 + \int_{\mathcal{B}} \frac{1}{n_1} m_s \cdot n\phi_x \,\mathrm{d}\mathcal{H}^2 \\
&- \int_I (\widetilde{\phi}\mu|n|)(0,s)\,\mathrm{d}s = 0.
\end{aligned} \tag{2.39}$$

Since we assumed that $n_1 > 0$, which means ∂_x is not a tangent vector on $\mathcal{B}$, by arbitrariness of the test function ϕ, we deduce that

$$\mu(0,s) = 0, \qquad \text{on } \ I, \tag{2.40}$$

$$\mathrm{div}\,\varpi = m_r \cdot \frac{n}{|n|} - W, \quad \text{in } \ \mathcal{B}, \tag{2.41}$$

$$m_s \cdot n = 0, \qquad \text{in } \ \mathcal{B}. \tag{2.42}$$

These could be considered as generalized Rankine-Hugoniot conditions of concentration layers, which totally characterize Radon measure solutions of problem (2.31), (2.33)–(2.34) with the form (2.36). Notice that (2.42) guarantees ϖ to be indeed a vector field on the surface $\mathcal{B}$. Note that (2.40)–(2.42) make sense for $n_1 \geq 0$.

We apply the ideas presented above to define and calculate Radon measure solutions of Problem (A).

Definition 2.2 *We say vector-valued Radon measures $m, m_1, m_2, m_3, m_4 : (\mathcal{G}, \mathcal{F}) \to \mathbb{R}^3$, and a locally $\mathcal{H}^2$-integrable function σ on $\mathcal{B}$ determine a Radon measure solution (ρ, U, H) to Problem* (A), *provides that*

(i) $\rho : (\mathcal{G}, \mathcal{F}) \to \mathbb{R}_+ \cup \{0\}$ *is a nonnegative Radon measure, and the vector field* U*, the function* H *are measurable with respect to* ρ*. Furthermore,* m, m_1, m_2, m_3, m_4 *are all absolutely continuous with respect to* ρ*, with the Radon-Nikodym derivatives satisfying that*

$$U \doteq (u, v, w) = \frac{\mathrm{d}m}{\mathrm{d}\rho}, \quad UH = \frac{\mathrm{d}m_4}{\mathrm{d}\rho}, \tag{2.43}$$

$$Uu = \frac{\mathrm{d}m_1}{\mathrm{d}\rho}, \quad Uv = \frac{\mathrm{d}m_2}{\mathrm{d}\rho}, \quad Uw = \frac{\mathrm{d}m_3}{\mathrm{d}\rho}; \tag{2.44}$$

(ii) *for any test function* $\phi \in C_c^1(\mathbb{R}^3)$*, there hold*

$$\langle m, \nabla\phi\rangle + \int_{\mathcal{I}} \rho_0 u_0 \phi \,\mathrm{d}\mathcal{L}^2 = 0, \tag{2.45}$$

$$\langle m_1, \nabla\phi\rangle + \int_{\mathcal{I}} \rho_0 u_0^2 \phi \,\mathrm{d}\mathcal{L}^2 = \Big\langle \sigma \frac{n_1}{|n|}\delta_{\mathcal{B}}, \phi \Big\rangle, \tag{2.46}$$

$$\langle m_2, \nabla\phi\rangle = \Big\langle \sigma \frac{n_2}{|n|}\delta_{\mathcal{B}}, \phi \Big\rangle, \tag{2.47}$$

$$\langle m_3, \nabla\phi\rangle = \Big\langle \sigma \frac{n_3}{|n|}\delta_{\mathcal{B}}, \phi \Big\rangle, \tag{2.48}$$

$$\langle m_4, \nabla\phi\rangle + \int_{\mathcal{I}} \rho_0 u_0 H_0 \phi \,\mathrm{d}\mathcal{L}^2 = 0. \tag{2.49}$$

We observe that σ is a pressure distribution on $\mathcal{B}$ which measures the impact of particles hitting the obstacle, and

$$\mathbf{F} \doteq \int_{\mathcal{B}} \sigma \frac{n}{|n|} \,\mathrm{d}\mathcal{H}^2 \tag{2.50}$$

is the force of lift/drag acting on the obstacle $\mathcal{O}$ in the limiting hypersonic flow, that is of fundamental importance in aerodynamics. We will present more explicit expressions of σ below, see (2.61). It is straightforward to see that, by (2.53)–(2.56) and the divergence theorem, one could calculate (2.50) by simply measuring the fluxes along $\partial\mathcal{B}$.

2.3 The Euler equations of concentration layers and generalized Newton-Busemann law

By (2.43)–(2.44), we suppose that

$$\begin{aligned}
\rho &= \varrho_0 \mathcal{L}^3 \lfloor(\mathcal{G}) + \varrho\delta_{\mathcal{B}},\\
m &= \varrho_0 U_0 \mathcal{L}^3 \lfloor(\mathcal{G}) + \varrho\widetilde{U}\delta_{\mathcal{B}},\\
m_1 &= \varrho_0 u_0 U_0 \mathcal{L}^3 \lfloor(\mathcal{G}) + \varrho\widetilde{u}\widetilde{U}\delta_{\mathcal{B}},\\
m_2 &= \varrho_0 v_0 U_0 \mathcal{L}^3 \lfloor(\mathcal{G}) + \varrho\widetilde{v}\widetilde{U}\delta_{\mathcal{B}},\\
m_3 &= \varrho_0 w_0 U_0 \mathcal{L}^3 \lfloor(\mathcal{G}) + \varrho\widetilde{w}\widetilde{U}\delta_{\mathcal{B}},\\
m_4 &= \varrho_0 H_0 U_0 \mathcal{L}^3 \lfloor(\mathcal{G}) + \varrho h\widetilde{U}\delta_{\mathcal{B}},
\end{aligned} \tag{2.51}$$

where (see (2.24) and (2.10))

$$U_0 = (u_0, v_0, w_0) = (1, 0, 0), \quad \varrho_0 = 1, \quad \widetilde{U} = (\widetilde{u}, \widetilde{v}, \widetilde{w}) = \varpi = \mu r_x + \nu r_s,$$

and by (2.18),

$$\widetilde{u} = \varpi(x), \quad \widetilde{v} = \varpi(b_1), \quad \widetilde{w} = \varpi(b_2). \tag{2.52}$$

Direct applications of (2.41) to (2.45)–(2.49) yield

$$\operatorname{div}(\varrho\varpi) = \frac{n_1}{|n|}, \tag{2.53}$$

$$\operatorname{div}(\varrho\widetilde{u}\varpi) = \frac{n_1}{|n|}(1-\sigma), \tag{2.54}$$

$$\operatorname{div}(\varrho\widetilde{v}\varpi) = -\frac{n_2}{|n|}\sigma, \tag{2.55}$$

$$\operatorname{div}(\varrho\widetilde{w}\varpi) = -\frac{n_3}{|n|}\sigma, \tag{2.56}$$

$$\operatorname{div}(\varrho h\varpi) = \frac{n_1}{|n|}H_0. \tag{2.57}$$

We also remember (2.40), which implies now that $\varrho\mu(0,s) = 0$ (note that for the moment, $m_s = \varrho\widetilde{U} = \varrho\mu r_x + \varrho\nu r_s$). By the formulation of Problem (A), it is expected that μ is nonzero, i.e., particles hitting at the tip of the body $\mathcal{B}\cap\{x=0\}$ and then moving transversally. This means that

$$\varrho(0,s) = 0 \quad \text{on} \quad I. \tag{2.58}$$

So there is no concentration initially at the tip, which is natural. Furthermore, by the initial value $H = H_0$ in (2.24), and conservation of mass (2.53), (2.57) holds for $h \equiv H_0$. Once we solve from (2.53)–(2.56) the unknowns ϖ, ϱ and σ, we could get a Radon measure solution to Problem (A), with ρ given by (2.51), $H = H_0$, and $U = U_0\lfloor(\overline{\mathcal{G}}\setminus\mathcal{B}) + \varpi\lfloor(\mathcal{B})$. Here $f\lfloor(A)$ is the function obtained by restricting a function f to the set A.

In the following, we try to write (2.54)–(2.56) in an intrinsic way. Recalling the following formula valid on Riemannian manifolds

$$\operatorname{div}(\phi\varpi) = \phi\operatorname{div}\varpi + \varpi(\phi), \quad \forall\,\phi\in C^1,$$

one has, utilizing (2.54)–(2.56),

$$\begin{aligned}&\operatorname{div}\Big[\frac{n_1}{|n|}\varrho\widetilde{u}\varpi + \frac{n_2}{|n|}\varrho\widetilde{v}\varpi + \frac{n_3}{|n|}\varrho\widetilde{w}\varpi\Big]\\ &= \frac{n_1^2}{|n|^2}(1-\sigma) - \frac{n_2^2}{|n|^2}\sigma - \frac{n_3^2}{|n|^2}\sigma + \varrho\Big[\widetilde{u}\varpi\Big(\frac{n_1}{|n|}\Big) + \widetilde{v}\varpi\Big(\frac{n_2}{|n|}\Big) + \widetilde{w}\varpi\Big(\frac{n_3}{|n|}\Big)\Big].\end{aligned} \tag{2.59}$$

Since

$$\frac{n_1}{|n|}\widetilde{u} + \frac{n_2}{|n|}\widetilde{v} + \frac{n_3}{|n|}\widetilde{w} = 0, \tag{2.60}$$

it follows that, recalling (2.18),

$$\begin{aligned}\sigma &= \frac{n_1^2}{|n|^2} + \varrho\Big[\varpi(x)\varpi\Big(\frac{n_1}{|n|}\Big) + \varpi(b_1)\varpi\Big(\frac{n_2}{|n|}\Big) + \varpi(b_2)\varpi\Big(\frac{n_3}{|n|}\Big)\Big]\\ &= \frac{n_1^2}{|n|^2} - \frac{\varrho|n|}{|n|^2}(n_1\varpi^2(x) + n_2\varpi^2(b_1) + n_3\varpi^2(b_2)).\end{aligned} \tag{2.61}$$

Here $\varpi^2(x)$ means $\varpi(\varpi(x))$, etc. The second equality holds by acting ϖ on (2.60) and then applying Leibniz rule. The right-hand sides of (2.61) depend only on ϱ, ϖ and the second-order derivatives of b_1, b_2. Hence the surface is required to be C^2. We call this generalized Newton-Busemann pressure law, as from it we could derive the Newton's sine-squared law for straight wedge/cone and Busemann's pressure formula for curved wedge/cone in hypersonic flows.

On the contrary, if (2.61) holds, then by (2.60), we still have (2.59), which implies that as long as two equations in (2.54)–(2.56) hold, then the third also holds. For example, if $n_1 > 0$, we could solve the closed system (2.53), (2.55)–(2.56) and (2.61).

2.4 Examples

2.4.1 Ramps

We firstly consider the special case of a ramp: Taking $s = z$, $b_1 = b(x,z)$ and $b_2 = z$. Then

$$n = (\partial_x b, -1, \partial_z b), \quad |n| = \sqrt{1 + (\partial_x b)^2 + (\partial_z b)^2}.$$

In the local coordinates (x,z) of the surface $\mathcal{B}$, we could write (2.53)–(2.54) and (2.56) as

$$\partial_x(\vartheta\mu) + \partial_z(\vartheta\nu) = \partial_x b, \tag{2.62}$$

$$\partial_x(\vartheta\mu^2) + \partial_z(\vartheta\mu\nu) = (\partial_x b)(1-\sigma), \tag{2.63}$$

$$\partial_x(\vartheta\mu\nu) + \partial_z(\vartheta\nu^2) = -(\partial_z b)\sigma, \tag{2.64}$$

where $\vartheta \doteq |n|\varrho$, $\varpi = \mu\partial_x + \nu\partial_z$, and some straightforward computation shows that (2.61) is reduced to

$$\sigma = \frac{(\partial_x b)^2}{|n|^2} + \frac{\vartheta}{|n|^2}(\mu^2\partial_{xx}b + 2\mu\nu\partial_{xz}b + \nu^2\partial_{zz}b). \tag{2.65}$$

Note that by (2.7), ϑ represents the density of mass measured by the Lebesgue measure $\mathcal{L}^2$ on the (x,z)-plane: The mass of gas contained in an infinitesimal element $\mathrm{d}x\mathrm{d}z$ in the concentration layer is $\vartheta\mathrm{d}x\mathrm{d}z$. Thus (2.62)–(2.64) constitute a pressureless Euler system with source terms on a half Euclidean-plane $\{x \geq 0,\ z \in \mathbb{R}\}$.

For a curved ramp, namely, $b = b(x)$ with $b(0) = 0$, it is natural to assume that $\nu \equiv 0$, and (2.62)–(2.64) are reduced to the following ordinary differential equations (ODE for short)

$$\frac{\mathrm{d}}{\mathrm{d}x}(\vartheta\mu - b) = 0, \tag{2.66}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\vartheta\mu^2) = b'(x)\cdot(1-\sigma). \tag{2.67}$$

Then by (2.65),

$$\sigma = \frac{(b'(x))^2}{1+(b'(x))^2} + \frac{\vartheta\mu^2}{1+(b'(x))^2}b''(x), \tag{2.68}$$

and one solves (2.66)–(2.67) to get

$$\varrho = \varrho(x) = \frac{b(x)^2}{\int_0^x \frac{b'(\tau)}{\sqrt{1+(b'(\tau))^2}}\,\mathrm{d}\tau}, \tag{2.69}$$

$$\mu = \mu(x) = \frac{1}{b(x)\sqrt{1+(b'(x))^2}}\int_0^x \frac{b'(\tau)}{\sqrt{1+(b'(\tau))^2}}\,\mathrm{d}\tau, \tag{2.70}$$

$$\sigma = \sigma(x) = \frac{(b'(x))^2}{1+(b'(x))^2} + \frac{b''(x)}{(1+(b'(x))^2)^{\frac{3}{2}}} \int_0^x \frac{b'(\tau)}{\sqrt{1+(b'(\tau))^2}}\, \mathrm{d}\tau. \tag{2.71}$$

They are the same as those results obtained in [13, (2.20), (2.22), (2.18)], and (2.71) is the Newton-Busemann pressure law (see [1, (3.29) in p.71] and [13, Remark 1.7]). Particularly, for $b(x) = x\tan\theta_0$, we have $\sigma = \sin^2\theta_0$, namely the Newton's sine-squared law (see [9, (3.1.1) in p. 132]).

2.4.2 Cones

Next we consider the case of a conical body. We use the cylindrical coordinates of $\mathbb{R}^3$, with $\theta \in [0, 2\pi)$ the angle between a vector on the (y,z)-plane and the positive y-axis. Then $b_1 = R(x,\theta)\cos\theta$ and $b_2 = R(x,\theta)\sin\theta$, where $R(x,\theta)$ is a nonnegative C^2 function for $x \geq 0,\ \theta \in [0, 2\pi)$, with period 2π for the θ-variable. There are two particularly interesting cases:

(i) Cylindrically symmetric cone, namely, R depends only on x.

(ii) Self-similar conical flows, namely, $R(x,\theta) = \Theta(\theta)x$.

For a general conical body, the normal vector on $\mathcal{B}$ is

$$n = (n_1, n_2, n_3) = (RR_x, -(R_\theta\sin\theta + R\cos\theta), R_\theta\cos\theta - R\sin\theta), \tag{2.72}$$

thus

$$|n| = \sqrt{R^2 + R_\theta^2 + (RR_x)^2}. \tag{2.73}$$

We assume that

$$\partial_x R(x,\theta) > 0, \tag{2.74}$$

then $n_1 > 0$ if $R > 0$, and by (2.52),

$$\widetilde{v} = (\mu R_x + \nu R_\theta)\cos\theta - R\nu\sin\theta, \quad \widetilde{w} = (\mu R_x + \nu R_\theta)\sin\theta + R\nu\cos\theta.$$

Recalling that $\vartheta = |n|\varrho$, (2.53) and (2.55)–(2.56) are reduced to

$$\partial_x(\vartheta\mu) + \partial_\theta(\vartheta\nu) = n_1, \tag{2.75}$$

$$\partial_x(\vartheta\widetilde{v}\mu) + \partial_\theta(\vartheta\widetilde{v}\nu) = -n_2\sigma, \tag{2.76}$$

$$\partial_x(\vartheta\widetilde{w}\mu) + \partial_\theta(\vartheta\widetilde{w}\nu) = -n_3\sigma, \tag{2.77}$$

and (2.61) reads now

$$\begin{aligned}\sigma = {} & \frac{R^2R_x^2}{R^2(1+R_x^2)+R_\theta^2} + \frac{\vartheta}{R^2(1+R_x^2)+R_\theta^2} \\ & \times(-\mu^2RR_{xx} + 2\mu\nu(R_xR_\theta - RR_{\theta x}) + \nu^2(-RR_{\theta\theta} + 2R_\theta^2 + R^2)).\end{aligned} \tag{2.78}$$

For cylindrically symmetric cones, namely, $R = R(x)$ with $R(0) = 0$, we may also assume that $\nu \equiv 0$, and obtain from (2.75)–(2.77) the ODE

$$\Big(\vartheta\mu - \frac{1}{2}R^2\Big)' = 0 \quad \Big(\Rightarrow \vartheta\mu = \frac{1}{2}R(x)^2\Big), \tag{2.79}$$

$$(\vartheta\mu^2)' + \vartheta\mu^2\frac{R''}{R'}\Big(1+\frac{1}{1+R'^2}\Big) = \frac{RR'}{1+R'^2}, \tag{2.80}$$

together with

$$\sigma = \frac{RR'^2 - \vartheta\mu^2 R''}{R(1+R'^2)}. \tag{2.81}$$

Note that

$$\int\frac{\mathrm{d}t}{t(1+t^2)} = \ln\Big(\frac{t}{\sqrt{1+t^2}}\Big) + C,$$

we solve (2.80) to get

$$\vartheta\mu^2 = \frac{\sqrt{1+R'^2}}{R'^2}\int_0^x \frac{R(\tau)R'(\tau)^3}{(\sqrt{1+R'(\tau)^2})^3}\,\mathrm{d}\tau,$$

hence

$$\varrho = \frac{1}{4}\frac{R^3R'^2}{1+R'^2}\frac{1}{\int_0^x \frac{R(\tau)R'(\tau)^3}{(\sqrt{1+R'(\tau)^2})^3}\,\mathrm{d}\tau}, \tag{2.82}$$

$$\mu = \frac{2}{R^2}\frac{\sqrt{1+R'^2}}{R'^2}\int_0^x \frac{R(\tau)R'(\tau)^3}{(\sqrt{1+R'(\tau)^2})^3}\,\mathrm{d}\tau \tag{2.83}$$

and

$$\sigma = \frac{R'^2}{1+R'^2} - \frac{R''}{RR'^2\sqrt{1+R'^2}}\int_0^x \frac{R(\tau)R'(\tau)^3}{(\sqrt{1+R'(\tau)^2})^3}\,\mathrm{d}\tau. \tag{2.84}$$

Particularly, for the straight symmetric cone with zero attacking angle, namely, $R(x) = (\tan\theta_0)x$, where θ_0 is the half open-angle of the cone, we have the Newton's sine-squared law (see [1, (3.3) in p. 59])

$$\sigma = \sin^2\theta_0. \tag{2.85}$$

For self-similar conical flows, namely $R = \Theta(\theta)x$, from (2.78), we have

$$n_1 = \Theta^2 x, \quad n_2 = -x(\Theta'\sin\theta + \Theta\cos\theta), \quad n_3 = x(\Theta'\cos\theta - \Theta\sin\theta),$$
$$|n| = x\sqrt{\Theta^2(1+\Theta^2)+\Theta'^2}$$

and

$$\sigma = \frac{\Theta^4}{\Theta^2(1+\Theta^2)+\Theta'^2} + \frac{\vartheta\nu^2}{\Theta^2(1+\Theta^2)+\Theta'^2}(\Theta^2 + 2\Theta'^2 - \Theta\Theta''). \tag{2.86}$$

The first term does not depend on x. It is reduced to (2.85) if $\Theta = \tan\theta_0$ and (then assuming naturally that) $\nu \equiv 0$ (no swirl). So generally the second term shall also be independent of x. This motivates us to suppose that

$$\mu = \widehat{\mu}(\theta), \quad \nu = \frac{1}{x}\widehat{\nu}(\theta), \quad \vartheta = x^2\widehat{\vartheta}(\theta). \tag{2.87}$$

Then

$$\widetilde{v} \doteq \widehat{v} = (\widehat{\mu}\Theta + \widehat{\nu}\Theta')\cos\theta - \Theta\widehat{\nu}\sin\theta,$$

$$\widetilde{w} \doteq \widehat{w} = (\widehat{\mu}\Theta + \widehat{\nu}\Theta')\sin\theta + \Theta\widehat{\nu}\cos\theta.$$

It follows that (2.75)–(2.77) read now

$$(\widehat{\nu}\widehat{\vartheta})' + 2\widehat{\mu}\widehat{\vartheta} = \Theta^2, \tag{2.88}$$

$$(\widehat{\nu}\widehat{\vartheta}\widehat{v})' + 2\widehat{\vartheta}\widehat{\mu}\widehat{v} = \sigma(\Theta'\sin\theta + \Theta\cos\theta), \tag{2.89}$$

$$(\widehat{\nu}\widehat{\vartheta}\widehat{w})' + 2\widehat{\vartheta}\widehat{\mu}\widehat{w} = -\sigma(\Theta'\cos\theta - \Theta\sin\theta). \tag{2.90}$$

Hence "$(2.89)\sin\theta - (2.90)\cos\theta$" implies that

$$(\widehat{\vartheta}\widehat{\nu}^2\Theta)' + 3\Theta\widehat{\vartheta}\widehat{\mu}\widehat{\nu} + \widehat{\vartheta}\widehat{\nu}^2\Theta' = -\sigma\Theta'. \tag{2.91}$$

Similarly "$(2.89)\cos\theta + (2.90)\sin\theta$" yields

$$[\widehat{\nu}\widehat{\vartheta}(\widehat{\mu}\Theta + \widehat{\nu}\Theta')]' + 2\widehat{\vartheta}\widehat{\mu}(\widehat{\mu}\Theta + \widehat{\nu}\Theta') - \widehat{\vartheta}\widehat{\nu}^2\Theta = \sigma\Theta. \tag{2.92}$$

Thus (2.88) and (2.91)–(2.92) constitute the ODE to determine the self-similar conical flows, see [17, (3.36)–(3.38), p. 513].

3 Conclusions and Discussions

For a three-dimensional obstacle $\mathcal{O}$ moving uniformly in hypersonic limit flows with constant density and total enthalpy, a concentration layer appears on the upwind boundary surface $\mathcal{B}$ where $n_1 > 0$. The distribution of density and velocity of the flow in the concentration layer are governed by a first-order hyperbolic system of balance laws, which resembles with the pressureless compressible Euler equations defined on a given curved surface (i.e., the upwind boundary of the obstacle), but there are source terms reflecting the fact that particles in the hypersonic flow hitting the obstacle everywhere on its upwind boundary, leading to changes of mass and momentum in the concentration layer. A formula (2.61) for distribution of pressure on the obstacle is presented, which in general depends on the state of the flow in concentration layer. For wedges or cones, we could derive as special cases the celebrated Newton's sine-squared law or Newton-Busemann law of lift/drag in hypersonic aerodynamics. This demonstrates the approach we took, namely studying Radon measure solutions of compressible Euler equations to incorporate different scales in the hypersonic flow fields, is not only mathematically interesting, but also physically significant.

As readers might notice, these ideas and results provoke more questions than what we answered. It is interesting to study classical or weak solutions of (2.62)–(2.65), under suitable small perturbations of the background surface (such as a straight wedge), or even construct Radon measure solution to this system which models concentration of mass on lower dimensional set, such as delta shocks. There are many interesting discussions and conjectures about this aspect in [9, Sections 5–6 in Chapter III]. Initial data $\varrho(0, s) = 0$ means that the 'vacuum' is initial data, which leads to a singular Cauchy problem. Efficient numerical methods would also be useful for applications.

The PDE for free concentration layers, and the case the upcoming flow is not uniform, are all necessary for further research. Just for the two-dimensional unsteady compressible Euler equations, there are lots of interesting untouched problems, such as pressureless jet interacting with supersonic polytropic gas, and studying its multidimensional stability, etc. We wish there could be more works in this direction of research in the near future.

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