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Stability of Rotation Relations of Three Unitaries with the Flip Action in C^*-Algebras*

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Abstract The authors show that if $\Theta = (\theta_{jk})$ is a 3×3 totally irrational real skew-symmetric matrix, where $\theta_{jk} \in [0, 1)$ for $j, k = 1, 2, 3$, then for any $\varepsilon > 0$, there exists $\delta > 0$ satisfying the following: For any unital C^*-algebra A with the cancellation property, strict comparison and nonempty tracial state space, any four unitaries $u_1, u_2, u_3, w \in A$ such that (1) $\|u_k u_j - \mathrm{e}^{2\pi \mathrm{i}\theta_{jk}} u_j u_k\| < \delta$, $w u_j w^{-1} = u_j^{-1}$, $w^2 = 1_A$ for $j, k = 1, 2, 3$; (2) $\tau(aw) = 0$ and $\tau((u_k u_j u_k^* u_j^*)^n) = \mathrm{e}^{2\pi \mathrm{i} n\theta_{jk}}$ for all $n \in \mathbb{N}$, all $a \in C^*(u_1, u_2, u_3)$, $j, k = 1, 2, 3$ and all tracial states τ on A, where $C^*(u_1, u_2, u_3)$ is the C^*-subalgebra generated by u_1, u_2 and u_3, there exists a 4-tuple of unitaries $\widetilde{u}_1, \widetilde{u}_2, \widetilde{u}_3, \widetilde{w}$ in A such that

$$\widetilde{u}_k \widetilde{u}_j = \mathrm{e}^{2\pi \mathrm{i}\theta_{jk}} \widetilde{u}_j \widetilde{u}_k, \quad \widetilde{w} \widetilde{u}_j \widetilde{w}^{-1} = \widetilde{u}_j^{-1}, \quad \widetilde{w}^2 = 1_A$$

and

$$\|u_j - \widetilde{u}_j\| < \varepsilon, \quad \|w - \widetilde{w}\| < \varepsilon$$

for $j, k = 1, 2, 3$. The above conclusion is also called that the rotation relations of three unitaries with the flip action is stable under the above conditions.

Keywords C^*-Algebras, Stability, Rotation relation, Unitary, Flip action
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1 Introduction

The notion of stability appears in many forms throughout mathematics. Following Hyers and Ulam (see [29]), a general sense of this notion can be expressed as follows: Are elements that "almost" satisfy some equations "close" to some elements that exactly satisfy the equations?

An example of a concrete stability problem is the following.

For a given $\varepsilon > 0$, is there a $\delta > 0$, depending only on ε, such that if a and b are two $n \times n$ self-adjoint matrices with $\|a\|, \|b\| \le 1$ satisfying

$$\|ab - ba\| < \delta,$$

then there exists a pair of self-adjoint matrices $\widetilde{a}$ and $\widetilde{b}$ in M_n such that

$$\widetilde{a}\widetilde{b} = \widetilde{b}\widetilde{a}, \quad \|a - \widetilde{a}\| < \varepsilon \quad \text{and} \quad \|b - \widetilde{b}\| < \varepsilon?$$

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This is an old and famous question in matrix and operator theory (see [2, 9, 39]), which popularized by Halmos (see [22]). In the 1990's Lin affirmatively solved this question (see [20, 31]). The corresponding questions for a pair of unitary matrices and for a triple of self-adjoint matrices are all false, as pointed out by Voiculescu [40–41]. However the story does not end here. An obstruction has been found by Exel and Loring [11] in the corresponding question for a pair of unitary matrices. The answer becomes yes if this obstruction vanishes. See also [11–13, 15–16].

A natural generalization for pairs of almost commuting unitary matrices is to see what happens for pairs of unitaries that almost commute up to a scalar with norm 1. It turns out that similar conclusion holds, and in fact one can deal with more general ambient C^*-algebras rather than just matrix algebras. More precisely, in [26] the third-named author and Lin proved the following.

Theorem 1.1 (see [26]) *Let θ be a real number in $(-\frac{1}{2}, \frac{1}{2})$. Then, for any $\varepsilon > 0$, there is a $\delta > 0$, depending only on ε and θ, such that if u and v are two unitaries in any unital simple separable C^*-algebra A with tracial rank zero satisfying*

$$\begin{aligned} &\|uv - \mathrm{e}^{2\pi \mathrm{i}\theta} vu\| < \delta \quad \textit{and} \\ &\tau(\log(uvu^*v^*)) = 2\pi \mathrm{i}\theta \end{aligned} \tag{1.1}$$

for all tracial state τ on A, then there exists a pair of unitaries $\widetilde{u}$ and $\widetilde{v}$ in A such that

$$\widetilde{u}\widetilde{v} = \mathrm{e}^{2\pi \mathrm{i}\theta}\widetilde{v}\widetilde{u}, \quad \|u - \widetilde{u}\| < \varepsilon \quad \text{and} \quad \|v - \widetilde{v}\| < \varepsilon.$$

Note that the trace condition (1.1) is also necessary.

Let $\theta \in \mathbb{R}$. We call a pair of unitaries $\mathfrak{u}, \mathfrak{v}$ with $\mathfrak{u}\mathfrak{v} = \mathrm{e}^{2\pi \mathrm{i}\theta}\mathfrak{v}\mathfrak{u}$ to satisfy the rotation relation with respect to θ, since the universal C^*-algebra generated by such unitaries is the rotation algebra A_θ. So another way to phrase Theorem 1.1 is to say that the rotation relation is stable in unital simple separable C^*-algebras with tracial rank zero, providing that the trace condition (1.1) is satisfied.

In [27], the third-named author and Wang further studied the stability of the rotation relations of three unitaries and proved the following theorem.

Theorem 1.2 (see [27]) *Let $\Theta = (\theta_{jk})_{3\times 3}$ be a non-degenerate real skew-symmetric matrix (here non-degeneracy is equivalent to $\dim_{\mathbb{Q}}(\mathrm{span}_{\mathbb{Q}}(1, \theta_{12}, \theta_{13}, \theta_{23})) \geq 3$, see [1, Lemma 3.1]), where $\theta_{jk} \in [0, 1)$ for $j, k = 1, 2, 3$. Then, for any $\varepsilon > 0$, there exists $\delta > 0$ satisfying the following: For any unital simple separable C^*-algebra A with tracial rank at most one, any three unitaries $u_1, u_2, u_3 \in A$ such that*

$$\|u_k u_j - \mathrm{e}^{2\pi \mathrm{i}\,\theta_{jk}} u_j u_k\| < \delta, \quad j, k = 1, 2, 3,$$

there exists a triple of unitaries $\widetilde{u}_1, \widetilde{u}_2, \widetilde{u}_3 \in A$ such that

$$\widetilde{u}_k\widetilde{u}_j = \mathrm{e}^{2\pi \mathrm{i}\,\theta_{jk}}\widetilde{u}_j\widetilde{u}_k \quad \textit{and} \quad \|\widetilde{u}_j - u_j\| < \varepsilon, \quad j, k = 1, 2, 3$$

if and only if

$$\tau(\log_{\theta_{jk}}(u_k u_j u_k^* u_j^*)) = 2\pi \mathrm{i}\theta_{jk} \quad \textit{for } j, k = 1, 2, 3 \quad \textit{and all tracial state } \tau \textit{ on } A,$$

where $\log_{\theta_{jk}}$ is defined as in Definition 3.4 of the present paper.

In [23], the third-named author extended Theorem 1.2 to the stability of more general relations of three unitaries in any unital simple separable C^*-algebra with tracial rank at most one.

Let $\alpha : \mathfrak{u} \to \mathfrak{u}^{-1}, \mathfrak{v} \to \mathfrak{v}^{-1}$ be the flip automorphism on A_θ. Notice that $A_\theta \rtimes_\alpha \mathbb{Z}_2$ is the universal C^*-algebra generated by a triple of unitaries $\mathfrak{u}, \mathfrak{v}$ and $\mathfrak{w}$ satisfying

$$\mathfrak{u}\mathfrak{v} = e^{2\pi i\theta}\mathfrak{v}\mathfrak{u}, \quad \mathfrak{w}\mathfrak{u}\mathfrak{w}^* = \mathfrak{u}^*, \quad \mathfrak{w}\mathfrak{v}\mathfrak{w}^* = \mathfrak{v}^* \quad \text{and} \quad \mathfrak{w}^2 = 1.$$

The crossed products $A_\theta \rtimes_\alpha \mathbb{Z}_2$ which are considered as the noncommutative spheres have been studied. See, for example, [3–5, 10, 18–19, 30, 42]. So in [24], the author studied the stability of rotation relation of two unitaries with the flip action. More precisely, the third-named author proved the following theorem.

Theorem 1.3 (see [24]) *Let $\theta \in (0,1)$ be an irrational number. Then, for any $\varepsilon > 0$, there exists $\delta > 0$ satisfying the following: For any unital C^*-algebra A with the cancellation property, strict comparison and nonempty tracial state space, any three unitaries $u, v, w \in A$ such that*

(1) $\|uv - e^{2\pi i\theta}vu\| < \delta$, $wuw^{-1} = u^{-1}$, $wvw^{-1} = v^{-1}$, $w^2 = 1_A$,

(2) $\tau(aw) = 0$ and $\tau((uvu^*v^*)^n) = e^{2\pi i n\theta}$ for all $n \in \mathbb{N}$, all $a \in C^*(u,v)$ and all tracial state τ on A, where $C^*(u,v)$ is the C^*-subalgebra generated by u and v,
there exists a triple of unitaries $\widetilde{u}, \widetilde{v}, \widetilde{w} \in A$ such that

$$\widetilde{u}\widetilde{v} = e^{2\pi i\theta}\widetilde{v}\widetilde{u}, \quad \widetilde{w}^2 = 1_A, \quad \widetilde{w}\widetilde{u}\widetilde{w}^{-1} = \widetilde{u}^{-1}, \quad \widetilde{w}\widetilde{v}\widetilde{w}^{-1} = \widetilde{v}^{-1} \quad \textit{and}$$
$$\|u - \widetilde{u}\| < \varepsilon, \quad \|v - \widetilde{v}\| < \varepsilon, \quad \|w - \widetilde{w}\| < \varepsilon.$$

In Theorem 1.3, the C^*-algebra A does not need to be simple. In the meantime the cancellation property and strict comparison are more general conditions, many C^*-algebras have the cancellation property and strict comparison. For example, a unital simple C^*-algebra with tracial rank at most one as in Theorem 1.2 has the cancellation property and strict comparison (see [32]). Readers can refer to [6, 25, 28] for more stability problems.

Let $\Theta = (\theta_{jk})$ be 3×3 real skew-symmetric matrix. Let A_Θ be the universal C^*-algebra generated by unitaries $\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3$ subject to the relations

$$\mathfrak{u}_k\mathfrak{u}_j = e^{2\pi i\theta_{jk}}\mathfrak{u}_j\mathfrak{u}_k$$

for $j, k = 1, 2, 3$. Let $\alpha : \mathfrak{u}_j \to \mathfrak{u}_j^{-1}$, $j = 1, 2, 3$ be the flip automorphism on A_Θ (it is worth mentioning that the only canonical action by a nontrivial finite cyclic group on a simple 3-dimensional torus A_Θ is the flip action by $\mathbb{Z}_2$ by [21, Theorem 1.4]). Notice that $A_\Theta \rtimes_\alpha \mathbb{Z}_2$ is the universal C^*-algebra generated by a 4-tuple of unitaries $\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3$ and $\mathfrak{w}$ satisfying

$$\mathfrak{u}_k\mathfrak{u}_j = e^{2\pi i\theta_{jk}}\mathfrak{u}_j\mathfrak{u}_k, \quad \mathfrak{w}\mathfrak{u}_j\mathfrak{w}^* = \mathfrak{u}_j^* \quad \text{and} \quad \mathfrak{w}^2 = 1 \quad \textit{for} \quad j, k = 1, 2, 3.$$

The crossed products $A_\Theta \rtimes_\alpha \mathbb{Z}_2$ have been studied by many researchers. See, for example, [7, 10]. So in this paper, we will study the stability of rotation relations of three unitaries with the flip action. Specifically, we prove the following theorem.

Theorem 1.4 *Let $\Theta = (\theta_{jk})$ be a 3×3 totally irrational real skew-symmetric matrix, where $\theta_{jk} \in [0,1)$ for $j, k = 1, 2, 3$. Then, for any $\varepsilon > 0$, there exists $\delta > 0$ satisfying the following:*

For any unital C^*-algebra A with the cancellation property, strict comparison and nonempty tracial state space, any four unitaries $u_1, u_2, u_3, w \in A$ such that

(1) $\|u_k u_j - \mathrm{e}^{2\pi\mathrm{i}\theta_{jk}} u_j u_k\| < \delta$, $w u_j w^{-1} = u_j^{-1}$, $w^2 = 1_A$ *for* $j, k = 1, 2, 3$,

(2) $\tau(aw) = 0$ *and* $\tau((u_k u_j u_k^* u_j^*)^n) = \mathrm{e}^{2\pi\mathrm{i}n\theta_{jk}}$ *for all* $n \in \mathbb{N}$, *all* $a \in C^*(u_1, u_2, u_3)$, $j, k = 1, 2, 3$ *and all tracial state* τ *on* A, *where* $C^*(u_1, u_2, u_3)$ *is the* C^**-subalgebra generated by* u_1, u_2 *and* u_3, *there exists a 4-tuple of unitaries* $\widetilde{u}_1, \widetilde{u}_2, \widetilde{u}_3, \widetilde{w} \in A$ *such that*

$$\widetilde{u}_k \widetilde{u}_j = \mathrm{e}^{2\pi\mathrm{i}\theta_{jk}} \widetilde{u}_j \widetilde{u}_k, \quad \widetilde{w}\widetilde{u}_j\widetilde{w}^{-1} = \widetilde{u}_j^{-1}, \quad \widetilde{w}^2 = 1_A \quad \text{and} \quad \|u_j - \widetilde{u}_j\| < \varepsilon, \quad \|w - \widetilde{w}\| < \varepsilon$$

for $j, k = 1, 2, 3$.

The above theorem can be regarded as a generalization of Theorems 1.2–1.3.

This paper is organized as follows. In Section 2, we list some notations and known results. In Section 3, by using the flip invariant projections on A_Θ, we obtain four projections in the crossed product C^*-algebra $A_\Theta \rtimes_\alpha \mathbb{Z}_2$, which are four of the twelve generators of K_0-group of $A_\Theta \rtimes_\alpha \mathbb{Z}_2$. In Section 4, we consider some other projections which are other generators of K_0-group of $A_\Theta \rtimes_\alpha \mathbb{Z}_2$. In the last section, we prove our main results by using the existence theorem and the uniqueness theorem in the theory of C^*-algebras classification.

2 Preliminaries

In this section, we will give some symbols and definitions to be used later.

Let $n \geq 2$ be an integer and $\mathcal{T}_n$ denote the space of $n \times n$ real skew-symmetric matrices.

Definition 2.1 (see [37]) *Let $\Theta = (\theta_{jk})_{n\times n} \in \mathcal{T}_n$. The noncommutative torus A_Θ is the universal C^*-algebra generated by unitaries $\mathfrak{u}_1, \mathfrak{u}_2, \cdots, \mathfrak{u}_n$ subject to the relations*

$$\mathfrak{u}_k \mathfrak{u}_j = \mathrm{e}^{2\pi\mathrm{i}\theta_{jk}} \mathfrak{u}_j \mathfrak{u}_k$$

for $1 \leq j, k \leq n$. (*Of course, if all θ_{jk} are integers, it is not really noncommutative.*) *Throughout this paper, we will use $\mathfrak{u}_1, \mathfrak{u}_2, \cdots, \mathfrak{u}_n$ to represent the n generators of A_Θ, sometimes without special emphasis. In particular, given $\theta \in \mathbb{R}$, we also let A_θ denote the universal C^*-algebra generated by a pair of unitaries $\mathfrak{u}$ and $\mathfrak{v}$ subject to $\mathfrak{u}\mathfrak{v} = \mathrm{e}^{2\pi\mathrm{i}\theta}\mathfrak{v}\mathfrak{u}$.*

For any $\Theta = (\theta_{jk})_{n\times n}$ in $\mathcal{T}_n$, A_Θ has a canonical tracial state τ_Θ given by the integration over the canonical action of $\widehat{\mathbb{Z}^n}$ (see [38, page 4] for more details). We denote this trace by τ_Θ or τ_{A_Θ}.

Definition 2.2 *A skew symmetric real $n \times n$ matrix Θ is nondegenerate if whenever $x \in \mathbb{Z}^n$ satisfies $\mathrm{e}^{2\pi\mathrm{i}\langle x, \Theta y\rangle} = 1$ for all $y \in \mathbb{Z}^n$, then $x = 0$. Otherwise, we say Θ is degenerate.*

The following theorem shows the structure and the K-theory of A_Θ when Θ is nondegenerate.

Theorem 2.1 (see [33]) *Let Θ be in $\mathcal{T}_n$ with $n \geq 2$. The C^*-algebra A_Θ is simple if and only if Θ is nondegenerate. Moreover, if A_Θ is simple, then it is a unital $A\mathbb{T}$ algebra and has the unique tracial state τ_Θ, and $K_0(A_\Theta) \cong K_1(A_\Theta) = \mathbb{Z}^{2^{n-1}}$.*

Definition 2.3 *Let $\Theta = (\theta_{jk}) \in \mathcal{T}_n$. We say Θ is totally irrational if θ_{jk}, $1 \leq j < k \leq n$ are irrational and rational independent.*

Note that if Θ is totally irrational, then Θ is nondegenerate.

Definition 2.4 (see [34]) *Let $\theta \in (0,1)$. Choose ε such that $0 < \varepsilon \le \theta < \theta + \varepsilon \le 1$. Set*

$$f(\mathrm{e}^{2\pi \mathrm{i} t}) = \begin{cases} \varepsilon^{-1} t, & 0 \le t \le \varepsilon, \\ 1, & \varepsilon \le t \le \theta, \\ \varepsilon^{-1}(\theta + \varepsilon - t), & \theta \le t \le \theta + \varepsilon, \\ 0, & \theta + \varepsilon \le t \le 1 \end{cases}$$

and

$$g(\mathrm{e}^{2\pi \mathrm{i} t}) = \begin{cases} 0, & 0 \le t \le \theta, \\ [f(\mathrm{e}^{2\pi \mathrm{i} t})(1 - f(\mathrm{e}^{2\pi \mathrm{i} t}))]^{\frac{1}{2}}, & \theta \le t \le \theta + \varepsilon, \\ 0, & \theta + \varepsilon \le t \le 1. \end{cases}$$

Then f and g are the real-valued functions on the circle which satisfy

(1) $g(\mathrm{e}^{2\pi \mathrm{i} t}) \cdot g(\mathrm{e}^{2\pi \mathrm{i}(t-\theta)}) = 0$,

(2) $g(\mathrm{e}^{2\pi \mathrm{i} t}) \cdot [f(\mathrm{e}^{2\pi \mathrm{i} t}) + f(\mathrm{e}^{2\pi \mathrm{i}(t+\theta)})] = g(\mathrm{e}^{2\pi \mathrm{i} t})$ *and*

(3) $f(\mathrm{e}^{2\pi \mathrm{i} t}) = [f(\mathrm{e}^{2\pi \mathrm{i} t})]^2 + [g(\mathrm{e}^{2\pi \mathrm{i} t})]^2 + [g(\mathrm{e}^{2\pi \mathrm{i}(t-\theta)})]^2$.

Let $\mathfrak{u}$, $\mathfrak{v}$ be the canonical generators of A_θ. The Rieffel projection in A_θ is the projection

$$p = g(\mathfrak{u})\mathfrak{v}^* + f(\mathfrak{u}) + \mathfrak{v} g(\mathfrak{u}).$$

Theorem 2.2 and Proposition 2.1 below are surely well known.

Theorem 2.2 *Let $\theta \in [0,1)$. Let $C(\mathbb{T})$ denote the C^*-algebra of all continuous complex functions on the circle and let z be the identity function of $C(\mathbb{T})$. Let $\sigma\colon \mathbb{Z} \curvearrowright C(\mathbb{T})$ be the action determined by $\sigma(z) = \mathrm{e}^{2\pi \mathrm{i}\theta} z$. Then A_θ is naturally isomorphic to the crossed product $C(\mathbb{T}) \rtimes_\sigma \mathbb{Z}$. Moreover, there is a short exact sequence induced from the Pimsner-Voiculsecu six-term exact sequence from this crossed product*:

$$0 \to K_0(C(\mathbb{T})) \xrightarrow{i_{*0}} K_0(A_\theta) \xrightarrow{\partial} K_1(C(\mathbb{T})) \to 0.$$

Definition 2.5 (see [27, Definition 3.2]) *Let $\theta \in [0,1)$. Let $\mathfrak{u}$, $\mathfrak{v}$ be the canonical generators of A_θ. If $\theta \ne 0$, we define $b_{\mathfrak{u},\mathfrak{v}} \in K_0(A_\theta)$ to be the equivalent class of the Rieffel projection as constructed in Definition* 2.4. *If $\theta = 0$, we let $b_{\mathfrak{u},\mathfrak{v}} \in K_0(A_\theta)$ be the bott element* (*see* [26, *Definition* 2.7]).

Proposition 2.1 *Let $\theta \in [0,1)$ and τ_{A_θ} be the canonical tracial state on A_θ. Let $b_{\mathfrak{u},\mathfrak{v}} \in K_0(A_\theta)$ be defined as in Definition* 2.5. *Then $\tau_{A_\theta}(b_{\mathfrak{u},\mathfrak{v}}) = \theta$. Moreover, if $\partial\colon K_0(A_\theta) \to K_1(C(\mathbb{T}))$ is the homomorphism defined as in Theorem* 2.2, *then $\partial(b_{\mathfrak{u},\mathfrak{v}}) = [z]$, where z is the identity function of $C(\mathbb{T})$.*

Definition 2.6 (see [17]) *For $0 \le \varepsilon \le 2$ and $\theta \in [0,1)$, the soft rotation algebras $\mathcal{S}_{\varepsilon,\theta}$ is defined to be the universal C^*-algebra generated by a pair of unitaries $\mathfrak{u}_{\varepsilon,\theta}$ and $\mathfrak{v}_{\varepsilon,\theta}$ subject to $\|\mathfrak{u}_{\varepsilon,\theta}\mathfrak{v}_{\varepsilon,\theta} - \mathrm{e}^{2\pi \mathrm{i}\theta}\mathfrak{v}_{\varepsilon,\theta}\mathfrak{u}_{\varepsilon,\theta}\| \le \varepsilon$. In particular, we have $\mathcal{S}_{0,\theta} = A_\theta$.*

Definition 2.7 *For $0 \le \varepsilon \le 2$ and $\theta \in [0,1)$, let $B_{\varepsilon,\theta}$ be the universal C^*-algebra generated by unitaries $x_n, n \in \mathbb{Z}$, subject to the relations $\|x_{n+1} - \mathrm{e}^{2\pi \mathrm{i}\theta} x_n\| \le \varepsilon$. Let $\sigma_{\varepsilon,\theta}$ be the automorphism of $B_{\varepsilon,\theta}$ specified by $\sigma_{\varepsilon,\theta}(x_n) = x_{n+1}$.*

Proposition 2.2 (see [17, Proposition 2.2]) *For $0<\varepsilon<2$, $B_{\varepsilon,\theta}\rtimes_{\sigma_{\varepsilon,\theta}}\mathbb{Z}$ is isomorphic to $\mathcal{S}_{\varepsilon,\theta}$.*

Theorem 2.3 (see [17, Theorem 2.3], [14, Theorem 2.4]) *Assume $0\le\varepsilon<2$. Let z denote the canonical generator of C^*-algebra $C(\mathbb{T})$. Identify A_θ as the crossed product of $C(\mathbb{T})$ by the action σ of $\mathbb{Z}$ induced by $\sigma(z)=\mathrm{e}^{2\pi\mathrm{i}\theta}z$. Then*

(1) *Let $\psi_\varepsilon^\theta: B_{\varepsilon,\theta}\to C(\mathbb{T})$ be the unique homomorphism such that $\psi_\varepsilon^\theta(x_n)=\mathrm{e}^{2n\pi\mathrm{i}\theta}z$ for all $n\in\mathbb{Z}$. Then ψ_ε^θ induces a homotopy equivalence between $B_{\varepsilon,\theta}$ and $C(\mathbb{T})$.*

(2) *Let $\varphi_\varepsilon^\theta$ be the homomorphism defined by*

$$\varphi_\varepsilon^\theta:\mathcal{S}_{\varepsilon,\theta}\to\mathcal{S}_{0,\theta}=A_\theta,\quad \varphi_\varepsilon^\theta(\mathfrak{u}_{\varepsilon,\theta})=\mathfrak{u}_{0,\theta}=\mathfrak{u},\quad \varphi_\varepsilon^\theta(\mathfrak{v}_{\varepsilon,\theta})=\mathfrak{v}_{0,\theta}=\mathfrak{v}.$$

Then we have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & K_0(B_{\epsilon,\theta}) \arrow[r] \arrow[d, "\psi^\theta_{\epsilon*}"'] & K_0(\mathcal{S}_{\epsilon,\theta}) \arrow[r, "\partial"] \arrow[d, "\varphi^\theta_{\epsilon*}"'] & K_1(B_{\epsilon,\theta}) \arrow[r] \arrow[d, "\psi^\theta_{\epsilon*}"'] & 0, \\
0 \arrow[r] & K_0(C(\mathbb{T})) \arrow[r] & K_0(A_\theta) \arrow[r, "\partial"] & K_1(C(\mathbb{T})) \arrow[r] & 0,
\end{tikzcd}$$

where all vertical maps are isomorphisms and all rows are derived from the Pimsner-Voiclescu exact sequences.

Definition 2.8 *Let $\theta\in[0,1)$. Let $\mathfrak{u}$, $\mathfrak{v}$ be the canonical generators of A_θ. We define b_ϵ^θ to be the element in $K_0(\mathcal{S}_{\epsilon,\theta})$ given by $b_\epsilon^\theta=(\varphi^\theta_{\epsilon*})^{-1}(b_{\mathfrak{u},\mathfrak{v}})$, where $\varphi_\varepsilon^\theta$ is defined as in Theorem 2.3.*

It follows immediately from the definition that $\partial(b_\epsilon^\theta)=[x_0]$ in $K_1(B_{\epsilon,\theta})$.

Definition 2.9 *Let A be a unital C^*-algebra and let u and v be two unitaries in A such that $\|uv-\mathrm{e}^{2\pi\mathrm{i}\theta}vu\|\le\varepsilon<2$. There is a homomorphism $\phi^\theta_{u,v}:\mathcal{S}_{\varepsilon,\theta}\to A$ such that $\phi^\theta_{u,v}(\mathfrak{u}_{\varepsilon,\theta})=u$ and $\phi^\theta_{u,v}(\mathfrak{v}_{\varepsilon,\theta})=v$. We define $b^\theta_{u,v}=(\phi^\theta_{u,v})_{*0}(b^\theta_\varepsilon)$. Note that $b^\theta_{u,v}$ does not depend on ε as long as $\|uv-\mathrm{e}^{2\pi\mathrm{i}\theta}vu\|<\varepsilon<2$.*

Definition 2.10 *An AF-algebra is a C^*-algebra which is (isomorphic to) the inductive limit of a sequence of finite dimensional C^*-algebras.*

Given $\Theta\in\mathcal{T}_n$. Let α denote the flip automorphism on A_Θ satisfying $\alpha(\mathfrak{u}_j)=\mathfrak{u}_j^{-1}$ for $j=1,2,\cdots,n$. By abuse of notations, we still use α to denote the flip automorphism on the subalgebra $A_{\theta_{jk}}$ of A_Θ for $1\le j<k\le n$.

Next we recall that the structure of the crossed product algebra $A_\Theta\rtimes_\alpha\mathbb{Z}_2$.

Theorem 2.4 (see [10]) *Let $\Theta\in\mathcal{T}_n$ be nondegenerate. Let $\alpha:A_\Theta\to A_\Theta$ be the flip automorphism. Then $A_\Theta\rtimes_\alpha\mathbb{Z}_2$ is a unital simple AF-algebra with the unique tracial state, and*

$$K_0(A_\Theta\rtimes_\alpha\mathbb{Z}_2)\cong\mathbb{Z}^{3\cdot2^{n-1}},\quad K_1(A_\Theta\rtimes_\alpha\mathbb{Z}_2)=0.$$

Notation 2.1 Let A be a unital C^*-algebra. Denote by $T(A)$ the tracial state space of A. Denote by $\mathrm{Aff}(T(A))$ the space of all real affine continuous functions on $T(A)$. If $\tau\in T(A)$, we will use $\tau^{\oplus k}$ or even τ to denote the trace $\tau\otimes Tr$ on $M_k(A)$ for all integer $k\ge1$, where Tr is the unnormalized trace on the matrix algebra M_k. Denote by $\rho_A:K_0(A)\to\mathrm{Aff}(T(A))$ the order preserving map induced by $\rho_A([p])(\tau)=\tau^{\oplus n}(p)$ for all projections $p\in A\otimes M_n, n=1,2,\cdots$.

We define the dimension function d_τ associated to $\tau \in T(A)$ by $d_\tau(a) = \lim_{n\to\infty} \tau(a^{\frac{1}{n}})$ for any positive element $a \in M_k(A)$, where τ is regarded as an unnormalized trace on $M_k(A)$. In particular, if $a = p$ is a projection, then $d_\tau(p) = \tau(p)$.

Let $U(A)$ be the group of all unitary elements of A. Let $u \in A$ be a unitary, define $\mathrm{Ad}u(a) = u^*au$ for all $a \in A$. For any $a \in A$, denote by $\mathrm{spec}(a)$ the spectrum of a.

We always use $\chi_{(\frac{1}{2},+\infty)}$ to represent the characteristic function on $\left(\frac{1}{2}, +\infty\right)$.

Definition 2.11 *We say that a C^*-algebra A has strict comparison if for any two positive elements $a, b \in M_k(A)$ with $d_\tau(a) < d_\tau(b)$ for any $\tau \in T(A)$, there exist $r_n \in M_k(A), n \in \mathbb{N}$ such that $\lim_{n\to\infty} r_n^* b r_n = a$.*

Let A be a C^*-algebra with strict comparison and $p, q \in M_k(A)$ be two projections. If $\tau(p) < \tau(q)$ for all $\tau \in T(A)$, then there exists $s \in M_k(A)$ such that $p = s^*s$ and $ss^* \le q$.

Definition 2.12 *Let A and C be two C^*-algebras and let $A \to C$ be a linear map. Let $\delta > 0$ and $\mathcal{G} \subset A$ be a finite subset. We say L is $\mathcal{G}$-δ-multiplicative if*

$$\|L(ab) - L(a)L(b)\| < \delta \quad \text{for all } a, b \in \mathcal{G}.$$

For convenience, if $L\colon A \to C$ is a linear map, we will use the same symbol L to denote the induced map $L \otimes \mathrm{id}_n\colon A \otimes M_n \to C \otimes M_n$.

It is well known that if $a \in M_n(A)$ is an 'almost' projection, then it is norm close to a projection. Two norm close projections are unitarily equivalent. So $[a] \in K_0(A)$ is well-defined. If $L\colon A \to C$ is an 'almost' homomorphism, we shall use $[L]$ to denote the induced (partially defined) map on the K-theories. From [32, Remark 4.5.1], we can know that for any finite set $\mathcal{P} \subset K_0(A)$, there is a finite subset $\mathcal{G} \subset A$ and $\delta > 0$ such that, for any unital completely positive $\mathcal{G}$-δ-multiplicative linear map L, $[L]$ is well defined on $\mathcal{P}$.

3 From the Flip Invariant Projection to Some Obstacles to Stability

In this section, we first construct an α-invariant projection $p_\theta(\mathfrak{u}, \mathfrak{v})$ in A_θ for $\theta \in [\frac{1}{2}, 1)$, then by this α-invariant projection we can construct a projection $P_\theta(\mathfrak{u}, \mathfrak{v}, \mathfrak{w})$ in $A_\theta \rtimes_\alpha \mathbb{Z}_2$ which is one of the six generators of $K_0(A_\theta \rtimes_\alpha \mathbb{Z}_2)$ and $\tau_{A_\theta \rtimes_\alpha \mathbb{Z}_2}(P_\theta(\mathfrak{u}, \mathfrak{v}, \mathfrak{w})) = \frac{\theta}{2}$ for $\theta \in (0, 1)$, where $\tau_{A_\theta \rtimes_\alpha \mathbb{Z}_2}$ is the canonical tracial state on $A_\theta \rtimes_\alpha \mathbb{Z}_2$. Let $\Theta = (\theta_{jk}) \in \mathcal{T}_3$. Then we think of $A_{\theta_{jk}} \rtimes_\alpha \mathbb{Z}_2, 1 \le j < k \le 3$, as subalgebras of $A_\Theta \rtimes_\alpha \mathbb{Z}_2$ to obtain four projections in $A_\Theta \rtimes_\alpha \mathbb{Z}_2$ which are four of the twelve generators of $K_0(A_\Theta \rtimes_\alpha \mathbb{Z}_2)$. Finally, we obtain some obstacles to the stability of rotation relations of three unitaries with the flip action.

Next we begin with the construction of the α-invariant projection, which is taken from [42].

The functions f_θ and g_θ chosen here will be those constructed by Connes [8].

Set

$$f_\theta(\mathrm{e}^{2\pi \mathrm{i}t}) = \begin{cases} (1-\theta)^{-1}t, & 0 \le t \le 1-\theta, \\ 1, & 1-\theta \le t \le \theta, \\ (1-\theta)^{-1}(1-t), & \theta \le t \le 1 \end{cases}$$

and

$$g_\theta(\mathrm{e}^{2\pi \mathrm{i}t}) = \begin{cases} 0, & 0 \le t \le \theta, \\ [f_\theta(\mathrm{e}^{2\pi \mathrm{i}t})(1 - f_\theta(\mathrm{e}^{2\pi \mathrm{i}t}))]^{\frac{1}{2}}, & \theta \le t \le 1. \end{cases}$$

Then f_θ and g_θ are the real-valued continuous functions on the circle which satisfy

(1) $g_\theta(\mathrm{e}^{2\pi\mathrm{i}t}) \cdot g_\theta(\mathrm{e}^{2\pi\mathrm{i}(t-\theta)}) = 0$,
(2) $g_\theta(\mathrm{e}^{2\pi\mathrm{i}t}) \cdot [f_\theta(\mathrm{e}^{2\pi\mathrm{i}t}) + f_\theta(\mathrm{e}^{2\pi\mathrm{i}(t-\theta)})] = g_\theta(\mathrm{e}^{2\pi\mathrm{i}t})$ and
(3) $f_\theta(\mathrm{e}^{2\pi\mathrm{i}t}) = [f_\theta(\mathrm{e}^{2\pi\mathrm{i}t})]^2 + [g_\theta(\mathrm{e}^{2\pi\mathrm{i}t})]^2 + [g_\theta(\mathrm{e}^{2\pi\mathrm{i}(t+\theta)})]^2$.

Lemma 3.1 (see [42], [24, Lemma 3.1]) *Given $\theta \in [\frac{1}{2}, 1)$. Let $\mathfrak{u}, \mathfrak{v}$ and $\mathfrak{w}$ be the generators of $A_\theta \rtimes_\alpha \mathbb{Z}_2$ satisfying*

$$\mathfrak{u}\mathfrak{v} = \mathrm{e}^{2\pi\mathrm{i}\theta}\mathfrak{v}\mathfrak{u}, \quad \mathfrak{w}\mathfrak{u}\mathfrak{w}^* = \mathfrak{u}^*, \quad \mathfrak{w}\mathfrak{v}\mathfrak{w}^* = \mathfrak{v}^*, \quad \mathfrak{w}^2 = 1.$$

Then there exists a projection $p_\theta(\mathfrak{u}, \mathfrak{v}) = \mathfrak{u}g_\theta(\mathfrak{v}) + f_\theta(\mathfrak{v}) + g_\theta(\mathfrak{v})\mathfrak{u}^*$ in A_θ satisfying $\mathfrak{w}p_\theta(\mathfrak{u}, \mathfrak{v}) = p_\theta(\mathfrak{u}, \mathfrak{v})\mathfrak{w}$ and $\tau_{A_\theta}(p_\theta(\mathfrak{u}, \mathfrak{v})) = \theta$, where f_θ, g_θ are real-valued continuous functions on the circle and τ_{A_θ} is the canonical tracial state on A_θ.

For $\theta \in \left(0, \frac{1}{2}\right)$ and $\mathfrak{u}\mathfrak{v} = \mathrm{e}^{2\pi\mathrm{i}\theta}\mathfrak{v}\mathfrak{u}$, we have $1 - \theta \in (\frac{1}{2}, 1)$ and $\mathfrak{v}\mathfrak{u} = \mathrm{e}^{2\pi i(1-\theta)}\mathfrak{u}\mathfrak{v}$. By applying Lemma 3.1, $p_{1-\theta}(\mathfrak{v}, \mathfrak{u})$ is a projection in A_θ. So when $\theta \in (0, \frac{1}{2})$, we define $p_\theta(\mathfrak{u}, \mathfrak{v}) = 1 - p_{1-\theta}(\mathfrak{v}, \mathfrak{u})$. Then we also have $\tau_{A_\theta}(p_\theta(\mathfrak{u}, \mathfrak{v})) = \tau_{A_\theta}(1 - p_{1-\theta}(\mathfrak{v}, \mathfrak{u})) = 1 - (1 - \theta) = \theta$.

Remark 3.1 Let $\theta \in (0, 1)$ is an irrational number. So $(\tau_{A_\theta})_*$ is injective map from $K_0(A_\theta)$ to $\mathbb{R}$ by [36]. Note that

$$(\tau_{A_\theta})_*(b_{\mathfrak{u},\mathfrak{v}}) = (\tau_{A_\theta})_*([p_\theta(\mathfrak{u}, \mathfrak{v})]) = \theta,$$

we have $b_{\mathfrak{u},\mathfrak{v}} = [p_\theta(\mathfrak{u}, \mathfrak{v})]$ in $K_0(A_\theta)$.

Proposition 3.1 (see [24, Proposition 3.3]) *For $\theta \in [\frac{1}{2}, 1)$, let $P_\theta(\mathfrak{u}, \mathfrak{v}, \mathfrak{w}) = \frac{1}{2}p_\theta(\mathfrak{u}, \mathfrak{v}) + \frac{1}{2}p_\theta(\mathfrak{u}, \mathfrak{v})\mathfrak{w}$, where $\mathfrak{u}, \mathfrak{v}, \mathfrak{w}$ and $p_\theta(\mathfrak{u}, \mathfrak{v})$ are as in Lemma* 3.1*. Then $P_\theta(\mathfrak{u}, \mathfrak{v}, \mathfrak{w})$ is a projection in C^*-algebra $A_\theta \rtimes_\alpha \mathbb{Z}_2$.*

Proposition 3.2 (see [24, Proposition 3.4]) *For $\theta \in (0, \frac{1}{2})$, let $\mathfrak{u}, \mathfrak{v}$ and $\mathfrak{w}$ be the generators of $A_\theta \rtimes_\alpha \mathbb{Z}_2$ satisfying*

$$\mathfrak{u}\mathfrak{v} = \mathrm{e}^{2\pi\mathrm{i}\theta}\mathfrak{v}\mathfrak{u}, \quad \mathfrak{w}\mathfrak{u}\mathfrak{w}^* = \mathfrak{u}^*, \quad \mathfrak{w}\mathfrak{v}\mathfrak{w}^* = \mathfrak{v}^*, \quad \mathfrak{w}^2 = 1.$$

Let $P_\theta(\mathfrak{u}, \mathfrak{v}, \mathfrak{w}) = \frac{1}{2} + \frac{1}{2}\mathfrak{w} - P_{1-\theta}(\mathfrak{v}, \mathfrak{u}, \mathfrak{w})$, where $P_{1-\theta}(\mathfrak{v}, \mathfrak{u}, \mathfrak{w})$ is as in Proposition 3.1*. Then $P_\theta(\mathfrak{u}, \mathfrak{v}, \mathfrak{w})$ is also a projection in C^*-algebra $A_\theta \rtimes_\alpha \mathbb{Z}_2$.*

Let $\Theta = (\theta_{jk}) \in \mathcal{T}_3$. Notice that $A_\Theta \rtimes_\alpha \mathbb{Z}_2$ is the universal C^*-algebra generated by a 4-tuple of unitaries $\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3$ and $\mathfrak{w}$ satisfying

$$\mathfrak{u}_k\mathfrak{u}_j = \mathrm{e}^{2\pi\mathrm{i}\theta_{jk}}\mathfrak{u}_j\mathfrak{u}_k, \quad \mathfrak{w}\mathfrak{u}_j\mathfrak{w}^* = \mathfrak{u}_j^* \quad \text{and} \quad \mathfrak{w}^2 = 1 \quad \text{for } j, k = 1, 2, 3.$$

Now we think of $A_{\theta_{jk}} \rtimes_\alpha \mathbb{Z}_2, 1 \le j < k \le 3$, as subalgebras of $A_\Theta \rtimes_\alpha \mathbb{Z}_2$, and then we have the following theorem.

Theorem 3.1 *Let $\Theta = (\theta_{jk}) \in \mathcal{T}_3$ be totally irrational, where $\theta_{jk} \in [0, 1)$ for $j, k = 1, 2, 3$. Then $K_0(A_\Theta \rtimes_\alpha \mathbb{Z}_2)$ is isomorphic to $\mathbb{Z}^{12}$, which is generated by the K-theory classes of the elements*:

$$1, \quad Q_1 = \frac{1}{2}(1 + \mathfrak{w}), \quad Q_2 = \frac{1}{2}(1 - \mathfrak{u}_1\mathfrak{w}), \quad Q_3 = \frac{1}{2}(1 - \mathfrak{u}_2\mathfrak{w}), \quad Q_4 = \frac{1}{2}(1 - \mathrm{e}^{\pi\mathrm{i}\theta_{12}}\mathfrak{u}_1\mathfrak{u}_2\mathfrak{w}),$$
$$Q_5 = \frac{1}{2}(1 + \mathfrak{u}_3\mathfrak{w}), \quad Q_6 = \frac{1}{2}(1 - \mathrm{e}^{\pi\mathrm{i}\theta_{13}}\mathfrak{u}_1\mathfrak{u}_3\mathfrak{w}), \quad Q_7 = \frac{1}{2}(1 - \mathrm{e}^{\pi\mathrm{i}\theta_{23}}\mathfrak{u}_2\mathfrak{u}_3\mathfrak{w}),$$
$$P_{\theta_{12}}(\mathfrak{u}_2, \mathfrak{u}_1, \mathfrak{w}), \quad P_{\theta_{12}}(\mathrm{e}^{\pi\mathrm{i}\theta_{23}}\mathfrak{u}_2, \mathrm{e}^{\pi\mathrm{i}\theta_{13}}\mathfrak{u}_1, \mathfrak{u}_3\mathfrak{w}), \quad P_{\theta_{13}}(\mathfrak{u}_3, \mathfrak{u}_1, \mathfrak{w}), \quad P_{\theta_{23}}(\mathfrak{u}_3, \mathfrak{u}_2, \mathfrak{w}).$$

Moreover, $\tau_{A_\Theta \rtimes_\alpha \mathbb{Z}_2}$ *takes the following values* (*in this order*) *on these classes*:

$$1,\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{\theta_{12}}{2},\ \frac{\theta_{12}}{2},\ \frac{\theta_{13}}{2},\ \frac{\theta_{23}}{2}.$$

Proof By [7, Corollary 7.2], we know that

$$[1], [Q_1], [Q_2], [Q_3], [Q_4], [Q_5], [Q_6], [Q_7], [P_{\theta_{12}}(\mathfrak{u}_2, \mathfrak{u}_1, \mathfrak{w})], [P_{\theta_{12}}(\mathrm{e}^{\pi\mathrm{i}\theta_{23}}\mathfrak{u}_2, \mathrm{e}^{\pi\mathrm{i}\theta_{13}}\mathfrak{u}_1, \mathfrak{u}_3\mathfrak{w})],$$

$$[P_{\theta_{13}}(\mathfrak{u}_3, \mathfrak{u}_1, \mathfrak{w})] \quad \text{and} \quad [P_{\theta_{23}}(\mathfrak{u}_3, \mathfrak{u}_2, \mathfrak{w})]$$

are generators of $K_0(A_\Theta \rtimes_\alpha \mathbb{Z}_2)$. (The expression of the last four generators here is slightly different from that in [7], we can refer to [24, Theorem 3.6] for our expression.)

Note that $\tau_{A_\Theta \rtimes_\alpha \mathbb{Z}_2}$ is the canonical tracial state on $A_\Theta \rtimes_\alpha \mathbb{Z}_2$, we have

$$\tau_{A_\Theta \rtimes_\alpha \mathbb{Z}_2}(a\mathfrak{w}) = 0 \quad \text{for any } a \in C^*(\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3).$$

So

$$\tau_{A_\Theta \rtimes_\alpha \mathbb{Z}_2}(Q_j) = \frac{1}{2} \quad \text{for } j = 1, 2, \cdots, 7.$$

For $\theta_{12} \in \left[\frac{1}{2}, 1\right)$,

$$\begin{aligned}\tau_{A_\Theta \rtimes_\alpha \mathbb{Z}_2}(P_{\theta_{12}}(\mathfrak{u}_2, \mathfrak{u}_1, \mathfrak{w})) &= \tau_{A_\Theta \rtimes_\alpha \mathbb{Z}_2}\Big(\frac{1}{2}p_{\theta_{12}}(\mathfrak{u}_2, \mathfrak{u}_1) + \frac{1}{2}p_{\theta_{12}}(\mathfrak{u}_2, \mathfrak{u}_1)\mathfrak{w}\Big)\\ &= \tau_{A_\Theta \rtimes_\alpha \mathbb{Z}_2}\Big(\frac{1}{2}p_{\theta_{12}}(\mathfrak{u}_2, \mathfrak{u}_1)\Big)\\ &= \frac{\theta_{12}}{2}.\end{aligned} \tag{3.1}$$

For $\theta_{12} \in \left(0, \frac{1}{2}\right)$,

$$\begin{aligned}\tau_{A_\Theta \rtimes_\alpha \mathbb{Z}_2}(P_{\theta_{12}}(\mathfrak{u}_2, \mathfrak{u}_1, \mathfrak{w})) &= \tau_{A_\Theta \rtimes_\alpha \mathbb{Z}_2}\Big(\frac{1}{2} + \frac{1}{2}\mathfrak{w} - P_{1-\theta_{12}}(\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{w})\Big)\\ &= \tau_{A_\Theta \rtimes_\alpha \mathbb{Z}_2}\Big(\frac{1}{2} - P_{1-\theta_{12}}(\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{w})\Big)\\ &= \frac{1}{2} - \frac{1-\theta_{12}}{2} = \frac{\theta_{12}}{2}.\end{aligned} \tag{3.2}$$

Similar calculations show that

$$\begin{aligned}&\tau_{A_\Theta \rtimes_\alpha \mathbb{Z}_2}(P_{\theta_{12}}(\mathrm{e}^{\pi\mathrm{i}\theta_{23}}\mathfrak{u}_2, \mathrm{e}^{\pi\mathrm{i}\theta_{13}}\mathfrak{u}_1, \mathfrak{u}_3\mathfrak{w})) = \frac{\theta_{12}}{2},\\ &\tau_{A_\Theta \rtimes_\alpha \mathbb{Z}_2}(P_{\theta_{13}}(\mathfrak{u}_3, \mathfrak{u}_1, \mathfrak{w})) = \frac{\theta_{13}}{2}\end{aligned}$$

and

$$\tau_{A_\Theta \rtimes_\alpha \mathbb{Z}_2}(P_{\theta_{23}}(\mathfrak{u}_3, \mathfrak{u}_2, \mathfrak{w})) = \frac{\theta_{23}}{2}.$$

Next we will obtain some obstacles to the stability of rotation relations of three unitaries with the flip action.

Definition 3.1 *Let $\theta \in (0,1)$. Let A be a unital C^*-algebra and u, v be a pair of unitaries in A. We define*

$$e_\theta(u,v) = \begin{cases} \frac{1}{2}[(1 - vg_{1-\theta}(u) - f_{1-\theta}(u) - g_{1-\theta}(u)v^*) \\ +(1 - v^*g_{1-\theta}(u^*) - f_{1-\theta}(u^*) - g_{1-\theta}(u^*)v)], & 0 < \theta < \frac{1}{2}, \\ \frac{1}{2}[(ug_\theta(v) + f_\theta(v) + g_\theta(v)u^*) \\ +(u^*g_\theta(v^*) + f_\theta(v^*) + g_\theta(v^*)u)], & \frac{1}{2} \le \theta < 1. \end{cases}$$

Notice that if $uv = \mathrm{e}^{2\pi\mathrm{i}\theta}vu$, then $e_\theta(u,v)$ is a projection. In particular, we have $e_\theta(\mathfrak{u},\mathfrak{v}) = p_\theta(\mathfrak{u},\mathfrak{v})$.

It is clear that $e_\theta(u,v)$ is always self-adjoint.

Proposition 3.3 (see [24, Proposition 3.8]) *Let $\theta \in (0,1)$. There is a $\delta_0 > 0$ such that, for any unital C^*-algebra A, any pair of unitaries u, v in A with*

$$\|uv - \mathrm{e}^{2\pi\mathrm{i}\theta}vu\| < \delta_0,$$

we have

$$\|(e_\theta(u,v))^2 - e_\theta(u,v)\| < \frac{1}{4}.$$

In particular, the spectrum of $e_\theta(u,v)$ has a gap at $\frac{1}{2}$.

Definition 3.2 *Let $\theta \in (0,1)$. Let $\delta_0 > 0$ be chosen as in Proposition 3.3. Let A be a unital C^*-algebra and let u, v be a pair of unitaries in A with $\|uv - \mathrm{e}^{2\pi\mathrm{i}\theta}vu\| < \delta_0$. We define $R_\theta(u,v) = \chi_{(\frac{1}{2},+\infty)}(e_\theta(u,v))$.*

Proposition 3.4 (see [24, Propostion 3.10]) *Let $\theta \in (0,1)$ be an irrational number. Let $\delta_0 > 0$ be chosen as in Proposition 3.3. Let A be a unital C^*-algebra and let u, v be a pair of unitaries in A with*

$$\|uv - \mathrm{e}^{2\pi\mathrm{i}\theta}vu\| < \delta_0.$$

Then $b^\theta_{u,v} = [R_\theta(u,v)]$.

Definition 3.3 *Let $\theta \in (0,1)$. Let $\delta_0 > 0$ be chosen as in Proposition 3.3. Let A be a unital C^*-algebra and let u, v and w be a triple of unitaries in A such that*

$$\|uv - \mathrm{e}^{2\pi\mathrm{i}\theta}vu\| < \delta_0$$

and

$$wuw^{-1} = u^{-1}, \quad wvw^{-1} = v^{-1}, \quad w^2 = 1_A.$$

We define $R_\theta(u,v,w) = \frac{1}{2}R_\theta(u,v) + \frac{1}{2}R_\theta(u,v)w$. We know from the following discussion of [24, Definition 3.11] that w commutes with $R_\theta(u,v)$. So $R_\theta(u,v,w)$ is a projection. In particular, if

$$uv = \mathrm{e}^{2\pi\mathrm{i}\theta}vu, \quad wuw^{-1} = u^{-1}, \quad wvw^{-1} = v^{-1} \quad \text{and} \quad w^2 = 1_A,$$

then $R_\theta(u,v,w) = P_\theta(u,v,w)$.

Definition 3.4 *Let $\theta \in [0, 1)$. Denote by $\log_\theta$ the continuous branch of logarithm defined on $F_\theta = \{e^{it} : t \in (2\pi\theta - \pi, 2\pi\theta + \pi)\}$ with values in $\{ri : r \in (2\pi\theta - \pi, 2\pi\theta + \pi)\}$ such that $\log_\theta(e^{2\pi i\,\theta}) = 2\pi i\,\theta$. Note that if u is any unitary in some C^*-algebra A such that $\|u - e^{2\pi i\theta}\| < 2$, then $\operatorname{spec}(u)$ has a gap at $e^{2\pi i\theta + \pi i}$, thus $\log_\theta(u)$ is well defined. In particular, if $\theta = 0$, we simply write $\log(u)$ for $\log_0(u)$.*

Theorem 3.2 (see [27, Theorem 4.14]) *Let A be a unital C^*-algebra with $T(A) \neq \emptyset$ and $\theta \in [0, 1)$. Then for any $u, v \in U(A)$ with $\|uv - e^{2\pi i\theta}vu\| < 2$ and with $b^\theta_{u,v}$ defined as in Definition 2.9, we have*

$$\rho_A(b^\theta_{u,v})(\tau) = \frac{1}{2\pi i}\tau(\log_\theta(uvu^*v^*)) \quad \text{for all } \tau \in T(A). \tag{3.3}$$

The formula (3.3) is called the generalized Exel trace formula. The case where A is the matrix algebra and $\theta = 0$ is proved in [14]. The case where A is an arbitrary unital C^*-algebra and $\theta = 0$ is proved in [26, Theorem 3.7]. Next, we use the generalized Exel trace formula to describe some obstacles to stability.

Lemma 3.2 *Let $\Theta = (\theta_{jk}) \in \mathcal{T}_3$ be totally irrational, where $\theta_{jk} \in [0, 1)$ for $j, k = 1, 2, 3$. Let A be a unital C^*-algebra with $T(A) \neq \emptyset$, let δ_0 be chosen as in Proposition 3.3 (select the smallest δ_0 according to $\theta_{12}, \theta_{13}, \theta_{23}$). For any u_1, u_2, u_3 and $w \in U(A)$ satisfying*

(1) $\|u_k u_j - e^{2\pi i\theta_{jk}}u_j u_k\| < \delta_0 < 2$, $wu_jw^{-1} = u_j^{-1}$, $w^2 = 1_A$ for all $j, k = 1, 2, 3$,

(2) $\tau(aw) = 0$ for all $a \in C^*(u_1, u_2, u_3)$ and all $\tau \in T(A)$,

we have

$\rho_A([R_{\theta_{12}}(u_2, u_1, w)])(\tau) = \frac{1}{2}\rho_A(b^{\theta_{12}}_{u_2,u_1})(\tau) = \frac{1}{4\pi i}\tau(\log_{\theta_{12}}(u_2u_1u_2^*u_1^*))$,

$\rho_A([R_{\theta_{12}}(e^{\pi i\theta_{23}}u_2, e^{\pi i\theta_{13}}u_1, u_3w)])(\tau) = \frac{1}{2}\rho_A(b^{\theta_{12}}_{e^{\pi i\theta_{23}}u_2, e^{\pi i\theta_{13}}u_1})(\tau) = \frac{1}{4\pi i}\tau(\log_{\theta_{12}}(u_2u_1u_2^*u_1^*))$,

$\rho_A([R_{\theta_{13}}(u_3, u_1, w)])(\tau) = \frac{1}{2}\rho_A(b^{\theta_{13}}_{u_3,u_1})(\tau) = \frac{1}{4\pi i}\tau(\log_{\theta_{13}}(u_3u_1u_3^*u_1^*))$, *and*

$\rho_A([R_{\theta_{23}}(u_3, u_2, w)])(\tau) = \frac{1}{2}\rho_A(b^{\theta_{23}}_{u_3,u_2})(\tau) = \frac{1}{4\pi i}\tau(\log_{\theta_{23}}(u_3u_2u_3^*u_2^*))$

for all $\tau \in T(A)$, where $b^\theta_{u,v}$ is defined as in Definition 2.9, and $R_\theta(u, v, w)$ is defined as in Definition 3.3.

Proof Since $\|u_k u_j - e^{2\pi i\theta_{jk}}u_j u_k\| < \delta_0$, we have

$$\rho_A([R_{\theta_{12}}(u_2, u_1, w)])(\tau) = \rho_A\Big(\Big[\frac{1}{2}R_{\theta_{12}}(u_2, u_1) + \frac{1}{2}R_{\theta_{12}}(u_2, u_1)w\Big]\Big)(\tau).$$

By assumption that $\tau(aw) = 0$ for all $a \in C^*(u_1, u_2, u_3)$ and all $\tau \in T(A)$, we have

$$\tau(R_{\theta_{12}}(u_2, u_1)w) = 0$$

for all $\tau \in T(A)$. So

$$\rho_A([R_{\theta_{12}}(u_2, u_1, w)])(\tau) = \rho_A\Big(\Big[\frac{1}{2}R_{\theta_{12}}(u_2, u_1)\Big]\Big)(\tau).$$

Furthermore, by the generalized Exel trace formula of Theorem 3.2 and Proposition 3.4, we conclude that

$$\rho_A([R_{\theta_{12}}(u_2, u_1, w)])(\tau) = \frac{1}{2}\rho_A(b^{\theta_{12}}_{u_2,u_1})(\tau) = \frac{1}{4\pi i}\tau(\log_{\theta_{12}}(u_2u_1u_2^*u_1^*)) \quad \text{for all } \tau \in T(A).$$

Similarly, we have

$$
\begin{aligned}
&\rho_A([R_{\theta_{12}}(\mathrm{e}^{\pi\mathrm{i}\theta_{23}}u_2, \mathrm{e}^{\pi\mathrm{i}\theta_{13}}u_1, u_3w)])(\tau)\\
&= \rho_A\Big(\Big[\frac{1}{2}R_{\theta_{12}}(\mathrm{e}^{\pi\mathrm{i}\theta_{23}}u_2, \mathrm{e}^{\pi\mathrm{i}\theta_{13}}u_1) + \frac{1}{2}R_{\theta_{12}}(\mathrm{e}^{\pi\mathrm{i}\theta_{23}}u_2, \mathrm{e}^{\pi\mathrm{i}\theta_{13}}u_1)u_3w\Big]\Big)(\tau)\\
&= \rho_A\Big(\Big[\frac{1}{2}R_{\theta_{12}}(\mathrm{e}^{\pi\mathrm{i}\theta_{23}}u_2, \mathrm{e}^{\pi\mathrm{i}\theta_{13}}u_1)\Big]\Big)(\tau)\\
&= \frac{1}{4\pi\mathrm{i}}\tau(\log_{\theta_{12}}((\mathrm{e}^{\pi\mathrm{i}\theta_{23}}u_2)(\mathrm{e}^{\pi\mathrm{i}\theta_{13}}u_1)(\mathrm{e}^{\pi\mathrm{i}\theta_{23}}u_2)^*(\mathrm{e}^{\pi\mathrm{i}\theta_{13}}u_1)^*))\\
&= \frac{1}{4\pi\mathrm{i}}\tau(\log_{\theta_{12}}(u_2u_1u_2^*u_1^*)),\\
&\rho_A([R_{\theta_{13}}(u_3, u_1, w)])(\tau)\\
&= \frac{1}{2}\rho_A(b^{\theta_{13}}_{u_3,u_1})(\tau) = \frac{1}{4\pi\mathrm{i}}\tau(\log_{\theta_{13}}(u_3u_1u_3^*u_1^*))
\end{aligned}
$$

and

$$
\rho_A([R_{\theta_{23}}(u_3, u_2, w)])(\tau) = \frac{1}{2}\rho_A(b^{\theta_{23}}_{u_3,u_2})(\tau) = \frac{1}{4\pi\mathrm{i}}\tau(\log_{\theta_{23}}(u_3u_2u_3^*u_2^*))
$$

for all $\tau \in T(A)$.

Lemma 3.3 (see [24, Lemma 4.3]) *Let $\theta \in [0, 1)$. For any unital C^*-algebra A with $T(A) \neq \emptyset$, any two unitaries $u, v \in A$ with $\|uv - \mathrm{e}^{2\pi\mathrm{i}\theta}vu\| < 2$, if $\tau((uvu^*v^*)^n) = \mathrm{e}^{2\pi\mathrm{i}n\theta}$ for all $n \in \mathbb{N}$ and all $\tau \in T(A)$, then $\tau(\log_\theta(uvu^*v^*)) = 2\pi\mathrm{i}\theta$ for all $\tau \in T(A)$.*

Lemma 3.4 *Let $\Theta = (\theta_{jk}) \in \mathcal{T}_3$ be totally irrational, where $\theta_{jk} \in [0, 1)$ for $j, k = 1, 2, 3$. Let A be a unital C^*-algebra with $T(A) \neq \emptyset$, let δ_0 be chosen as in Proposition* 3.3 (*select the smallest δ_0 according to $\theta_{12}, \theta_{13}, \theta_{23}$*). *For any u_1, u_2, u_3 and $w \in U(A)$ satisfying*

(1) *$\|u_ku_j - \mathrm{e}^{2\pi\mathrm{i}\theta_{jk}}u_ju_k\| < \delta_0 < 2$, $wu_jw^{-1} = u_j^{-1}$, $w^2 = 1_A$ for all $j, k = 1, 2, 3$,*

(2) *$\tau(aw) = 0$ and $\tau((u_ku_ju_k^*u_j^*)^n) = \mathrm{e}^{2\pi\mathrm{i}n\theta_{jk}}$ for all $a \in C^*(u_1, u_2, u_3)$, all $n \in \mathbb{N}$, $j, k = 1, 2, 3$ and all $\tau \in T(A)$,*

we have

$$
\begin{aligned}
&\rho_A([R_{\theta_{12}}(u_2, u_1, w)])(\tau) = \frac{\theta_{12}}{2}, \quad \rho_A([R_{\theta_{12}}(\mathrm{e}^{\pi\mathrm{i}\theta_{23}}u_2, \mathrm{e}^{\pi\mathrm{i}\theta_{13}}u_1, u_3w)])(\tau) = \frac{\theta_{12}}{2},\\
&\rho_A([R_{\theta_{13}}(u_3, u_1, w)])(\tau) = \frac{\theta_{13}}{2} \quad \text{and} \quad \rho_A([R_{\theta_{23}}(u_3, u_2, w)])(\tau) = \frac{\theta_{23}}{2}
\end{aligned}
$$

for all $\tau \in T(A)$, where $R_\theta(u, v, w)$ is defined as in Definition 3.3.

Proof By combining Lemmas 3.2–3.3, we get the conclusion.

4 Some Other Projections

Lemma 4.1 *Let $\Theta = (\theta_{jk}) \in \mathcal{T}_3$, where $\theta_{jk} \in [0, 1)$ for $j, k = 1, 2, 3$. For any unital C^*-algebra A, any four unitaries $u_1, u_2, u_3, w \in A$, if $\|u_ku_j - \mathrm{e}^{2\pi\mathrm{i}\theta_{jk}}u_ju_k\| < 2$, $wu_jw^{-1} = u_j^{-1}$ and $w^2 = 1_A$ for $j, k = 1, 2, 3$, then*

(1) *$\widetilde{Q}_1 = \frac{1}{2}(1+w)$, $\widetilde{Q}_2 = \frac{1}{2}(1-u_1w)$, $\widetilde{Q}_3 = \frac{1}{2}(1-u_2w)$ and $\widetilde{Q}_5 = \frac{1}{2}(1+u_3w)$ are projections in A,*

(2) $\widetilde{Q}_4 = \frac{1}{4}((1-\mathrm{e}^{\pi\mathrm{i}\theta_{12}}u_1u_2w)+(1-\mathrm{e}^{\pi\mathrm{i}\theta_{12}}u_1u_2w)^*)$, $\widetilde{Q}_6 = \frac{1}{4}((1-\mathrm{e}^{\pi\mathrm{i}\theta_{13}}u_1u_3w)+(1-\mathrm{e}^{\pi\mathrm{i}\theta_{13}}u_1u_3w)^*)$ *and* $\widetilde{Q}_7 = \frac{1}{4}((1-\mathrm{e}^{\pi\mathrm{i}\theta_{23}}u_2u_3w)+(1-\mathrm{e}^{\pi\mathrm{i}\theta_{23}}u_2u_3w)^*)$ *are self-adjoint elements and the spectra of* $\widetilde{Q}_4,\widetilde{Q}_6,\widetilde{Q}_7$ *have a gap at* $\frac{1}{2}$.

Proof (1) It is easy to verify that $\widetilde{Q}_1,\widetilde{Q}_2,\widetilde{Q}_3,\widetilde{Q}_5$ are projections.

(2) We only prove that the case of $\widetilde{Q}_4$. The proofs that the cases of $\widetilde{Q}_6$ and $\widetilde{Q}_7$ are similar. We compute that

$$\begin{aligned}(\widetilde{Q}_4)^* &= \Big(\frac{1}{4}((1-\mathrm{e}^{\pi\mathrm{i}\theta_{12}}u_1u_2w)+(1-\mathrm{e}^{\pi\mathrm{i}\theta_{12}}u_1u_2w)^*)\Big)^*\\ &= \frac{1}{4}((1-\mathrm{e}^{\pi\mathrm{i}\theta_{12}}u_1u_2w)+(1-\mathrm{e}^{\pi\mathrm{i}\theta_{12}}u_1u_2w)^*)\\ &= \widetilde{Q}_4.\end{aligned}$$

So $\widetilde{Q}^4$ is a self-adjoint element.

Since

$$\begin{aligned}(\widetilde{Q}_4)^2 &= \Big(\frac{1}{4}((1-\mathrm{e}^{\pi\mathrm{i}\theta_{12}}u_1u_2w)+(1-\mathrm{e}^{\pi\mathrm{i}\theta_{12}}u_1u_2w)^*)\Big)^2\\ &= \Big(\frac{1}{4}(2-\mathrm{e}^{\pi\mathrm{i}\theta_{12}}u_1u_2w-\mathrm{e}^{-\pi\mathrm{i}\theta_{12}}u_2u_1w)\Big)^2\\ &= \frac{1}{16}(4-2\mathrm{e}^{\pi\mathrm{i}\theta_{12}}u_1u_2w-2\mathrm{e}^{-\pi\mathrm{i}\theta_{12}}u_2u_1w-2\mathrm{e}^{\pi\mathrm{i}\theta_{12}}u_1u_2w+\mathrm{e}^{\pi\mathrm{i}\theta_{12}}u_1u_2w\mathrm{e}^{\pi\mathrm{i}\theta_{12}}u_1u_2w\\ &\quad+\mathrm{e}^{\pi\mathrm{i}\theta_{12}}u_1u_2w\mathrm{e}^{-\pi\mathrm{i}\theta_{12}}u_2u_1w-2\mathrm{e}^{-\pi\mathrm{i}\theta_{12}}u_2u_1w+\mathrm{e}^{-\pi\mathrm{i}\theta_{12}}u_2u_1w\mathrm{e}^{\pi\mathrm{i}\theta_{12}}u_1u_2w\\ &\quad+\mathrm{e}^{-\pi\mathrm{i}\theta_{12}}u_2u_1w\mathrm{e}^{-\pi\mathrm{i}\theta_{12}}u_2u_1w)\\ &= \frac{1}{16}(6-4\mathrm{e}^{\pi\mathrm{i}\theta_{12}}u_1u_2w-4\mathrm{e}^{-\pi\mathrm{i}\theta_{12}}u_2u_1w+\mathrm{e}^{2\pi\mathrm{i}\theta_{12}}u_1u_2u_1^{-1}u_2^{-1}\\ &\quad+\mathrm{e}^{-2\pi\mathrm{i}\theta_{12}}u_2u_1u_2^{-1}u_1^{-1}),\end{aligned}$$

we have

$$\begin{aligned}&\|(\widetilde{Q}_4)^2-\widetilde{Q}_4\|\\ &= \Big\|\Big(\frac{1}{4}(2-\mathrm{e}^{\pi\mathrm{i}\theta_{12}}u_1u_2w-\mathrm{e}^{-\pi\mathrm{i}\theta_{12}}u_2u_1w)\Big)^2-\frac{1}{4}(2-\mathrm{e}^{\pi\mathrm{i}\theta_{12}}u_1u_2w-\mathrm{e}^{-\pi\mathrm{i}\theta_{12}}u_2u_1w)\Big\|\\ &= \Big\|\frac{1}{16}(-2+\mathrm{e}^{2\pi\mathrm{i}\theta_{12}}u_1u_2u_1^{-1}u_2^{-1}+\mathrm{e}^{-2\pi\mathrm{i}\theta_{12}}u_2u_1u_2^{-1}u_1^{-1})\Big\|\\ &\le \frac{1}{8}\|u_2u_1-\mathrm{e}^{2\pi\mathrm{i}\theta_{12}}u_1u_2\|\\ &< \frac{1}{4}.\end{aligned}$$

So $\frac{1}{2}$ is not in the spectrum of $\widetilde{Q}_4$ and $\mathrm{spec}(\widetilde{Q}_4)\subset\big(-\frac{1}{2},\frac{1}{2}\big)\bigcup\big(\frac{1}{2},\frac{3}{2}\big)$.

Lemma 4.2 *Let* $\Theta=(\theta_{jk})\in\mathcal{T}_3$, *where* $\theta_{jk}\in[0,1)$ *for* $j,k=1,2,3$. *For any unital* C^*-*algebra* A *with* $T(A)\neq\emptyset$, *any four unitaries* $u_1,u_2,u_3,w\in A$, *if the following are satisfied*:

(1) $\|u_ku_j-\mathrm{e}^{2\pi\mathrm{i}\theta_{jk}}u_ju_k\|<2$, $wu_jw^{-1}=u_j^{-1}$ *and* $w^2=1_A$ *for* $j,k=1,2,3$,

(2) $\tau(aw)=0$ *and* $\tau((u_ku_ju_k^*u_j^*)^n)=\mathrm{e}^{2\pi\mathrm{i}n\theta_{jk}}$ *for all* $n\in\mathbb{N}$, *all* $a\in C^*(u_1,u_2,u_3)$, *all* $\tau\in T(A)$ *and* $j,k=1,2,3$,

then $\chi_{(\frac{1}{2},+\infty)}(\widetilde{Q}_j)$ *is a projection in* A *and* $\tau(\chi_{(\frac{1}{2},+\infty)}(\widetilde{Q}_j))=\frac{1}{2}$ *for all* $\tau\in T(A)$ *and* $j=4,6,7$, *where* $\widetilde{Q}_j$ *is defined as in Lemma* 4.1.

Proof Since $\|u_k u_j - \mathrm{e}^{2\pi \mathrm{i}\theta_{jk}} u_j u_k\| < 2$, $w u_j w^{-1} = u_j^{-1}$ and $w^2 = 1_A$ for $j, k = 1, 2, 3$, by applying Lemma 4.1, $\frac{1}{2}$ is not in the spectrum of $\widetilde{Q}_j$ for $j = 4, 6, 7$. So $\chi_{(\frac{1}{2}, +\infty)}(\widetilde{Q}_j)$ is a projection in A.

We only prove that $\tau(\chi_{(\frac{1}{2},+\infty)}(\widetilde{Q}_4)) = \frac{1}{2}$ for all $\tau \in T(A)$. The proofs that $\tau(\chi_{(\frac{1}{2},+\infty)}(\widetilde{Q}_6)) = \frac{1}{2}$ and $\tau(\chi_{(\frac{1}{2},+\infty)}(\widetilde{Q}_7)) = \frac{1}{2}$ are similar.

Next we first show that $\tau((\widetilde{Q}_4)^n) = \frac{1}{2}$ for all $n \in \mathbb{N}$ and all $\tau \in T(A)$.

For $n = 2k + 1$, $k = 0, 1, 2, \cdots$, by assumption and

$$(\mathrm{e}^{\pi \mathrm{i}\theta_{12}} u_1 u_2 w + \mathrm{e}^{-\pi \mathrm{i}\theta_{12}} u_2 u_1 w)^2 = \mathrm{e}^{2\pi \mathrm{i}\theta_{12}} u_1 u_2 u_1^* u_2^* + 2 + \mathrm{e}^{-2\pi \mathrm{i}\theta_{12}} u_2 u_1 u_2^* u_1^*,$$

we have

$$\begin{aligned}
&\tau((\mathrm{e}^{\pi \mathrm{i}\theta_{12}} u_1 u_2 w + \mathrm{e}^{-\pi \mathrm{i}\theta_{12}} u_2 u_1 w)^n) \\
&= \tau((\mathrm{e}^{2\pi \mathrm{i}\theta_{12}} u_1 u_2 u_1^* u_2^* + 2 + \mathrm{e}^{-2\pi \mathrm{i}\theta_{12}} u_2 u_1 u_2^* u_1^*)^k (\mathrm{e}^{\pi \mathrm{i}\theta_{12}} u_1 u_2 w + \mathrm{e}^{-\pi \mathrm{i}\theta_{12}} u_2 u_1 w)) \\
&= 0
\end{aligned}$$

and

$$\tau((\mathrm{e}^{\pi \mathrm{i}\theta_{12}} u_1 u_2 w + \mathrm{e}^{-\pi \mathrm{i}\theta_{12}} u_2 u_1 w)^{2j}) = 2^{2j} \quad \text{for } j = 1, 2, \cdots.$$

So

$$\begin{aligned}
\tau((\widetilde{Q}_4)^n) &= \frac{1}{4^n} \tau((2 - \mathrm{e}^{\pi \mathrm{i}\theta_{12}} u_1 u_2 w - \mathrm{e}^{-\pi \mathrm{i}\theta_{12}} u_2 u_1 w)^n) \\
&= \frac{1}{4^n} \sum_{j=0}^{n} C_n^j 2^{n-j} (-1)^j \tau((\mathrm{e}^{\pi \mathrm{i}\theta_{12}} u_1 u_2 w + \mathrm{e}^{-\pi \mathrm{i}\theta_{12}} u_2 u_1 w)^j) \\
&= \frac{1}{4^n} \sum_{j=0}^{k} C_n^{2j} 2^{n-2j} \tau((\mathrm{e}^{\pi \mathrm{i}\theta_{12}} u_1 u_2 w + \mathrm{e}^{-\pi \mathrm{i}\theta_{12}} u_2 u_1 w)^{2j}) \\
&= \frac{1}{4^n} \sum_{j=0}^{k} C_n^{2j} 2^n \\
&= \frac{1}{2^n} \sum_{j=0}^{k} C_n^{2j} = \frac{1}{2},
\end{aligned} \tag{4.1}$$

where C_n^j are the coefficients of the binomial expansion for $j = 0, \cdots, n$.

For $n = 2k$, $k = 1, 2, \cdots$, by assumption we have

$$\tau((\mathrm{e}^{\pi \mathrm{i}\theta_{12}} u_1 u_2 w + \mathrm{e}^{-\pi \mathrm{i}\theta_{12}} u_2 u_1 w)^n) = 4^k = 2^n.$$

Now for any $0 < \varepsilon < 1$, by applying the Stone-Weierstrass theorem, there exists a polynomial $P(x) = a_0 + a_1 x + \cdots + a_n x^n$ on the spectrum of $\widetilde{Q}_4$ such that

$$\|\chi_{(\frac{1}{2},+\infty)} - P\|_{\mathrm{spec}(\widetilde{Q}_4)} < \frac{\varepsilon}{8}.$$

Note that

$$\|a_0\| = \|a_0 - 0\| = \|P(0) - \chi_{(\frac{1}{2},+\infty)}(0)\| < \frac{\varepsilon}{8},$$

let $P_1(x) = a_1x + \cdots + a_nx^n$, then we have

$$\|P(x) - P_1(x)\|_{\mathrm{spec}(\widetilde{Q}^4)} = \|a_0\| < \frac{\varepsilon}{8}$$

and

$$\|P_1(1) - 1\| \le \|P_1(1) - P(1)\| + \|\chi_{(\frac{1}{2},+\infty)}(1) - P(1)\| < \frac{\varepsilon}{4}.$$

Next we let $\widetilde{P}(x) = P_1(x)/P_1(1)$, then $\widetilde{P}(0) = 0$, $\widetilde{P}(1) = 1$ and

$$\begin{aligned}
&\|\widetilde{P} - \chi_{(\frac{1}{2},+\infty)}\|_{\mathrm{spec}(\widetilde{Q}_4)}\\
&= \|P_1(x)/P_1(1) - \chi_{(\frac{1}{2},+\infty)}(x)\|_{\mathrm{spec}(\widetilde{Q}_4)}\\
&\le \frac{\|(P_1(x) - P(x)) + (P(x) - \chi_{(\frac{1}{2},+\infty)}(x)) + (\chi_{(\frac{1}{2},+\infty)}(x) - \chi_{(\frac{1}{2},+\infty)}(x)P_1(1))\|}{\|P_1(1)\|}\\
&< \frac{\frac{\varepsilon}{2}}{\|P_1(1)\|} < \varepsilon.
\end{aligned} \tag{4.2}$$

It follows from (4.1) and (4.2) that

$$\tau(\widetilde{P}(\widetilde{Q}_4)) = \frac{1}{P_1(1)}(a_1\tau(\widetilde{Q}_4) + \cdots + a_n\tau((\widetilde{Q}_4)^n)) = \frac{1}{2}\widetilde{P}(1) = \frac{1}{2}.$$

By (4.2) and the arbitrariness of ε, we can get $\tau(\chi_{(\frac{1}{2},+\infty)}(\widetilde{Q}_4)) = \frac{1}{2}$ for all $\tau \in T(A)$.

5 Stability of Rotation Relations of Three Unitaries with the Flip Action

Let A be a C^*-algebra. Suppose that p is a projection in $M_n(A)$ and q is a projection in $M_m(A)$. Then $p \sim_0 q$ if there is an element v in $M_{m,n}(A)$ with $p = v^*v$ and $q = vv^*$.

Definition 5.1 (see [38, Definition 7.3.1]) *A C^*-algebra A is said to have the cancellation property if for every pair of projections p, q in $\bigcup_{n=1}^{\infty} M_n(A)$,*

$$[p] = [q] \quad \textit{in } K_0(A) \textit{ if and only if } p \sim_0 q.$$

It is known that many C^*-algebras have the cancellation property, for example, every C^*-algebra of stable rank one has the cancellation property by [35]. (A unital C^*-algebra A is said to have stable rank one, if the group of invertible elements in A is dense in A.)

Let us give a brief outline of our strategy of proving the stability of rotation relations of three unitaries with the flip action. Suppose u_1, u_2, u_3 and w are four unitaries in a unital C^*-algebra A, where u_1, u_2, u_3 almost satisfy the rotation relation with respect to Θ, and $wu_jw^{-1} = u_j^{-1}$ for $j = 1, 2, 3$, $w^2 = 1_A$. Then there is an almost homomorphism from $A_\Theta \rtimes_\alpha \mathbb{Z}_2$ to A. Now the stability of the rotation relations of three unitaries with the flip action is equivalent to that this almost homomorphism is close to an actual homomorphism.

The latter problem is usually divided into two parts: The existence part and the uniqueness part. An almost homomorphism will induce an 'almost' homomorphism between the invariants of the two C^*-algebras, where the invariant consists of the K-theories. It is usually easier to show that an 'almost' homomorphism of the invariants is close to an actual homomorphism.

The existence part says that a homomorphism at the invariant level lifts to a homomorphism at the C^*-algebra level. The uniqueness part says that, two almost homomorphisms which induces 'almost' the same maps on the invariants are almost unitary equivalent. Therefore, conjugating suitable unitaries, one shows that an almost homomorphism is close to an actual homomorphism.

The following is sometime called the existence and uniqueness theorem for homomorphisms from AF-algebras to unital C^*-algebras with the cancellation property.

Theorem 5.1 (see [24, Theorem 5.3]) *Let A be a unital AF-algebra, and let C be a unital C^*-algebra with the cancellation property. Suppose that $\psi : K_0(A) \to K_0(C)$ is a unital positive homomorphism. Then there is a unital homomorphism $h : A \to C$ such that $h_* = \psi$.*

Theorem 5.2 (see [24, Theorem 5.4]) *Let A be a unital AF-algebra. Then, for any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$, there exists $\delta > 0$, a finite subset $\mathcal{P} \subset K_0(A)$ and a finite subset $\mathcal{G} \subset A$ satisfying the following: If $L_1, L_2 : A \to C$, where C is a unital C^*-algebra with the cancellation property, are two $\mathcal{G}$-δ-multiplicative contractive completely positive linear maps such that*

$$[L_1]|_{G(\mathcal{P})} = [L_2]|_{G(\mathcal{P})},$$

where $G(\mathcal{P})$ is the subgroup generated by $\mathcal{P}$, then there is a unitary $u \in C$ such that

$$\mathrm{Ad}u \circ L_1 \approx_\varepsilon L_2 \quad \text{on } \mathcal{F}.$$

Proposition 5.1 *Let $\Theta = (\theta_{jk}) \in \mathcal{T}_3$. Let $\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3$ and $\mathfrak{w}$ be the canonical generators of $A_\Theta \rtimes_\alpha \mathbb{Z}_2$ as in Lemma 3.1. Then, for any finite subset $\mathcal{G} \subset A_\Theta \rtimes_\alpha \mathbb{Z}_2$, any $\eta > 0$ and any $\varepsilon > 0$, there is a $\delta > 0$ such that: For any unital C^*-algebra A, any four unitaries u_1, u_2, u_3 and w in A satisfying*

$$\|u_k u_j - \mathrm{e}^{2\pi \mathrm{i}\theta_{jk}} u_j u_k\| < \delta, \quad w u_j w^{-1} = u_j^{-1} \quad \textit{and} \quad w^2 = 1_A \quad \textit{for } j,k = 1,2,3,$$

there is a unital $\mathcal{G}$-η-multiplicative c.p.c. (completely positive contractive) map $L : A_\Theta \rtimes_\alpha \mathbb{Z}_2 \to A$ such that

$$\|L(\mathfrak{u}_j) - u_j\| < \varepsilon \quad \textit{and} \quad \|L(\mathfrak{w}) - w\| < \varepsilon \quad \textit{for } j = 1,2,3.$$

Proof Assume that the proposition is false. Let $\{\delta_m\}_{m=1}^\infty$ be a sequence of positive numbers decreasing to 0. Then there is a finite subset $\mathcal{G} \subset A_\Theta \rtimes_\alpha \mathbb{Z}_2$, some $\varepsilon, \eta > 0$ such that for any m, there is a unital C^*-algebra A_m and four unitaries $u_1^{(m)}, u_2^{(m)}, u_3^{(m)}, w^{(m)}$ in A_m satisfying

$$\|u_k^{(m)} u_j^{(m)} - \mathrm{e}^{2\pi \mathrm{i}\theta_{jk}} u_j^{(m)} u_k^{(m)}\| < \delta_m, \quad w^{(m)} u_j^{(m)} (w^{(m)})^{-1} = (u_j^{(m)})^{-1} \quad \text{and} \quad (w^{(m)})^2 = 1_A$$

for $j,k = 1,2,3$, but for any unital $\mathcal{G}$-η-multiplicative c.p.c. map $\phi_m : A_\Theta \rtimes_\alpha \mathbb{Z}_2 \to A_m$, we have

$$\|\phi_m(\mathfrak{u}_j) - u_j^{(m)}\| \ge \varepsilon \quad \text{for some } j = 1,2,3 \quad \text{or} \quad \|\phi_m(\mathfrak{w}) - w^{(m)}\| \ge \varepsilon.$$

Set $C = \prod\limits_{m=1}^\infty A_m \Big/ \bigoplus\limits_{m=1}^\infty A_m$. Let $\pi : \prod\limits_{m=1}^\infty A_m \to C$ be the canonical quotient map. Let $u_j = (u_j^{(m)}), w = (w^{(m)}) \in \prod\limits_{m=1}^\infty A_m$ for $j = 1,2,3$. Then $\pi(u_1), \pi(u_2), \pi(u_3)$ and $\pi(w)$ are unitaries satisfying

$$\pi(u_k)\pi(u_j) = \mathrm{e}^{2\pi\mathrm{i}\theta_{jk}}\pi(u_j)\pi(u_k), \quad \pi(w)\pi(u_j)\pi(w)^{-1} = \pi(u_j)^{-1} \quad \text{and} \quad \pi(w)^2 = 1_A$$

for $j, k = 1, 2, 3$. Therefore there is a unital homomorphism $\phi : A_\Theta \rtimes_\alpha \mathbb{Z}_2 \to C$. By the Choi-Effros lifting theorem, we can lift ϕ to a unital c.p.c. map

$$\widetilde{\phi} = (\phi_1, \phi_2, \cdots, \phi_m, \cdots) : A_\Theta \rtimes_\alpha \mathbb{Z}_2 \to \prod_{m=1}^{\infty} A_m.$$

In particular, each coordinate map ϕ_m is unital completely positive, and we can also assume that they are contractive by normalization. By choosing m large enough, we can make sure that ϕ_m are $\mathcal{G}$-η-multiplicative. From our construction,

$$\lim_{m\to\infty} \|\phi_m(\mathfrak{u}_j) - u_j^{(m)}\| = 0 \quad \text{for } j = 1, 2, 3 \quad \text{and} \quad \lim_{m\to\infty} \|\phi_m(\mathfrak{w}) - w^{(m)}\| = 0.$$

This is a contradiction.

The following follows from functional calculus and the fact that norm close projections are equivalent.

Lemma 5.1 *Let $\Theta = (\theta_{jk}) \in \mathcal{T}_3$ be totally irrational, where $\theta_{jk} \in [0, 1)$ for $j, k = 1, 2, 3$. Let $\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3$ and $\mathfrak{w}$ be the canonical generators of $A_\Theta \rtimes_\alpha \mathbb{Z}_2$ as in Theorem* 3.1. *Let $[Q_j]$ for $j = 1, \cdots, 7$, $[P_{\theta_{12}}(\mathfrak{u}_2, \mathfrak{u}_1, \mathfrak{w})]$, $[P_{\theta_{12}}(\mathrm{e}^{\pi\mathrm{i}\theta_{23}}\mathfrak{u}_2, \mathrm{e}^{\pi\mathrm{i}\theta_{13}}\mathfrak{u}_1, \mathfrak{u}_3\mathfrak{w})]$, $[P_{\theta_{13}}(\mathfrak{u}_3, \mathfrak{u}_1, \mathfrak{w})]$, $[P_{\theta_{23}}(\mathfrak{u}_3, \mathfrak{u}_2, \mathfrak{w})]$ be the elements of $K_0(A_\Theta \rtimes_\alpha \mathbb{Z}_2)$ as defined in Theorem* 3.1. *Then there exists a finite subset $\mathcal{G} \subset A_\Theta \rtimes_\alpha \mathbb{Z}_2$, $\delta > 0$ and $\varepsilon > 0$, such that: For any unital C^*-algebra A, any four unitaries u_1, u_2, u_3 and w in A, if $L : A_\Theta \rtimes_\alpha \mathbb{Z}_2 \to A$ is a $\mathcal{G}$-δ-multiplicative contractive completely positive linear map such that*

$$\|L(\mathfrak{u}_j) - u_j\| < \varepsilon \quad \textit{for } j = 1, 2, 3 \quad \textit{and} \quad \|L(\mathfrak{w}) - w\| < \varepsilon,$$

then

$$\begin{aligned}
&[L]([1]) = [1_A],\ [L]([Q_j]) = [\widetilde{Q}_j] \quad \textit{for } j = 1, \cdots, 7,\ [L]([P_{\theta_{12}}(\mathfrak{u}_2, \mathfrak{u}_1, \mathfrak{w})]) = [R_{\theta_{12}}(u_2, u_1, w)],\\
&[L]([P_{\theta_{12}}(\mathrm{e}^{\pi\mathrm{i}\theta_{23}}\mathfrak{u}_2, \mathrm{e}^{\pi\mathrm{i}\theta_{13}}\mathfrak{u}_1, \mathfrak{u}_3\mathfrak{w})]) = [R_{\theta_{12}}(\mathrm{e}^{\pi\mathrm{i}\theta_{23}}u_2, \mathrm{e}^{\pi\mathrm{i}\theta_{13}}u_1, u_3w)],\\
&[L]([P_{\theta_{13}}(\mathfrak{u}_3, \mathfrak{u}_1, \mathfrak{w})]) = [R_{\theta_{13}}(u_3, u_1, w)],\ [L]([P_{\theta_{23}}(\mathfrak{u}_3, \mathfrak{u}_2), \mathfrak{w}]) = [R_{\theta_{23}}(u_3, u_2, w)].
\end{aligned}$$

Proof We only prove that $[L]([Q_1]) = [\widetilde{Q}_1]$, the remaining proofs are similar, and we ignore them.

Note that $Q_1 = \frac{1}{2}(1+\mathfrak{w})$, then $L(Q_1) = \frac{1}{2}(L(1)+L(\mathfrak{w}))$. Since $\widetilde{Q}_1 = \frac{1}{2}(1+w)$, for $\mathfrak{w}, \mathfrak{w}^* \in \mathcal{G}$ we have

$$\begin{aligned}
L(Q_1) - \widetilde{Q}_1 &= \frac{1}{2}(L(1) + L(\mathfrak{w})) - \frac{1}{2}(1 + w)\\
&= \frac{1}{2}((L(1) - 1) + (L(\mathfrak{w}) - w))\\
&= \frac{1}{2}((L(\mathfrak{w}\mathfrak{w}^*) - ww^*) + (L(\mathfrak{w}) - w))\\
&\approx_{\frac{\delta}{2}} \frac{1}{2}((L(\mathfrak{w})L(\mathfrak{w}^*) - ww^*) + (L(\mathfrak{w}) - w))\\
&= \frac{1}{2}((L(\mathfrak{w}) - w)L(\mathfrak{w}^*) + w(L(\mathfrak{w}^*) - w^*) + (L(\mathfrak{w}) - w)).
\end{aligned}$$

Thus

$$\|L(Q_1) - \widetilde{Q}_1\| < \frac{\delta}{2} + \frac{3\varepsilon}{2}.$$

Now we take sufficiently small δ and ε such that $\|L(Q_1) - \widetilde{Q}_1\| < 1$. Since $L(Q_1)$ and $\widetilde{Q}_1$ are projections, we get that $[L]([Q_1]) = [\widetilde{Q}_1]$.

Next we will prove our main theorem.

Theorem 5.3 *Let $\Theta = (\theta_{jk}) \in \mathcal{T}_3$ be totally irrational, where $\theta_{jk} \in [0,1)$ for $j,k = 1,2,3$. Then, for any $\varepsilon > 0$, there exists $\delta > 0$ satisfying the following: For any unital C^*-algebra A with the cancellation property, strict comparison and $T(A) \neq \emptyset$, any four unitaries $u_1, u_2, u_3, w \in A$ such that*

(1) $\|u_k u_j - \mathrm{e}^{2\pi \mathrm{i}\theta_{jk}} u_j u_k\| < \delta$, $w u_j w^{-1} = u_j^{-1}$, $w^2 = 1_A$ for $j,k = 1,2,3$,

(2) $\tau(aw) = 0$ and $\tau((u_k u_j u_k^* u_j^*)^n) = \mathrm{e}^{2\pi \mathrm{i} n\theta_{jk}}$ for all $a \in C^*(u_1,u_2,u_3)$, all $n \in \mathbb{N}$, $j,k = 1,2,3$ and all $\tau \in T(A)$,

there exists a 4-tuple of unitaries $\widetilde{u}_1, \widetilde{u}_2, \widetilde{u}_3, \widetilde{w} \in A$ such that

$$\widetilde{u}_k \widetilde{u}_j = \mathrm{e}^{2\pi \mathrm{i}\theta_{jk}} \widetilde{u}_j \widetilde{u}_k, \quad \widetilde{w}\widetilde{u}_j \widetilde{w}^{-1} = \widetilde{u}_j^{-1}, \quad \widetilde{w}^2 = 1_A$$

and

$$\|u_j - \widetilde{u}_j\| < \varepsilon, \quad \|w - \widetilde{w}\| < \varepsilon$$

for $j,k = 1,2,3$.

Proof Let $B = A_\Theta \rtimes_\alpha \mathbb{Z}_2$. Let τ_B denote the canonical tracial state on B and let $\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3, \mathfrak{w}$ be the canonical generators of B as in Lemma 3.1. By Theorems 2.4 and 3.1, B is a unital simple AF-algebra with

$$\begin{aligned} K_0(B) = \mathbb{Z}[1_B] &+ \mathbb{Z}[Q_1] + \mathbb{Z}[Q_2] + \mathbb{Z}[Q_3] + \mathbb{Z}[Q_4] + \mathbb{Z}[Q_5] + \mathbb{Z}[Q_6] + \mathbb{Z}[Q_7] + \mathbb{Z}[P_{\theta_{12}}(\mathfrak{u}_2, \mathfrak{u}_1, \mathfrak{w})] \\ &+ \mathbb{Z}[P_{\theta_{12}}(\mathrm{e}^{\pi \mathrm{i}\theta_{23}}\mathfrak{u}_2, \mathrm{e}^{\pi \mathrm{i}\theta_{13}}\mathfrak{u}_1, \mathfrak{u}_3\mathfrak{w})] + \mathbb{Z}[P_{\theta_{13}}(\mathfrak{u}_3, \mathfrak{u}_1, \mathfrak{w})] + \mathbb{Z}[P_{\theta_{23}}(\mathfrak{u}_3, \mathfrak{u}_2, \mathfrak{w})] \end{aligned} \tag{5.1}$$

and

$$K_0(B)_+ = \{c \in K_0(B) \mid (\tau_B)_*(c) > 0\}.$$

Let $\varepsilon > 0$ be given. For $\frac{\varepsilon}{2} > 0$ and $\mathcal{F} = \{1_B, \mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3, \mathfrak{w}\}$. Let $\delta_1 > 0$, $\mathcal{G}_1 \subset B$ be a finite subset, and let $\mathcal{P} = \{[1_B], [Q_1], [Q_2], [Q_3], [Q_4], [Q_5], [Q_6], [Q_7], [P_{\theta_{12}}(\mathfrak{u}_2, \mathfrak{u}_1, \mathfrak{w})], [P_{\theta_{12}}(\mathrm{e}^{\pi \mathrm{i}\theta_{23}}\mathfrak{u}_2, \mathrm{e}^{\pi \mathrm{i}\theta_{13}}\mathfrak{u}_1, \mathfrak{u}_3\mathfrak{w})], [P_{\theta_{13}}(\mathfrak{u}_3, \mathfrak{u}_1, \mathfrak{w})], [P_{\theta_{23}}(\mathfrak{u}_3, \mathfrak{u}_2, \mathfrak{w})]\} \subset K_0(B)$ be a finite subset required by Theorem 5.2.

By applying Lemma 5.1, we can choose finite subset $\mathcal{G}_2 \subset B$, $0 < \varepsilon_0 < \frac{\varepsilon}{2}$ and $\delta_2 > 0$, so that whenever u_1, u_2, u_3 and w are unitaries in A and $L : B \to A$ is a $\mathcal{G}_2$-δ_2-multiplicative completely positive contractive linear map such that

$$\|L(\mathfrak{u}_j) - u_j\| < \varepsilon_0 \quad \text{for } j = 1,2,3 \quad \text{and} \quad \|L(\mathfrak{w}) - w\| < \varepsilon_0,$$

then

$$[L]([1]) = [1_A], \quad [L]([Q_j]) = [\widetilde{Q}_j] \quad \text{for } j = 1, \cdots, 7, \tag{5.2}$$

$$[L]([P_{\theta_{12}}(\mathfrak{u}_2, \mathfrak{u}_1, \mathfrak{w})]) = [R_{\theta_{12}}(u_2, u_1, w)], \tag{5.3}$$

$$[L]([P_{\theta_{12}}(\mathrm{e}^{\pi \mathrm{i}\theta_{23}}\mathfrak{u}_2, \mathrm{e}^{\pi \mathrm{i}\theta_{13}}\mathfrak{u}_1, \mathfrak{u}_3\mathfrak{w})]) = [R_{\theta_{12}}(\mathrm{e}^{\pi \mathrm{i}\theta_{23}}u_2, \mathrm{e}^{\pi \mathrm{i}\theta_{13}}u_1, u_3 w)], \tag{5.4}$$

$$[L]([P_{\theta_{13}}(\mathfrak{u}_3,\mathfrak{u}_1,\mathfrak{w})])=[R_{\theta_{13}}(u_3,u_1,w)],\quad [L]([P_{\theta_{23}}(\mathfrak{u}_3,\mathfrak{u}_2,\mathfrak{w})])=[R_{\theta_{23}}(u_3,u_2,w)]. \tag{5.5}$$

Let δ_0 be chosen as in Proposition 3.3 (select the smallest δ_0 according to $\theta_{12},\theta_{13},\theta_{23}$). Let $\mathcal{G}=\mathcal{G}_1\cup\mathcal{G}_2$ and $\delta_3=\min\{\delta_0,\delta_1,\delta_2\}$. Find a positive number $\delta\le\delta_3$ according to $\mathcal{G}$, δ_3 (in place of η) and $\varepsilon_0>0$ (in place of ε) as in Proposition 5.1.

Now suppose that A is a unital C^*-algebra with the cancellation property and strict comparison. Let $u_1,u_2,u_3,w\in A$ be unitaries such that

(1) $\|u_ku_j-\mathrm{e}^{2\pi\mathrm{i}\theta_{jk}}u_ju_k\|<\delta$, $wu_jw^{-1}=u_j^{-1}$ and $w^2=1_A$ for $j,k=1,2,3$,

(2) $\tau(aw)=0$ and $\tau((u_ku_ju_k^*u_j^*)^n)=\mathrm{e}^{2\pi\mathrm{i}n\theta_{jk}}$ for all $n\in\mathbb{N}$, all $a\in C^*(u,v)$, all $\tau\in T(A)$ and $j,k=1,2,3$.

Define $\kappa:K_0(B)\to K_0(A)$ by

$$\begin{aligned}
&\kappa([1])=[1_A],\quad \kappa([Q_j])=[\widetilde{Q}_j]\quad\text{for } j=1,\cdots,7,\\
&\kappa([P_{\theta_{12}}(\mathfrak{u}_2,\mathfrak{u}_1,\mathfrak{w})])=[R_{\theta_{12}}(u_2,u_1,w)],\\
&\kappa([P_{\theta_{12}}(\mathrm{e}^{\pi\mathrm{i}\theta_{23}}\mathfrak{u}_2,\mathrm{e}^{\pi\mathrm{i}\theta_{13}}\mathfrak{u}_1,\mathfrak{u}_3\mathfrak{w})])=[R_{\theta_{12}}(\mathrm{e}^{\pi\mathrm{i}\theta_{23}}u_2,\mathrm{e}^{\pi\mathrm{i}\theta_{13}}u_1,u_3w)],\\
&\kappa([P_{\theta_{13}}(\mathfrak{u}_3,\mathfrak{u}_1,\mathfrak{w})])=[R_{\theta_{13}}(u_3,u_1,w)],\quad \kappa([P_{\theta_{23}}(\mathfrak{u}_3,\mathfrak{u}_2,\mathfrak{w})])=[R_{\theta_{23}}(u_3,u_2,w)].
\end{aligned}$$

We claim that this is a positive homomorphism. Indeed, let $[p]\in K_0(B)$ be a positive element. Then $(\tau_B)_*([p])>0$. There are integers n_j, $j=1,\cdots,12$ such that

$$\begin{aligned}
[p]=\;&n_1[1_B]+n_2[Q_1]+n_3[Q_2]+n_4[Q_3]+n_5[Q_4]+n_6[Q_5]+n_7[Q_6]+n_8[Q_7]\\
&+n_9[P_{\theta_{12}}(\mathfrak{u}_2,\mathfrak{u}_1,\mathfrak{w})]+n_{10}[P_{\theta_{12}}(\mathrm{e}^{\pi\mathrm{i}\theta_{23}}\mathfrak{u}_2,\mathrm{e}^{\pi\mathrm{i}\theta_{13}}\mathfrak{u}_1,\mathfrak{u}_3\mathfrak{w})]+n_{11}[P_{\theta_{13}}(\mathfrak{u}_3,\mathfrak{u}_1,\mathfrak{w})]\\
&+n_{12}[P_{\theta_{23}}(\mathfrak{u}_3,\mathfrak{u}_2,\mathfrak{w})].
\end{aligned}$$

It follows from Lemma 3.4 that

$$\begin{aligned}
\tau_*([R_{\theta_{12}}(u_2,u_1,w)])&=\frac{1}{4\pi\mathrm{i}}\tau(\log_{\theta_{12}}(u_2u_1u_2^*u_1^*))=\frac{\theta_{12}}{2},\\
\tau_*([R_{\theta_{12}}(\mathrm{e}^{\pi\mathrm{i}\theta_{23}}u_2,\mathrm{e}^{\pi\mathrm{i}\theta_{13}}u_1,u_3w)])&=\frac{1}{4\pi\mathrm{i}}\tau(\log_{\theta_{12}}(u_2u_1u_2^*u_1^*))=\frac{\theta_{12}}{2},\\
\tau_*([R_{\theta_{13}}(u_3,u_1,w)])&=\frac{1}{4\pi\mathrm{i}}\tau(\log_{\theta_{13}}(u_3u_1u_3^*u_1^*))=\frac{\theta_{13}}{2},\\
\tau_*([R_{\theta_{23}}(u_3,u_2,w)])&=\frac{1}{4\pi\mathrm{i}}\tau(\log_{\theta_{23}}(u_3u_2u_3^*u_2^*))=\frac{\theta_{23}}{2}
\end{aligned}$$

for all $\tau\in T(A)$. By Lemma 4.2, $\tau_*([\widetilde{Q}_4])=\tau_*([\widetilde{Q}_6])=\tau_*([\widetilde{Q}_7])=\frac{1}{2}$ for all $\tau\in T(A)$. By the assumption that $\tau(aw)=0$ for all $a\in C^*(u_1,u_2,u_3)$ and all $\tau\in T(A)$, we have

$$\tau_*([\widetilde{Q}_1])=\tau_*([\widetilde{Q}_2])=\tau_*([\widetilde{Q}_3])=\tau_*([\widetilde{Q}_5])=\frac{1}{2}$$

for all $\tau\in T(A)$.

Now for any $\tau\in T(A)$, we can compute that

$$\begin{aligned}
\tau_*(\kappa([p])=\;&n_1\tau_*([1_A])+n_2\tau_*([\widetilde{Q}_1])+n_3\tau_*([\widetilde{Q}_2])+n_4\tau_*([\widetilde{Q}_3])+n_5\tau_*([\widetilde{Q}_4])\\
&+n_6\tau_*([\widetilde{Q}_5])+n_7\tau_*([\widetilde{Q}_6])+n_8\tau_*([\widetilde{Q}_7])+n_9\tau_*([R_{\theta_{12}}(u_2,u_1,w)])\\
&+n_{10}\tau_*([R_{\theta_{12}}(\mathrm{e}^{\pi\mathrm{i}\theta_{23}}u_2,\mathrm{e}^{\pi\mathrm{i}\theta_{13}}u_1,u_3w)])+n_{11}\tau_*([R_{\theta_{13}}(u_3,u_1,w)])\\
&+n_{12}\tau_*([R_{\theta_{23}}(u_3,u_2,w)])
\end{aligned}$$

$$\begin{aligned}
&= n_1 \cdot 1 + n_2 \cdot \frac{1}{2} + n_3 \cdot \frac{1}{2} + n_4 \cdot \frac{1}{2} + n_5 \cdot \frac{1}{2} + n_6 \cdot \frac{1}{2} + n_7 \cdot \frac{1}{2} + n_8 \cdot \frac{1}{2} \\
&\quad + n_9 \cdot \frac{\theta_{12}}{2} + n_{10} \cdot \frac{\theta_{12}}{2} + n_{11} \cdot \frac{\theta_{13}}{2} + n_{12} \cdot \frac{\theta_{23}}{2} \\
&= n_1(\tau_B)_*([1_B]) + n_2(\tau_B)_*([Q_1]) + n_3(\tau_B)_*([Q_2]) + n_4(\tau_B)_*([Q_3]) \\
&\quad + n_5(\tau_B)_*([Q_4]) + n_6(\tau_B)_*([Q_5]) + n_7(\tau_B)_*([Q_6]) + n_8(\tau_B)_*([Q_7]) \\
&\quad + n_9(\tau_B)_*([P_{\theta_{12}}(\mathfrak{u}_2, \mathfrak{u}_1, \mathfrak{w})]) + n_{10}(\tau_B)_*([P_{\theta_{12}}(\mathrm{e}^{\pi \mathrm{i}\theta_{23}}\mathfrak{u}_2, \mathrm{e}^{\pi \mathrm{i}\theta_{13}}\mathfrak{u}_1, \mathfrak{u}_3\mathfrak{w})]) \\
&\quad + n_{11}(\tau_B)_*([P_{\theta_{13}}(\mathfrak{u}_3, \mathfrak{u}_1, \mathfrak{w})]) + n_{12}(\tau_B)_*([P_{\theta_{23}}(\mathfrak{u}_3, \mathfrak{u}_2, \mathfrak{w})]) \\
&= (\tau_B)_*([p]) > 0.
\end{aligned}$$

Since A has strict comparison, this shows that $\kappa([p])$ is positive. Therefore, by Theorem 5.1 there is a unital homomorphism $h : B \to A$ such that

$$h_{*0} = \kappa. \tag{5.6}$$

By Proposition 5.1, there is a unital $\mathcal{G}$-δ_3-multiplicative c.p.c. map $L : B \to A$ such that

$$\|L(\mathfrak{u}_j) - u_j\| < \varepsilon_0 \quad \text{for} \quad j = 1, 2, 3 \quad \text{and} \quad \|L(\mathfrak{w}) - w\| < \varepsilon_0. \tag{5.7}$$

It follows from (5.2) and (5.4)–(5.6) that $[h]|_{K_0(B)} = [L]|_{K_0(B)}$. Therefore, by Theorem 5.2 there exists a unitary $s \in A$ such that

$$\|s^* h(\mathfrak{u}_j)s - L(\mathfrak{u}_j)\| < \frac{\varepsilon}{2} \quad \text{for } j = 1, 2, 3 \quad \text{and} \quad \|s^* h(\mathfrak{w})s - L(\mathfrak{w})\| < \frac{\varepsilon}{2}. \tag{5.8}$$

Let

$$\widetilde{u}_j = s^* h(\mathfrak{u}_j)s \quad \text{for } j = 1, 2, 3 \quad \text{and} \quad \widetilde{w} = s^* h(\mathfrak{w})s.$$

Then, since h is a homomorphism,

$$\widetilde{u}_k \widetilde{u}_j = \mathrm{e}^{2\pi \mathrm{i}\theta_{jk}} \widetilde{u}_j \widetilde{u}_k, \quad \widetilde{w}\widetilde{u}_j \widetilde{w}^{-1} = \widetilde{u}_j^{-1} \quad \text{and} \quad \widetilde{w}^2 = 1_A \quad \text{for } j, k = 1, 2, 3.$$

By (5.7)–(5.8), we have

$$\|u_j - \widetilde{u}_j\| < \varepsilon \quad \text{for } j = 1, 2, 3 \quad \text{and } \|w - \widetilde{w}\| < \varepsilon.$$

Remark 5.1 The condition (2) of Theorem 5.3 is only a sufficient condition to get the conclusion. We will study the necessary and sufficient condition in the subsequent paper.

Remark 5.2 It is natural to ask what happens if there are $n+1$ unitary elements $u_1, u_2, \cdots, u_n, w$ in A for $n \geq 4$ and $\|u_k u_j - \mathrm{e}^{2\pi \mathrm{i}\theta_{jk}} u_j u_k\| < \delta,\ w u_j w^{-1},\ w^2 = 1_A$ for $j, k = 1, 2, \cdots, n$ as in Theorem 5.3. Notice that when $\Theta \in \mathcal{T}_n$ for $n \geq 4$, the generators of $K_0(A_\Theta \rtimes_\alpha \mathbb{Z}_2)$ are more complicated, the conditions of our Theorem 5.3 seem to be insufficient to get the conclusion in the case $n \geq 4$.

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