Chin. Ann. Math. Ser. B
44(4), 2023, 599–614
DOI: 10.1007/s11401-023-0034-9

Chinese Annals of Mathematics, Series B

On Energy Gap Phenomena of the Whitney Spheres in $\mathbb{C}^n$ or $\mathbb{CP}^n$ *

Yong LUO[1] Liuyang ZHANG[1]

Abstract Zhang (2021), Luo and Yin (2022) initiated the study of Lagrangian submanifolds satisfying $\nabla^*T = 0$ or $\nabla^*\nabla^*T = 0$ in $\mathbb{C}^n$ or $\mathbb{CP}^n$, where $T = \nabla^*\widetilde{h}$ and $\widetilde{h}$ is the Lagrangian trace-free second fundamental form. They proved several rigidity theorems for Lagrangian surfaces satisfying $\nabla^*T = 0$ or $\nabla^*\nabla^*T = 0$ in $\mathbb{C}^2$ under proper small energy assumption and gave new characterization of the Whitney spheres in $\mathbb{C}^2$. In this paper, the authors extend these results to Lagrangian submanifolds in $\mathbb{C}^n$ of dimension $n \geq 3$ and to Lagrangian submanifolds in $\mathbb{CP}^n$.

Keywords Lagrangian submanifolds, Whitney spheres, Gap theorem, Conformal maslov form
2000 MR Subject Classification 53C24, 53C42

1 Introduction

Assume that $N^n(4c)$ is the standard complex space form with standard complex structure J, Kähler form ω and metric $\langle,\rangle$, i.e., $N^n(0) = \mathbb{C}^n$ and $N^n(4) = \mathbb{CP}^n$. A real n-dimensional submanifold of $N^n(4c)$ is a Lagrangian submanifold if J is an isometric map between its tangent bundle and normal bundle. The most canonical and important examples of Lagrangian submanifolds of $\mathbb{C}^n$ or $\mathbb{CP}^n$ are the Lagrangian subspaces and Whitney spheres. The Whitney spheres in $\mathbb{C}^n$ are defined by the following example (cf. [28]).

Example 1.1

$$\begin{aligned}\phi_{r,A} : \mathbb{S}^n &\to \mathbb{C}^n\\ (x_1, \cdots, x_{n+1}) &\mapsto \frac{r}{1 + x_{n+1}^2}(x_1, x_1x_{n+1}, \cdots, x_n, x_nx_{n+1}) + A,\end{aligned}$$

where $\mathbb{S}^n = \{(x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} | x_1^2 + \cdots + x_{n+1}^2 = 1\}$, r is a positive number and A is a vector of $\mathbb{C}^n$.

The Whitney spheres in $\mathbb{CP}^n$ are defined by the following example (cf. [8, 11]).

Manuscript received September 1, 2021, Revised September 30, 2022.
[1]Mathematical Science Research Center, Chongqing University of Technology, Chongqing 400054, China.
E-mail: yongluo-math@cqut.edu.cn zhangliuyang@cqut.edu.cn
*This work was supported by the National Natural Science Foundation of China (No. 12271069), the Natural Science Foundation of Chongqing (No. cstc2021jcyj-msxmX0443), the Chongqing "Zhitongche" foundation for doctors (No. CSTB2022BSXM-JCX0101) and the Scientific and Technological Research Program of Chongqing Municipal Education Commission (No. KJQN202201138).

Example 1.2

$$\phi_\theta : \mathbb{S}^n \to \mathbb{CP}^n, \quad \theta > 0$$
$$(x_1, \cdots, x_{n+1}) \mapsto \Big[\Big(\frac{(x_1, \cdots, x_n)}{\cosh\theta + \mathrm{i}\sinh\theta x_{n+1}}, \frac{\sinh\theta\cosh\theta(1 + x_{n+1}^2) + \mathrm{i}x_{n+1}}{\cosh^2\theta + \sinh^2\theta x_{n+1}^2}\Big)\Big],$$

where $\mathbb{S}^n = \{(x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \cdots + x_{n+1}^2 = 1\}$.

The Lagrangian subspaces and Whitney spheres $\phi_{r,A}$ in $\mathbb{C}^n$ or the real projective space $\mathbb{RP}^n$ and Whitney spheres ϕ_θ in $\mathbb{CP}^n$ play a similar role with that of totally umbilical hypersurfaces in a real Euclidean space $\mathbb{R}^{n+1}$ or in the unit sphere $\mathbb{S}^{n+1}$, and they are locally characterized by vanishing of the following so called Lagrangian trace free second fundamental form the following example (cf. [7–8, 11, 24]).

$$\widetilde{h}(V, W) := h(V, W) - \frac{n}{n + 2}\{\langle V, W\rangle H + \langle JV, H\rangle JW + \langle JW, H\rangle JV\}, \tag{1.1}$$

where h denotes the second fundamental form and $H = \frac{1}{n}h$ denotes the mean curvature vector field.

Various characterizations of the Lagrangian subspaces, $\mathbb{RP}^n$ or Whitney spheres in $\mathbb{C}^n$ or $\mathbb{CP}^n$ were obtained in [2, 6–7, 9–10, 12–13, 18, 20, 24]. In particular, Castro, Montealegre, Ros and Urbano [6, 9, 24] introduced and studied Lagrangian submanifolds with conformal Maslov form in $\mathbb{C}^n$ or $\mathbb{CP}^n$, that is Lagrangian submanifolds in $\mathbb{C}^n$ or $\mathbb{CP}^n$ with the 2-form $T = 0$, where in local orthonormal basis

$$T_{ij} := \frac{1}{n}\sum_m \widetilde{h}^{m^*}_{ij,m} = \frac{1}{n + 2}\Big(nH^{i^*}_{,j} - \sum_m H^{m^*}_{,m}\, g_{ij}\Big). \tag{1.2}$$

They proved that the only compact(nonminimal) Lagrangian submanifolds in $\mathbb{C}^n$ or $\mathbb{CP}^n$ with conformal Maslov form (i.e., $T = 0$) and null first Betti number are the Whitney spheres. The Whitney spheres in $\mathbb{C}^n$ also play an important role in the study of Lagrangian mean curvature flow (cf. [5, 25]).

Recently, Zhang [32], Luo and Yin [21] initiated the study of Lagrangian submanifolds in $\mathbb{C}^n$ or $\mathbb{CP}^n$ satisfying $\nabla^*T = 0$ or $\nabla^*\nabla^*T = 0$. In particular, they proved the following results.

Theorem 1.1 (cf. [32]) *Assume that $\Sigma \hookrightarrow \mathbb{C}^2$ is a properly immersed complete Lagrangian surface satisfying $\nabla^*T = 0$. Then there exists a constant $\varepsilon_0 > 0$ such that if*

$$\int_\Sigma |\widetilde{h}|^2 \mathrm{d}\mu \le \varepsilon_0 \quad \textit{and} \quad \lim_{R\to+\infty} \frac{1}{R^2}\int_{\Sigma_R} |h|^2 \mathrm{d}\mu = 0,$$

where $\Sigma_R := \Sigma \cap B_R(0)$ and $B_R(0)$ denotes the ball centered at 0 in $\mathbb{C}^2$ with radius R, then Σ is either a Lagrangian plane or a 2-dimensional Whitney sphere.

Remark 1.1 Though it was assumed properness in the above theorem, we see from the proof in [32] that we only need assume that Σ is complete.

Theorem 1.2 (cf. [21]) *Assume that $\Sigma \hookrightarrow \mathbb{C}^2$ is a Lagrangian sphere satisfying $\nabla^*\nabla^*T = 0$. Then there exists a constant $\varepsilon_0 > 0$ such that if*

$$\int_\Sigma |\widetilde{h}|^2 \mathrm{d}\mu \le \varepsilon_0,$$

then Σ is a 2-dimensional Whitney sphere.

The aim of this paper is to extend the above results to higher dimensional Lagrangian submanifolds in $\mathbb{C}^n$ and to Lagrangian submanifolds in $\mathbb{CP}^n$. In fact we have the following theorem.

Theorem 1.3 *Assume that $M^n \hookrightarrow \mathbb{C}^n (n \geq 3)$ is a complete Lagrangian submanifold. We have*

(i) *if M^n satisfies $\nabla^* T = 0$, then there exists a constant $\varepsilon_0 > 0$ such that if*

$$\int_M |\widetilde{h}|^n \mathrm{d}\mu \leq \varepsilon_0 \quad \text{and} \quad \lim_{R \to +\infty} \frac{1}{R^2} \int_{M_R} |h|^2 \mathrm{d}\mu = 0,$$

where M_R denotes the geodesic ball in M^n with radius R, then M^n is either a Lagrangian subspace or a Whitney sphere;

(ii) *if M^n is a Lagrangian sphere satisfying $\nabla^* \nabla^* T = 0$, then there exists a constant $\varepsilon_0 > 0$ such that if*

$$\int_M |\widetilde{h}|^n \mathrm{d}\mu \leq \varepsilon_0,$$

then M^n is a Whitney sphere.

We would like to point out that compared with the 2-dimensional case, the proof of Theorem 1.3 is much more complicated. Firstly, in the 2-dimensional case we just need to test over a simple Simons' type identity, but in the case of dimension $n \geq 3$ we need to estimate the nonlinear terms in a much more complicated Simons' type equality to get a Simons' type inequality (cf. (3.17)) and then test over it. Secondly, in the higher dimensional case we need to adapt the original Michael-Simon inequality to get (4.4) and use it to absorb the "bad term" at the right hand of (3.17).

Similarly, for Lagrangian submanfiods in $\mathbb{CP}^n$, we have the following theorem.

Theorem 1.4 *Assume that $M^n \hookrightarrow \mathbb{CP}^n (n \geq 2)$ is a complete Lagrangian submanifold. We have*

(i) *if M^n satisfies $\nabla^* T = 0$, then there exists a constant $\varepsilon_0 > 0$ such that if*

$$\int_M |\widetilde{h}|^n \mathrm{d}\mu \leq \varepsilon_0 \quad \text{and} \quad \lim_{R \to +\infty} \frac{1}{R^2} \int_{M_R} |h|^2 \mathrm{d}\mu = 0,$$

where M_R denotes the geodesic ball in M^n with radius R, then M^n is the real projective space $\mathbb{RP}^n$ or a Whitney sphere;

(ii) *if M^n is a Lagrangian sphere satisfying $\nabla^* \nabla^* T = 0$, then there exists a constant $\varepsilon_0 > 0$ such that if*

$$\int_M |\widetilde{h}|^n \mathrm{d}\mu \leq \varepsilon_0,$$

then M^n is the real projective space $\mathbb{RP}^n$ or a Whitney sphere.

Note that similar $L^{\frac{n}{2}}$ pinching theorems for minimal submanifolds in a unit sphere were initiated by Shen [26], and later investigated by Wang [27], Lin and Xia [19]. $L^{\frac{n}{2}}$ pinching theorems for minimal submanifolds in a Euclidean space was investigated by Ni [23] and Yun [31]. Generalizations of $L^{\frac{n}{2}}$ pinching theorems to submanifolds with parallel mean curvature vector field in a sphere or in a Euclidean space were obtained by Xu [29] and Xu and Gu [30].

Our results could be seen as extensions of their results to more general submanifolds in the Lagrangian setting.

The rest of this paper is organized as follows. In Section 2 we give some preliminaries on Lagrangian submanifolds in $N^n(4c)$. In Section 3 we prove a Simons' type inequality for Lagrangian submanifolds in $N^n(4c)$, which plays a crucial role in the proof of Theorems 1.3–1.4. Theorem 1.3 is proved in Section 4 and Theorem 1.4 is proved in Section 5.

2 Preliminaries

In this section we collect some basic formulas and results of the Lagrangian submanifolds in a complex space form (cf. [1, 3]).

Let $N^n(4c)$ be a complete, simply connected, n-dimensional Kähler manifold with constant holomorphic sectional curvature $4c$. Let M^n be an n-dimensional Lagrangian submanifolds in $N^n(4c)$. We denote also by g the metric on M^n. Let ∇ (resp. $\overline{\nabla}$) be the Levi-Civita connection of M^n (resp. $N^n(4c)$). The Gauss and Weingarten formulas of $M^n \hookrightarrow N^n(4c)$ are given, respectively, by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y) \quad \text{and} \quad \overline{\nabla}_X V = -A_V X + \nabla^{\perp}_X V, \tag{2.1}$$

where $X, Y \in TM^n$ are tangent vector fields, $V \in T^{\perp}M^n$ is a normal vector field; $\nabla^{\perp}$ is the normal connection in the normal bundle $T^{\perp}M^n$; h is the second fundamental form and A_V is the shape operator with respect to V. From (2.1), we easily get

$$\langle h(X,Y), V\rangle = \langle A_V X, Y)\rangle. \tag{2.2}$$

The mean curvature vector H of M^n is defined by $H = \frac{1}{n}\mathrm{trace}\, h$.

For Lagrangian submanifolds in $N^n(4c)$, we have

$$\nabla^{\perp}_X JY = J\nabla_X Y, \tag{2.3}$$

$$A_{JX} Y = -Jh(X,Y) = A_{JY} X. \tag{2.4}$$

The above formulas immediately imply that $g(h(X,Y), JZ)$ is totally symmetric.

To utilize the moving frame method, we will use the following range convention of indices:

$$i, j, k, l, m, p, s = 1, \cdots, n; \quad i^* = i + n \text{ etc..}$$

Now, we choose a local adapted Lagrangian frame $\{e_1, \cdots, e_n, e_{1^*}, \cdots, e_{n^*}\}$ in $N^n(4c)$ in such a way that, restricted to M^n, $\{e_1, \cdots, e_n\}$ is an orthonormal frame of M^n, and $\{e_{1^*} = Je_1, \cdots, e_{n^*} = Je_n\}$ is a orthonormal frame of $M^n \hookrightarrow N^n(4c)$. Let $\{\theta_1, \cdots, \theta_n\}$ be the dual frame of $\{e_1, \cdots, e_n\}$. Let θ_{ij} and $\theta_{i^*j^*}$ denote the connection 1-forms of TM^n and $T^{\perp}M^n$, respectively.

Put $h^{k^*}_{ij} = g(h(e_i, e_j), Je_k)$. It is easily seen that

$$h^{k^*}_{ij} = h^{j^*}_{ik} = h^{i^*}_{jk}, \quad \forall\, i, j, k. \tag{2.5}$$

Denote by $R_{ijkl} := g(R(e_i, e_j)e_l, e_k)$ and $R_{ijk^*l^*} := g(R(e_i, e_j)e_{l^*}, e_{k^*})$ the components of the curvature tensors of ∇ and $\nabla^\perp$ with respect to the adapted Lagrangian frame, respectively. Then, we get the Gauss, Ricci and Codazzi equations, respectively,

$$R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \sum_m (h_{ik}^{m^*} h_{jl}^{m^*} - h_{il}^{m^*} h_{jk}^{m^*}), \tag{2.6}$$

$$R_{ijk^*l^*} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \sum_m (h_{ik}^{m^*} h_{jl}^{m^*} - h_{il}^{m^*} h_{jk}^{m^*}), \tag{2.7}$$

$$h_{ij,k}^{m^*} = h_{ik,j}^{m^*}, \tag{2.8}$$

where $h_{ij,k}^{m^*}$ is the components of the covariant differentiation of h, defined by

$$\sum_{l=1}^n h_{ij,l}^{m^*}\theta_l := \mathrm{d}h_{ij}^{m^*} + \sum_{l=1}^n h_{il}^{m^*}\theta_{lj} + \sum_{l=1}^n h_{jl}^{m^*}\theta_{li} + \sum_{l=1}^n h_{ij}^{l^*}\theta_{l^*m^*}. \tag{2.9}$$

Then from (2.5) and (2.8), we have

$$h_{ij,k}^{m^*} = h_{jk,m}^{i^*} = h_{km,i}^{j^*} = h_{mi,j}^{k^*}. \tag{2.10}$$

We also have Ricci identity

$$h_{ij,lp}^{m^*} - h_{ij,pl}^{m^*} = \sum_{k=1}^n h_{kj}^{m^*} R_{kilp} + \sum_{k=1}^n h_{ik}^{m^*} R_{kjlp} + \sum_{k=1}^n h_{ij}^{k^*} R_{k^*m^*lp}, \tag{2.11}$$

where $h_{ij,lp}^{m^*}$ is defined by

$$\sum_p h_{ij,lp}^{m^*}\theta_p = \mathrm{d}h_{ij,l}^{m^*} + \sum_p h_{pj,l}^{m^*}\theta_{pi} + \sum_p h_{ip,l}^{m^*}\theta_{pj} + \sum_p h_{ij,p}^{m^*}\theta_{pl} + \sum_p h_{ij,l}^{p^*}\theta_{p^*m^*}.$$

The mean curvature vector H of $M^n \hookrightarrow N^n(4c)$ is

$$H = \frac{1}{n}\sum_{i=1}^n h(e_i, e_i) = \sum_{k=1}^n H^{k^*} e_{k^*}, \quad H^{k^*} = \frac{1}{n}\sum_i h_{ii}^{k^*}.$$

Letting $i = j$ in (2.9) and carrying out summation over i, we have

$$H_{,l}^{k^*}\theta_l = \mathrm{d}H^{k^*} + \sum_l H^{l^*}\theta_{l^*k^*},$$

and we further have

$$H_{,i}^{k^*} = H_{,k}^{i^*} \tag{2.12}$$

for any i, k.

3 A Simons' Type Inequality

In this section, inspired by Chao and Dong [10], we will derive a new Simons' type inequality for Lagrangian submanifolds in $N^n(4c)$.

We assume that $M^n \hookrightarrow N^n(4c)$ is a Lagrangian submanifold and $n \geq 2$, where $N^n(4c)$ is the the standard complex space form of constant holomorphic sectional curvature $4c$ with standard complex structure J, Kähler form ω and metric $\langle,\rangle$.

Firstly, we define a trace-free tensor $\widetilde{h}(X,Y)$ defined by

$$\widetilde{h}(X,Y) = h(X,Y) - \frac{n}{n+2}\{\langle X,Y\rangle H + \langle JX,H\rangle JY + \langle JY,H\rangle JX\} \tag{3.1}$$

for any tangent vector fields X, Y on M^n.

With respect to Lagrangian frame $\{e_1,\cdots,e_n,e_{1^*},\cdots,e_{n^*}\}$ in $N^n(4c)$, we have

$$\begin{aligned}\widetilde{h}_{ij}^{m^*} &= h_{ij}^{m^*} - \frac{n}{n+2}\big(H^{m^*}\delta_{ij} + H^{i^*}\delta_{jm} + H^{j^*}\delta_{im}\big)\\ &= h_{ij}^{m^*} - c_{ij}^{m^*},\end{aligned} \tag{3.2}$$

where $c_{ij}^{m^*} = \frac{n}{n+2}\{H^{m^*}\delta_{ij} + H^{i^*}\delta_{jm} + H^{j^*}\delta_{im}\}$.

The first covariant derivatives of $\widetilde{h}_{ij}^{m^*}$ are defined by

$$\sum_{l=1}^n \widetilde{h}_{ij,l}^{m^*}\theta_l := \mathrm{d}\widetilde{h}_{ij}^{m^*} + \sum_{l=1}^n \widetilde{h}_{il}^{m^*}\theta_{lj} + \sum_{l=1}^n \widetilde{h}_{jl}^{m^*}\theta_{li} + \sum_{l=1}^n \widetilde{h}_{ij}^{l^*}\theta_{l^*m^*}. \tag{3.3}$$

The second covariant derivatives of $\widetilde{h}_{ij}^{m}$ are defined by

$$\sum_{l=1}^n \widetilde{h}_{ij,kl}^{m^*}\theta_l := \mathrm{d}\widetilde{h}_{ij,k}^{m^*} + \sum_{l=1}^n \widetilde{h}_{lj,k}^{m^*}\theta_{li} + \sum_{l=1}^n \widetilde{h}_{il,k}^{m^*}\theta_{lj} + \sum_{l=1}^n \widetilde{h}_{ij,l}^{m^*}\theta_{lk} + \sum_{l=1}^n \widetilde{h}_{ij,k}^{l^*}\theta_{l^*m^*}. \tag{3.4}$$

On the other hand, we have the following Ricci identities

$$\widetilde{h}_{ij,kp}^{m^*} - \widetilde{h}_{ij,pk}^{m^*} = \sum_l \widetilde{h}_{lj}^{m^*}R_{likp} + \sum_l \widetilde{h}_{il}^{m^*}R_{ljkp} + \sum_l \widetilde{h}_{ij}^{l^*}R_{l^*m^*kp}. \tag{3.5}$$

The following proposition links those geometric quantities together.

Lemma 3.1 *Let $M^n \hookrightarrow N^n(4c)$ be a Lagrangian submanifold. Then the Lagrangian trace-free second fundamental form $\widetilde{h}$ satisfies*

$$|\widetilde{h}|^2 = |h|^2 - \frac{3n^2}{n+2}|H|^2, \tag{3.6}$$

$$\sum_m \widetilde{h}_{ij,m}^{m^*} = \frac{n}{n+2}(nH_{,j}^{i^*} - \mathrm{div}JH\, g_{ij}). \tag{3.7}$$

Proof (3.6) and (3.7) can be immediately obtained from (3.2).

Definition 3.1 (cf. [6–7, 24]) *We define a $(0,2)$-tensor T in local orthonormal basis as follows*:

$$T_{ij} = \frac{1}{n}\sum_m \widetilde{h}_{ij,m}^{m^*} = \frac{1}{n+2}\Big(nH_{,j}^{i^*} - \sum_m H_{,m}^{m^*}\, g_{ij}\Big). \tag{3.8}$$

Remark 3.1 T is a trace-free tensor and symmetric. $T = 0$ if and only if JH is a conformal vector field.

In the following we will derive a Simons' type identity for $\Delta|\widetilde{h}|^2$. First we have the following lemma.

Lemma 3.2 *Let $M^n \hookrightarrow N^n(4c)$ be a Lagrangian immersion. Then*

$$\sum_{ijmk}\widetilde{h}_{ij}^{m^*}\widetilde{h}_{ij,kk}^{m^*}=(n+2)\langle\widetilde{h},\nabla T\rangle+\sum_{i,j,m,k,l}\widetilde{h}_{ij}^{m^*}(\widetilde{h}_{lk}^{m^*}R_{lijk}+\widetilde{h}_{il}^{m^*}R_{lkjk}+\widetilde{h}_{ik}^{l^*}R_{l^*m^*jk}). \tag{3.9}$$

Proof By using the Codazzi equation (2.8) and (3.2), the definition of $\widetilde{h}$ under local coordinates is just

$$\widetilde{h}_{ij,k}^{m^*}=\widetilde{h}_{ik,j}^{m^*}+\frac{n}{n+2}(\delta_{ik}H_{,j}^{m^*}+\delta_{km}H_{,j}^{i^*}-\delta_{ij}H_{,k}^{m^*}-\delta_{jm}H_{,k}^{i^*}). \tag{3.10}$$

With the help of Ricci identity (3.5), (3.8) and (3.10), we have

$$\begin{aligned}\sum_k\widetilde{h}_{ij,kk}^{m^*}&=\sum_k\widetilde{h}_{ik,jk}^{m^*}+\sum_k\frac{n}{n+2}(\delta_{ik}H_{,jk}^{m^*}+\delta_{km}H_{,jk}^{i}-\delta_{ij}H_{,kk}^{m^*}-\delta_{jm}H_{,kk}^{i^*})\\&=\sum_k\widetilde{h}_{ik,kj}^{m^*}+\sum_{k,l}\widetilde{h}_{lk}^{m^*}R_{lijk}+\sum_{k,l}\widetilde{h}_{il}^{m^*}R_{lkjk}+\sum_{k,l}\widetilde{h}_{ik}^{l^*}R_{l^*m^*jk}\\&\quad+\sum_k\frac{n}{n+2}(\delta_{ik}H_{,jk}^{m^*}+\delta_{km}H_{,jk}^{i}-\delta_{ij}H_{,kk}^{m^*}-\delta_{jm}H_{,kk}^{i^*})\\&=\sum_k\widetilde{h}_{kk,ij}^{m^*}+\sum_{k,l}\widetilde{h}_{lk}^{m^*}R_{lijk}+\sum_{k,l}\widetilde{h}_{il}^{m^*}R_{lkjk}+\sum_{k,l}\widetilde{h}_{ik}^{l^*}R_{l^*m^*jk}\\&\quad+\sum_k\frac{n}{n+2}(\delta_{ik}H_{,jk}^{m^*}+\delta_{km}H_{,jk}^{i^*}-\delta_{ij}H_{,kk}^{m^*}-\delta_{jm}H_{,kk}^{i^*})+nT_{im,j}.\end{aligned} \tag{3.11}$$

Then, by using (3.8) and the fact that $\widetilde{h}$ is trace free and tri-symmetric, we have

$$\begin{aligned}\sum_{i,j,m,k}\widetilde{h}_{ij}^{m^*}\widetilde{h}_{ij,kk}^{m^*}&=\sum_{i,j,m,k}\widetilde{h}_{ij}^{m^*}[\widetilde{h}_{lk}^{m^*}R_{lijk}+\widetilde{h}_{il}^{m^*}R_{lkjk}+\widetilde{h}_{ik}^{l^*}R_{l^*m^*jk}]\\&\quad+\sum_{m,i,j}\widetilde{h}_{ij}^{m^*}[T_{mj,i}+T_{ij,m}]+n\sum_{m,i,j}\widetilde{h}_{ij}^{m^*}T_{im,j}\\&=\sum_{i,j,m,k}\widetilde{h}_{ij}^{m^*}[\widetilde{h}_{lk}^{m^*}R_{lijk}+\widetilde{h}_{il}^{m^*}R_{lkjk}+\widetilde{h}_{ik}^{l^*}R_{l^*m^*jk}]\\&\quad+(n+2)\sum_{m,i,j}\widetilde{h}_{ij}^{m^*}T_{ij,m}.\end{aligned}$$

Thus, we obtain the assertion.

Next, by using Lemma 3.2,

$$\begin{aligned}\frac{1}{2}\Delta|\widetilde{h}|^2&=|\nabla\widetilde{h}|^2+\sum_{ijmk}\widetilde{h}_{ij}^{m^*}\widetilde{h}_{ij,kk}^{m^*}\\&=|\nabla\widetilde{h}|^2+(n+2)\langle\widetilde{h},\nabla T\rangle\\&\quad+\underbrace{\sum_{i,j,k,m,l}\widetilde{h}_{ij}^{m^*}\widetilde{h}_{lk}^{m^*}R_{lijk}}_{I}+\underbrace{\sum_{i,j,k,m,l}\widetilde{h}_{ij}^{m^*}\widetilde{h}_{il}^{m^*}R_{lkjk}}_{II}+\underbrace{\sum_{i,j,k,m,l}\widetilde{h}_{ij}^{m^*}\widetilde{h}_{ik}^{l^*}R_{lmjk}}_{III}.\end{aligned} \tag{3.12}$$

Note that by the symmetry of $\widetilde{h}_{ij}^{k^*}$, $I = III$. Hence we only need to compute I and II. Direct computations show that

$$\begin{aligned}
I &= c \sum_{i,j,k,m,l} \widetilde{h}_{ij}^{m^*}\widetilde{h}_{kl}^{m^*}(\delta_{lj}\delta_{ik} - \delta_{lk}\delta_{ij}) + \sum_{i,j,k,m,l,t} \widetilde{h}_{ij}^{m^*}\widetilde{h}_{kl}^{m^*}(\widetilde{h}_{lj}^{t^*}\widetilde{h}_{ik}^{t^*} - \widetilde{h}_{lk}^{t^*}\widetilde{h}_{ij}^{t^*}) \\
&\quad + \sum_{i,j,k,m,l,t} \widetilde{h}_{ij}^{m^*}\widetilde{h}_{kl}^{m^*}(\widetilde{h}_{lj}^{t^*}c_{ik}^{t^*} + c_{lj}^{t^*}\widetilde{h}_{ik}^{t^*} - \widetilde{h}_{lk}^{t^*}c_{ij}^{t^*} - c_{lk}^{t^*}\widetilde{h}_{ij}^{t^*} + c_{lj}^{t^*}c_{ik}^{t^*} - c_{lk}^{t^*}c_{ij}^{t^*}) \\
&= c|\widetilde{h}|^2 + \frac{n^2}{(n+2)^2}|\widetilde{h}|^2|H|^2 + \frac{2n}{n+2}\sum_{j,k,l,m,t} \widetilde{h}_{jk}^{m^*}\widetilde{h}_{kl}^{m^*}\widetilde{h}_{lj}^{t^*}H^{t^*} \\
&\quad + \sum_{i,j,k,m,l,t} \widetilde{h}_{ij}^{m^*}\widetilde{h}_{kl}^{m^*}(\widetilde{h}_{lj}^{t^*}\widetilde{h}_{ik}^{t^*} - \widetilde{h}_{lk}^{t^*}\widetilde{h}_{ij}^{t^*}) + \frac{2n^2}{(n+2)^2}\sum_{i,j,k,m} \widetilde{h}_{ij}^{m^*}\widetilde{h}_{jk}^{m^*}H^{i^*}H^{k^*},
\end{aligned} \tag{3.13}$$

where in the second equality we used the following identities derived by direct computations

$$\begin{aligned}
\sum_{i,j,k,m,l,t} \widetilde{h}_{ij}^{m^*}\widetilde{h}_{kl}^{m^*}\widetilde{h}_{lj}^{t^*}c_{ik}^{t^*} &= \sum_{i,j,k,m,l,t} \widetilde{h}_{ij}^{m^*}\widetilde{h}_{kl}^{m^*}c_{lj}^{t^*}\widetilde{h}_{ik}^{t^*} = \frac{3n}{n+2}\sum_{j,k,l,m,t} \widetilde{h}_{jk}^{m^*}\widetilde{h}_{kl}^{m^*}\widetilde{h}_{lj}^{t^*}H^{t^*}, \\
\sum_{i,j,k,m,l,t} \widetilde{h}_{ij}^{m^*}\widetilde{h}_{kl}^{m^*}\widetilde{h}_{lk}^{t^*}c_{ij}^{t^*} &= \sum_{i,j,k,m,l,t} \widetilde{h}_{ij}^{m^*}\widetilde{h}_{kl}^{m^*}c_{lk}^{t^*}\widetilde{h}_{ij}^{t^*} = \frac{2n}{n+2}\sum_{j,k,l,m,t} \widetilde{h}_{jk}^{m^*}\widetilde{h}_{kl}^{m^*}\widetilde{h}_{lj}^{t^*}H^{t^*},
\end{aligned}$$

and since

$$\begin{aligned}
\sum_t c_{lj}^{t^*}c_{ik}^{t^*} &= \frac{n^2}{(n+2)^2}\sum_t ((\mathfrak{S}_{t,l,j}H^{t^*}\delta_{lj})(\mathfrak{S}_{t,i,k}H^{t^*}\delta_{ik})) \\
&= \frac{n^2}{(n+2)^2}(\mathfrak{S}_{i,j,k,l}H^{l^*}H^{i^*}\delta_{jk} + 2H^{l^*}H^{j^*}\delta_{ik} \\
&\quad + 2H^{i^*}H^{k^*}\delta_{jl} + |H|^2\delta_{ik}\delta_{jl}),
\end{aligned}$$

where $\mathfrak{S}$ stands for the cyclic sum, thus

$$\begin{aligned}
\sum_{i,j,k,m,l,t} \widetilde{h}_{ij}^{m^*}\widetilde{h}_{kl}^{m^*}c_{lj}^{t^*}c_{ik}^{t^*} &= \frac{n^2}{(n+2)^2}|\widetilde{h}|^2|H|^2 + \frac{6n^2}{(n+2)^2}\sum_{i,j,k,m} \widetilde{h}_{ij}^{m^*}\widetilde{h}_{jk}^{m^*}H^{i^*}H^{k^*}, \\
\sum_{i,j,k,m,l,t} \widetilde{h}_{ij}^{m^*}\widetilde{h}_{kl}^{m^*}c_{lk}^{t^*}c_{ij}^{t^*} &= \frac{4n^2}{(n+2)^2}\sum_{i,j,k,m} \widetilde{h}_{ij}^{m^*}\widetilde{h}_{jk}^{m^*}H^{i^*}H^{k^*}.
\end{aligned}$$

Similarly we have

$$\begin{aligned}
II &= \sum_{i,j,k,m,l} \widetilde{h}_{ij}^{m^*}\widetilde{h}_{il}^{m^*}\Big[c(\delta_{lj}\delta_{kk} - \delta_{lk}\delta_{jk}) + \sum_t (h_{lj}^{t^*}h_{kk}^{t^*} - h_{lk}^{t^*}h_{kj}^{t^*})\Big] \\
&= (n-1)c|\widetilde{h}|^2 + \sum_{i,j,k,m,l,t} \widetilde{h}_{ij}^{m^*}\widetilde{h}_{li}^{m^*}(n\widetilde{h}_{lj}^{t^*}H^{t^*} + nc_{lj}^{t^*}H^{t^*} \\
&\quad - \widetilde{h}_{lk}^{t^*}\widetilde{h}_{kj}^{t^*} - \widetilde{h}_{lk}^{t^*}c_{kj}^{t^*} - c_{lk}^{t^*}\widetilde{h}_{kj}^{t^*} - c_{lk}^{t^*}c_{kj}^{t^*}) \\
&= (n-1)c|\widetilde{h}|^2 + \frac{n^3}{(n+2)^2}|\widetilde{h}|^2|H|^2 + \frac{n^2-2n}{n+2}\sum_{ijlmt} \widetilde{h}_{ij}^{m^*}\widetilde{h}_{li}^{m^*}\widetilde{h}_{lj}^{t^*}H^{t^*} \\
&\quad + \frac{n^2(n-2)}{(n+2)^2}\sum_{ijml} \widetilde{h}_{ij}^{m^*}\widetilde{h}_{li}^{m^*}H^{j^*}H^{l^*} - \sum_{i,j,k,m,l,t} \widetilde{h}_{ij}^{m^*}\widetilde{h}_{li}^{m^*}\widetilde{h}_{lk}^{t^*}\widetilde{h}_{kj}^{t^*},
\end{aligned} \tag{3.14}$$

where in the third equality we used the following identities derived by direct computations

$$n\sum_{i,j,k,m,l,t}\widetilde{h}_{ij}^{m^*}\widetilde{h}_{li}^{m^*}c_{lj}^{t^*}H^{t^*}=\frac{n^2}{n+2}|\widetilde{h}|^2|H|^2+\frac{2n^2}{n+2}\sum_{ijml}\widetilde{h}_{ij}^{m^*}\widetilde{h}_{li}^{m^*}H^{j^*}H^{l^*},$$

$$\sum_{i,j,k,m,l,t}\widetilde{h}_{ij}^{m^*}\widetilde{h}_{li}^{m^*}\widetilde{h}_{lk}^{t^*}c_{kj}^{t^*}=\sum_{i,j,k,m,l,t}\widetilde{h}_{ij}^{m^*}\widetilde{h}_{li}^{m^*}\widetilde{h}_{kj}^{t^*}c_{lk}^{t^*}=\frac{2n}{n+2}\sum_{ijlmt}\widetilde{h}_{ij}^{m^*}\widetilde{h}_{li}^{m^*}\widetilde{h}_{lj}^{t^*}H^{t^*},$$

and since

$$\begin{aligned}\sum_t c_{lk}^{t^*}c_{jk}^{t^*}&=\frac{n^2}{(n+2)^2}\sum_t((\mathfrak{S}_{t,l,k}H^{t^*}\delta_{lk})(\mathfrak{S}_{t,j,k}H^{t^*}\delta_{jk}))\\&=\frac{n^2}{(n+2)^2}(\mathfrak{S}_{j,k,k,l}H^{l^*}H^{j^*}\delta_{kk}+2H^{l^*}H^{k^*}\delta_{jk}\\&\quad+2H^{j^*}H^{k^*}\delta_{kl}+|H|^2\delta_{jk}\delta_{kl}),\end{aligned}$$

thus

$$\sum_{i,j,k,m,l,t}\widetilde{h}_{ij}^{m^*}\widetilde{h}_{li}^{m^*}c_{lk}^{t^*}c_{kj}^{t^*}=\frac{2n^2}{(n+2)^2}|\widetilde{h}|^2|H|^2+\frac{(n+6)n^2}{(n+2)^2}\sum_{ijml}\widetilde{h}_{ij}^{m^*}\widetilde{h}_{li}^{m^*}H^{j^*}H^{l^*}.$$

$$\begin{aligned}III=I&=c|\widetilde{h}|^2+\frac{n^2}{(n+2)^2}|\widetilde{h}|^2|H|^2+\frac{2n}{n+2}\sum_{j,k,l,m,t}\widetilde{h}_{jk}^{m^*}\widetilde{h}_{kl}^{m^*}\widetilde{h}_{lj}^{t^*}H^{t^*}\\&\quad+\sum_{i,j,k,m,l,t}\widetilde{h}_{ij}^{m^*}\widetilde{h}_{kl}^{m^*}(\widetilde{h}_{lj}^{t^*}\widetilde{h}_{ik}^{t^*}-\widetilde{h}_{lk}^{t^*}\widetilde{h}_{ij}^{t^*})\\&\quad+\frac{2n^2}{(n+2)^2}\sum_{i,j,k,m}\widetilde{h}_{ij}^{m^*}\widetilde{h}_{jk}^{m^*}H^{i^*}H^{k^*}.\end{aligned}\tag{3.15}$$

Set $A_{i*}=(\widetilde{h}_{jk}^{i^*})$. Then it follows from (3.9) and (3.12)–(3.15) that

$$\begin{aligned}\frac{1}{2}\Delta|\widetilde{h}|^2&=(n+2)\langle\widetilde{h},\nabla T\rangle+|\nabla\widetilde{h}|^2+(n+1)c|\widetilde{h}|^2+\frac{n^2}{(n+2)}|\widetilde{h}|^2|H|^2\\&\quad+\sum_{i,j}\mathrm{tr}(A_{i^*}A_{j^*}-A_{j^*}A_{i^*})^2-\sum_{i,j}(\mathrm{tr}A_{i^*}A_{j^*})^2\\&\quad+n\sum_{i,j,l,m,t}\widetilde{h}_{ji}^{m^*}\widetilde{h}_{jt}^{m^*}\widetilde{h}_{ti}^{l^*}H^{l^*}+\frac{n^2}{(n+2)}\sum_{i,j,k,m}\widetilde{h}_{ij}^{m^*}\widetilde{h}_{jk}^{m^*}H^{i^*}H^{k^*}.\end{aligned}\tag{3.16}$$

Next we estimate terms on the right hand side of (3.16). We will need the following lemma.

Lemma 3.3 (cf. [17]) *Let $B_1,\cdots,B_m$ be symmetric $(n\times n)$-matrices $(m\geq 2)$. Denote $S_{mk}=\mathrm{trace}(B_m^tB_k)$, where B^t is the transposal matrix of B, $N(B_m):=S_m:=S_{mm}$, $S=\sum_{i=1}^m S_i$. Then*

$$\sum_{m,k}N(B_mB_k-B_kB_m)+\sum_{m,k}S_{mk}^2\leq\frac{3}{2}S^2.$$

Now we are prepared to estimate the right hand side of (3.16), mainly the last two terms on the last line of (3.16).

We re-choose $\{e_i\}_{i=1}^n$ such that $\sum\limits_l \widetilde{h}_{ij}^{l^*} H^{l^*} = \lambda_i \delta_{ij}$, denote by

$$S_H = \sum_{j,i} \Big(\sum_l \widetilde{h}_{ji}^{l^*} H^{l^*} \Big)^2 = \sum_j \lambda_j^2, \quad S_{i^*} = \sum_{j,l} (\widetilde{h}_{jl}^{i^*})^2,$$

then $|\widetilde{h}|^2 = \sum\limits_i S_{i^*}$. Using Lemma 3.3 and the above fact, we have the following estimates for the right hand side of (3.16),

$$\begin{aligned}
\frac{1}{2}\Delta|\widetilde{h}|^2 \geq{}& (n+2)\langle \widetilde{h}, \nabla T\rangle + |\nabla \widetilde{h}|^2 + (n+1)c|\widetilde{h}|^2 + \frac{n^2}{(n+2)}|\widetilde{h}|^2|H|^2 - \frac{3}{2}|\widetilde{h}|^4 \\
&+ n\sum_i \lambda_i S_{i^*} + \frac{n^2}{n+2}\sum_i \lambda_i^2 \\
\geq{}& (n+2)\langle \widetilde{h}, \nabla T\rangle + |\nabla \widetilde{h}|^2 + (n+1)c|\widetilde{h}|^2 + \frac{n^2}{(n+2)}|\widetilde{h}|^2|H|^2 - \frac{3}{2}|\widetilde{h}|^4 \\
&+ \frac{n}{2}\sum_i (\lambda_i + S_{i^*})^2 - \frac{n}{2}\sum_i S_{i^*}^2 \\
\geq{}& (n+2)\langle \widetilde{h}, \nabla T\rangle + |\nabla \widetilde{h}|^2 + (n+1)c|\widetilde{h}|^2 + \frac{n^2}{(n+2)}|\widetilde{h}|^2|H|^2 - \frac{n+3}{2}|\widetilde{h}|^4 \\
&+ \frac{n}{2}\sum_i (|H|\lambda_i + S_{i^*})^2 \\
\geq{}& (n+2)\langle \widetilde{h}, \nabla T\rangle + |\nabla \widetilde{h}|^2 + (n+1)c|\widetilde{h}|^2 + \frac{n^2}{(n+2)}|\widetilde{h}|^2|H|^2 - \frac{3}{2}|\widetilde{h}|^4 \\
&+ \frac{n}{2}\sum_i (\lambda_i + S_{i^*})^2 - \frac{n}{2}\sum_i S_{i^*}^2 \\
\geq{}& (n+2)\langle \widetilde{h}, \nabla T\rangle + |\nabla \widetilde{h}|^2 + (n+1)c|\widetilde{h}|^2 + \frac{n^2}{(n+2)}|\widetilde{h}|^2|H|^2 - \frac{n+3}{2}|\widetilde{h}|^4, \qquad (3.17)
\end{aligned}$$

where in the last inequality we have used $|\widetilde{h}|^4 = \big(\sum\limits_i S_{i^*}\big)^2 \geq \sum\limits_i S_{i^*}^2$.

4 Proof of Theorem 1.3

In this section we will use C to denote constants depending only on n, which may vary line by line. We have the following lemma.

Lemma 4.1 *Assume that M^n is a complete Lagrangian submanifold in $\mathbb{C}^n$, and let γ be a cut-off function on M^n with $\|\nabla\gamma\|_{L_\infty} = \Gamma$, then we have*

$$\begin{aligned}
&\int_M (|\nabla\widetilde{h}|^2 + |H|^2|\widetilde{h}|^2)\gamma^2 \mathrm{d}\mu \\
&\leq C\int_M \langle \nabla^* T, H \lrcorner \omega\rangle \gamma^2 \mathrm{d}\mu + C\int_M |\widetilde{h}|^4\gamma^2 \mathrm{d}\mu + C\Gamma^2 \int_{\{\gamma>0\}} |h|^2 \mathrm{d}\mu. \qquad (4.1)
\end{aligned}$$

Proof Multiplying (3.17) by γ^2 we get

$$\frac{1}{2}\int_M \gamma^2 \Delta|\widetilde{h}|^2 \mathrm{d}\mu \geq (n+2)\int_M \langle \widetilde{h}\gamma^2, \nabla T\rangle \mathrm{d}\mu$$

$$+\int_M |\nabla\widetilde{h}|^2\gamma^2 \mathrm{d}\mu + \frac{n^2}{(n+2)}\int_M |\widetilde{h}|^2|H|^2\gamma^2\mathrm{d}\mu - \frac{n+3}{2}\int_M |\widetilde{h}|^4\gamma^2\mathrm{d}\mu. \quad (4.2)$$

Note that

$$\frac{1}{2}\int_M \gamma^2\Delta|\widetilde{h}|^2\mathrm{d}\mu = -\int_M \langle\nabla\widetilde{h}, \gamma\nabla\gamma\otimes\widetilde{h}\rangle\mathrm{d}\mu, \quad (4.3)$$

and by the definition of T and integral by parts

$$\begin{aligned}\int_M \langle\widetilde{h}\gamma^2, \nabla T\rangle\mathrm{d}\mu &= -\int_M \langle T, \gamma^2\nabla^*\widetilde{h}\rangle - 2\int_M \langle T\otimes\nabla\gamma, \gamma\widetilde{h}\rangle\mathrm{d}\mu \\ &= -n\int_M \langle T,T\rangle\gamma^2\mathrm{d}\mu - 2\int_M \langle T\otimes\nabla\gamma, \gamma\widetilde{h}\rangle\mathrm{d}\mu \\ &= -n\int_M \Big\langle T, \frac{1}{n+2}(n\nabla H\lrcorner\omega - \operatorname{div} JHg)\gamma^2\Big\rangle\mathrm{d}\mu - 2\int_M \langle T\otimes\nabla\gamma, \gamma\widetilde{h}\rangle\mathrm{d}\mu \\ &= -\frac{n^2}{n+2}\int_M \langle T, \nabla H\lrcorner\omega\rangle\gamma^2\mathrm{d}\mu - 2\int_M \langle T\otimes\nabla\gamma, \gamma\widetilde{h}\rangle\mathrm{d}\mu \\ &= \frac{n^2}{n+2}\int_M \langle\nabla^*T, H\lrcorner\omega\rangle\gamma^2\mathrm{d}\mu + \frac{2n^2}{n+2}\int_M \langle T, H\lrcorner\omega\otimes\nabla\gamma\rangle\gamma\mathrm{d}\mu \\ &\quad - 2\int_M \langle T\otimes\nabla\gamma, \gamma\widetilde{h}\rangle\mathrm{d}\mu.\end{aligned}$$

Therefore since $|T| \le C|\nabla\widetilde{h}|$, $|\widetilde{h}| \le C|h|$ and $|\nabla\gamma| \le \Gamma$ we have

$$\begin{aligned}&\int_M |\nabla\widetilde{h}|^2\gamma^2 + \frac{n^2}{(n+2)}|\widetilde{h}|^2|H|^2\gamma^2\mathrm{d}\mu \\ &\le -\int_M \langle\nabla\widetilde{h}, \gamma\nabla\gamma\otimes\widetilde{h}\rangle\mathrm{d}\mu - n^2\int_M \langle\nabla^*T, H\lrcorner\omega\rangle\gamma^2\mathrm{d}\mu - 2n^2\int_M \langle T, H\lrcorner\omega\otimes\nabla\gamma\rangle\gamma\mathrm{d}\mu \\ &\quad + 2(n+2)\int_M \langle T\otimes\nabla\gamma, \gamma\widetilde{h}\rangle\mathrm{d}\mu + \frac{n+3}{2}\int_M |\widetilde{h}|^4\gamma^2\mathrm{d}\mu \\ &\le -n^2\int_M \langle\nabla^*T, H\lrcorner\omega\rangle\gamma^2\mathrm{d}\mu + \frac{n+3}{2}\int_M |\widetilde{h}|^4\gamma^2\mathrm{d}\mu + \frac{1}{2}\int_M |\nabla\widetilde{h}|^2\gamma^2\mathrm{d}\mu + C\Gamma^2\int_{\{\gamma>0\}} |h|^2\mathrm{d}\mu.\end{aligned}$$

We will need the following Michael-Simon inequality.

Theorem 4.1 (cf. [15, 22]) *Assume that M^n is a compact submanifold of R^{n+p} with or without boundary. Assume that $v \in C^1(M^n)$ is a nonnegative function such that $v = 0$ on ∂M^n, if ∂M^n is not an empty set. Then*

$$\Big(\int_M v^{\frac{n}{n-1}}\mathrm{d}\mu\Big)^{\frac{n-1}{n}} \le C\int_M |\nabla v| + v|H|\mathrm{d}\mu,$$

where H is the mean curvature vector field of M^n.

Now assume that $n \ge 3$. In the above Michael-Simon inequality we let $v = f^{\frac{2(n-1)}{n-2}}$, then by Hölder inequality we easily get

$$\Big(\int_M f^{\frac{2n}{n-2}}\mathrm{d}\mu\Big)^{\frac{n-2}{n}} \le C\int_M |\nabla f|^2 + f^2|H|^2\mathrm{d}\mu. \quad (4.4)$$

Therefore by letting $f=|\widetilde{h}|\gamma$ in (4.4) we obtain

$$\begin{aligned}\int_M|\widetilde{h}|^4\gamma^2\mathrm{d}\mu&\le\Big(\int_M|\widetilde{h}|^n\mathrm{d}\mu\Big)^{\frac{2}{n}}\Big(\int_M(|\widetilde{h}|\gamma)^{\frac{2n}{n-2}}\mathrm{d}\mu\Big)^{\frac{n-2}{n}}\\&\le C\Big(\int_M|\widetilde{h}|^n\mathrm{d}\mu\Big)^{\frac{2}{n}}\Big(\int_M|\nabla(|\widetilde{h}|\gamma)|^2+|H|^2|\widetilde{h}|^2\gamma^2\mathrm{d}\mu\Big)\\&\le C\Big(\int_M|\widetilde{h}|^n\mathrm{d}\mu\Big)^{\frac{2}{n}}\Big(\int_M|\nabla\widetilde{h}|^2\gamma^2\mathrm{d}\mu+\Gamma^2\int_{\{\gamma>0\}}|\widetilde{h}|^2\mathrm{d}\mu+\int_M|H|^2|\widetilde{h}|^2\gamma^2\mathrm{d}\mu\Big)\\&\le C\Big(\int_M|\widetilde{h}|^n\mathrm{d}\mu\Big)^{\frac{2}{n}}(|\nabla\widetilde{h}|^2\gamma^2+|H|^2|\widetilde{h}|^2\gamma^2\mathrm{d}\mu)\\&\quad+C\Gamma^2\Big(\int_M|\widetilde{h}|^n\mathrm{d}\mu\Big)^{\frac{2}{n}}\int_{\{\gamma>0\}}|h|^2\mathrm{d}\mu.\end{aligned}\tag{4.5}$$

From (3.17) and (4.5) we have

$$\begin{aligned}&\int_M(|\nabla\widetilde{h}|^2+|H|^2|\widetilde{h}|^2)\gamma^2\mathrm{d}\mu\\&\le C\int_M\langle\nabla^*T,H\lrcorner\omega\rangle\gamma^2\mathrm{d}\mu+C\Big(\int_M|\widetilde{h}|^n\mathrm{d}\mu\Big)^{\frac{2}{n}}(|\nabla\widetilde{h}|^2\gamma^2+|H|^2|\widetilde{h}|^2\gamma^2\mathrm{d}\mu)+C\Gamma^2\int_{\{\gamma>0\}}|h|^2\mathrm{d}\mu,\end{aligned}$$

which implies that if there exists ε_0 sufficiently small such that

$$\int_M|\widetilde{h}|^n\mathrm{d}\mu\le\varepsilon_0,$$

we have

$$\int_M(|\nabla\widetilde{h}|^2+|H|^2|\widetilde{h}|^2)\gamma^2\mathrm{d}\mu\le C\int_M\langle\nabla^*T,H\lrcorner\omega\rangle\gamma^2\mathrm{d}\mu+C\Gamma^2\int_{\{\gamma>0\}}|h|^2\mathrm{d}\mu.\tag{4.6}$$

Case 1 If $\nabla^*T=0$, note that for any $R>0$ we can choose $\gamma\in C_c^1(M_R(p_0))$ such that $\gamma=1$ on $M_{\frac{R}{2}}(p_0)$ where $M_r(p_0)$ denotes geodesic ball of radius r with center $p_0\in M^n$, and $\Gamma\le\frac{C}{R}$, therefore by letting $R\to+\infty$ we get

$$\int_M|\nabla\widetilde{h}|^2+|H|^2|\widetilde{h}|^2\mathrm{d}\mu=0,$$

which implies that $\widetilde{h}=0$ and M^n is either a Lagrangian subspace or a Whitney sphere by [7, 24] or $H=0,\nabla h=0$ and M^n is a Lagrangian subspace by [16].

Case 2 If $\nabla^*\nabla^*T=0$ and M^n is a Lagrangian sphere, then by Dazord [14], there exists a smooth function f on M^n such that $H\lrcorner\omega=\mathrm{d}f$, and let $\gamma\equiv1$ on M^n, then we have

$$\begin{aligned}\int_M|\nabla\widetilde{h}|^2+|H|^2|\widetilde{h}|^2\mathrm{d}\mu&\le C\int_M\langle\nabla^*T,\mathrm{d}f\rangle\mathrm{d}\mu\\&=-C\int_M f\nabla^*\nabla^*T\mathrm{d}\mu\\&=0,\end{aligned}$$

which implies that $\widetilde{h}=0$ and M^n is either a Lagrangian subspace or a Whitney sphere by [7, 24] or $H=0,\nabla h=0$ and M^n is a Lagrangian subspace by [16].

This completes the proof of Theorem 1.3.

5 Proof of Theorem 1.4

The proof of Theorem 1.4 is quite similar with the proofs of Theorem 1.2 in [32], Theorem 1.5 in [21] and Theorem 1.3 in the present paper. Therefore we will only give a outline of the proof and omit some details. In this section we will use C to denote constants depending only on n which may vary line by line.

Letting $c = 1$ in (3.17), we have

$$\frac{1}{2}\Delta|\widetilde{h}|^2 \geq (n+2)\langle \widetilde{h}, \nabla T\rangle + |\nabla\widetilde{h}|^2 + (n+1)|\widetilde{h}|^2 + \frac{n^2}{(n+2)}|\widetilde{h}|^2|H|^2 - \frac{n+3}{2}|\widetilde{h}|^4. \tag{5.1}$$

Then similarly with Lemma 4.1 we can obtain the following lemma.

Lemma 5.1 *Assume that M^n is a complete Lagrangian submanifold in $\mathbb{CP}^n$ and γ is a cut off function on M^n with $\|\nabla\gamma\|_{L_\infty} = \Gamma$, then we have*

$$\begin{aligned}&\int_M (|\nabla\widetilde{h}|^2 + |\widetilde{h}|^2|H|^2 + |\widetilde{h}|^2)\gamma^2 \mathrm{d}\mu \\ &\leq C\int_M \langle \nabla^* T, H\lrcorner\omega\rangle\gamma^2\mathrm{d}\mu + C\int_M |\widetilde{h}|^4\gamma^2\mathrm{d}\mu + C\Gamma^2\int_{\{\gamma>0\}}|h|^2\mathrm{d}\mu.\end{aligned} \tag{5.2}$$

The same as the previous section, to absorb the "bad term" $\int_M |\widetilde{h}|^4\gamma^2\mathrm{d}\mu$ on the right hand side of (5.2), we will use the Michael-Simom inequality. In order to do this, we need isometrically immersed $\mathbb{CP}^n$ into some Euclidean space $\mathbb{R}^{n+p}$, which is possible by Nash's celebrated embedding theorem. Assume that $\mathbb{CP}^n$ has mean curvature H_0 as a submanifold in $\mathbb{R}^{n+p}$, and M^n has mean curvature $\overline{H}$ as a submanifold in $\mathbb{R}^{n+p}$. Then it is easy to see that $|\overline{H}|^2 \leq |H_0|^2 + |H|^2$.

If $n = 2$, from the original Michael-Simon inequality we see that if $M \hookrightarrow \mathbb{R}^{n+p}$ is compact with or without boundary then

$$\int_M f^2\mathrm{d}\mu \leq C\Big(\int_M |\nabla f|\mathrm{d}\mu + \int_M f|H|\mathrm{d}\mu\Big)^2 \tag{5.3}$$

for any nonnegative function $f \in C^1(M)$ with $f|_{\partial M} = 0$. Let $f = |\widetilde{h}|^2\gamma$ in the above inequality, we obtain

$$\begin{aligned}\int_M |\widetilde{h}|^4\gamma^2\mathrm{d}\mu &\leq C\Big(\int_M |\widetilde{h}||\nabla\widetilde{h}|\gamma + |\nabla\gamma||\widetilde{h}|^2\mathrm{d}\mu + \int_M |\widetilde{h}|^2|\overline{H}|\gamma\mathrm{d}\mu\Big)^2 \\ &\leq C\int_M |\widetilde{h}|^2\mathrm{d}\mu\Big(\int_M |\nabla\widetilde{h}|^2\gamma^2\mathrm{d}\mu + \int_M |\widetilde{h}|^2|\overline{H}|^2\gamma^2\mathrm{d}\mu\Big) + C\Gamma^2\Big(\int_{\{\gamma>0\}}|\widetilde{h}|^2\mathrm{d}\mu\Big)^2 \\ &\leq C\int_M |\widetilde{h}|^2\mathrm{d}\mu\Big(\int_M |\nabla\widetilde{h}|^2\gamma^2\mathrm{d}\mu + \int_M |\widetilde{h}|^2|H|^2\gamma^2\mathrm{d}\mu\Big) \\ &\quad + C\max_{\mathbb{CP}^n}|H_0|^2\Big(\int_M |\widetilde{h}|^2\gamma^2\mathrm{d}\mu\Big)^2 + C\Gamma^2\Big(\int_{\{\gamma>0\}}|\widetilde{h}|^2\mathrm{d}\mu\Big)^2 \\ &= C\int_M |\widetilde{h}|^2\mathrm{d}\mu\Big(\int_M |\nabla\widetilde{h}|^2\gamma^2\mathrm{d}\mu + \int_M |\widetilde{h}|^2|H|^2\gamma^2\mathrm{d}\mu\Big) \\ &\quad + C\Big(\int_M |\widetilde{h}|^2\gamma^2\mathrm{d}\mu\Big)^2 + C\Gamma^2\Big(\int_{\{\gamma>0\}}|\widetilde{h}|^2\mathrm{d}\mu\Big)^2.\end{aligned} \tag{5.4}$$

From (5.2) and (5.4) we see that

$$\begin{aligned}&\int_M(|\nabla\widetilde{h}|^2+|\widetilde{h}|^2|H|^2+|\widetilde{h}|^2)\gamma^2\mathrm{d}\mu\\ \le{}& C\int_M\langle\nabla^*T,H\lrcorner\omega\rangle\gamma^2\mathrm{d}\mu+C\int_M|\widetilde{h}|^2\mathrm{d}\mu\Big(\int_M|\nabla\widetilde{h}|^2\gamma^2\mathrm{d}\mu+\int_M|\widetilde{h}|^2|H|^2\gamma^2\mathrm{d}\mu\Big)\\ &+C\Big(\int_M|\widetilde{h}|^2\gamma^2\mathrm{d}\mu\Big)^2+C\Gamma^2\Big(\int_{\{\gamma>0\}}|\widetilde{h}|^2\mathrm{d}\mu\Big)^2+C\Gamma^2\int_{\{\gamma>0\}}|h|^2\mathrm{d}\mu.\end{aligned}$$

Therefore if M^n satisfies assumptions of Theorem 1.4, we can similarly with the previous section obtain that $\widetilde{h}=0$ and hence M^n is the real projective space $\mathbb{RP}^n$ or a Whitney sphere, by [11].

If $n\ge 3$, similarly with the proof of Theorem 1.3, we can obtain from the Michael-Simon inequality that

$$\Big(\int_M f^{\frac{2n}{n-2}}\mathrm{d}\mu\Big)^{\frac{n-2}{n}}\le C\Big(\int_M|\nabla f|^2\mathrm{d}\mu+\int_M f^2|\overline{H}|^2\mathrm{d}\mu\Big) \tag{5.5}$$

for any nonnegative function $f\in C^1(M^n)$ with $f|_{\partial M}=0$. Therefore by letting $f=|\widetilde{h}|\gamma$ in the above inequality we obtain

$$\begin{aligned}\int_M|\widetilde{h}|^4\gamma^2\mathrm{d}\mu&\le\Big(\int_M|\widetilde{h}|^n\mathrm{d}\mu\Big)^{\frac{2}{n}}\Big(\int_M(|\widetilde{h}|\gamma)^{\frac{2n}{n-2}}\mathrm{d}\mu\Big)^{\frac{n-2}{n}}\\ &\le C\Big(\int_M|\widetilde{h}|^n\mathrm{d}\mu\Big)^{\frac{2}{n}}\Big(\int_M|\nabla(|\widetilde{h}|\gamma)|^2+|\overline{H}|^2|\widetilde{h}|^2\gamma^2\mathrm{d}\mu\Big)\\ &\le C\Big(\int_M|\widetilde{h}|^n\mathrm{d}\mu\Big)^{\frac{2}{n}}\Big(\int_M|\nabla\widetilde{h}|^2\gamma^2\mathrm{d}\mu+\Gamma^2\int_{\{\gamma>0\}}|\widetilde{h}|^2\mathrm{d}\mu+\int_M|\overline{H}|^2|\widetilde{h}|^2\gamma^2\mathrm{d}\mu\Big)\\ &\le C\Big(\int_M|\widetilde{h}|^n\mathrm{d}\mu\Big)^{\frac{2}{n}}\Big(\int_M|\nabla\widetilde{h}|^2\gamma^2+|H|^2|\widetilde{h}|^2\gamma^2\mathrm{d}\mu\Big)\\ &\quad+C\max_{\mathbb{CP}^n}|H_0|^2\Big(\int_M|\widetilde{h}|^n\mathrm{d}\mu\Big)^{\frac{2}{n}}\int_M|\widetilde{h}|^2\gamma^2\mathrm{d}\mu+C\Gamma^2\Big(\int_M|\widetilde{h}|^n\mathrm{d}\mu\Big)^{\frac{2}{n}}\int_{\{\gamma>0\}}|\widetilde{h}|^2\mathrm{d}\mu\\ &=C\Big(\int_M|\widetilde{h}|^n\mathrm{d}\mu\Big)^{\frac{2}{n}}\Big(\int_M|\nabla\widetilde{h}|^2\gamma^2+|H|^2|\widetilde{h}|^2\gamma^2\mathrm{d}\mu\Big)+C\Big(\int_M|\widetilde{h}|^n\mathrm{d}\mu\Big)^{\frac{2}{n}}\int_M|\widetilde{h}|^2\gamma^2\mathrm{d}\mu\\ &\quad+C\Gamma^2\Big(\int_M|\widetilde{h}|^n\mathrm{d}\mu\Big)^{\frac{2}{n}}\int_{\{\gamma>0\}}|\widetilde{h}|^2\mathrm{d}\mu.\end{aligned} \tag{5.6}$$

From (5.2) and (5.6) we obtain

$$\begin{aligned}&\int_M(|\nabla\widetilde{h}|^2+|\widetilde{h}|^2|H|^2+|\widetilde{h}|^2)\gamma^2\mathrm{d}\mu\\ \le{}& C\int_M\langle\nabla^*T,H\lrcorner\omega\rangle\gamma^2\mathrm{d}\mu+C\Big(\int_M|\widetilde{h}|^n\mathrm{d}\mu\Big)^{\frac{2}{n}}\int_M(|\nabla\widetilde{h}|^2+|\widetilde{h}|^2|H|^2)\gamma^2\mathrm{d}\mu\\ &+C\Big(\int_M|\widetilde{h}|^n\mathrm{d}\mu\Big)^{\frac{2}{n}}\int_M|\widetilde{h}|^2\gamma^2\mathrm{d}\mu+C\Gamma^2\Big(\int_M|\widetilde{h}|^n\mathrm{d}\mu\Big)^{\frac{2}{n}}\int_{\{\gamma>0\}}|h|^2\mathrm{d}\mu+C\Gamma^2\int_{\{\gamma>0\}}|h|^2\mathrm{d}\mu.\end{aligned}$$

Therefore if M^n satisfies assumptions of Theorem 1.4, we can similarly with the previous section obtain that $\widetilde{h}=0$ and hence M^n is the real projective space $\mathbb{RP}^n$ or a Whitney sphere, by [11].

This completes the proof of Theorem 1.4.

Acknowledgements We would like to thank the referees for their critical reading and useful suggestions which made this paper more readable. After our paper was submitted, we note that Theorem 1.3(i) and Theorem 1.4(i) of this paper have been obtained independently by Shunjuan Cao and Entao Zhao in [4]. We are appreciated with Entao Zhao for his encouragement to publish our result too.

References

[1] Blair, D. E., Riemannian Geometry of Contact and Symplectic Manifolds, Progress in Math., **203**, Birkhäuser, Basel, 2002.

[2] Borreli, V., Chen, B. Y. and Morvan, J. M., Une caractérisation géométrique de la sphère de Whitney, *C. R. Acad. Sci. Paris Sér. I Math.*, **321**(11), 1995, 1485–1490.

[3] Cannas da Silva, A., Lectures on Symplectic Geometry, Lecture Notes in Mathematics, **1764**, Springer-Verlag, Berlin, 2001.

[4] Cao, S. J. and Zhao, E. T., Gap theorems for Lagrangian submanifolds in complex space forms, *Geom. Dedicata*, **215**, 2021, 315–331.

[5] Castro, I., Lerma, A. M. and Miquel, V., Evolution by mean curvature flow of Lagrangian spherical surfaces in complex Euclidean plane, *J. Math. Anal. Appl.*, **462**(1), 2018, 637–647.

[6] Castro, I., Montealegre, C. R. and Urbano, F., Closed conformal vector fields and Lagrangian submanifolds in complex space forms, *Pacific J. Math.*, **199**(2), 2001, 269–302.

[7] Castro, I. and Urbano, F., Lagrangian surfaces in the complex Euclidean plane with conformal Maslov form, *Tohoku Math. J.* (2), **45**(4), 1993, 565–582.

[8] Castro, I. and Urbano, F., Twistor holomorphic Lagrangian surfaces in the complex projective and hyperbolic planes, *Ann. Global Anal. Geom.*, **13**(1), 1995, 59–67.

[9] Castro, I. and Urbano, F., Willmore surfaces of $\mathbb{R}^4$ and the Whitney sphere, *Ann. Global Anal. Geom.*, **19**(2), 2001, 153–175.

[10] Chao, X. L. and Dong, Y. X., Rigidity theorems for Lagrangian submanifolds of complex space forms with conformal Maslov form, Recent developments in geometry and analysis, 17–25, Adv. Lect. Math. (ALM), **23**, Int. Press, Somerville, MA, 2012.

[11] Chen, B. Y., Jacobi's elliptic functions and Lagrangian immersions, *Proc. Royal Soc. Edinburgh*, **126**, 1996, 687–704.

[12] Chen, B. Y., On Ricci curvature of isotropic and Lagrangian submanifolds in complex space forms, *Arch. Math.* (*Basel*), **74**(2), 2000, 154–160.

[13] Chen, B. Y., Riemannian geometry of Lagrangian submanifolds, *Taiwanese J. Math.*, **5**(4), 2001, 681–723.

[14] Dazord, P., Sur la géométrie des sous-fibrés et des feuilletages lagrangiens, *Ann. Sci. École Norm. Sup.* (4), **14**(4), 1981, 465–480.

[15] Hoffman, D. and Spruck, J., Sobolev and isoperimetric inequalities for Riemannian submanifolds, *Comm. Pure Appl. Math.*, **27**, 1974, 715–727.

[16] Lawson, H. Blaine Jr., Local rigidity theorems for minimal hypersurfaces, *Ann. of Math.* (2), **89**, 1969, 187–197.

[17] Li, A. M. and Li, J. M., An intrinsic rigidity theorem for minimal submanifolds in a sphere, *Arch. Math.* (*Basel*), **58**(6), 1992, 582–594.

[18] Li, H. Z. and Vrancken, V., A bisic inequality and new characterization of Whitney spheres in a complex space form, *Israel J. Math.*, **146**, 2005, 223–242.

[19] Lin, J. M. and Xia, C. Y., Global pinching theorem for even dimensional minimal submanifolds in a unit sphere, *Math. Z.*, **201**, 1989, 381–389.

[20] Luo, Y. and Wang, G. F., On geometrically constrained variational problems of the Willmore functional I., The Lagrangian-Willmore problem, *Comm. Anal. Geom.*, **23**(1), 2015, 191–223.

[21] Luo, Y. and Yin, J. B., On energy gap phenomena of the Whitney sphere and related problems, *Pacific J. Math.*, **320**(2), 2022, 299–318.

[22] Michael, J. H. and Simon, L. M., Sobolev and mean-value inequalities on generalized submanifolds of $\mathbb{R}^n$, *Comm. Pure Appl. Math.*, **26**, 1973, 361–379.

[23] Ni, L., Gap theorems for minimal submanifolds in $\mathbb{R}^{n+1}$, *Comm. Anal. Geom.*, **9**(3), 2001, 641–656.

[24] Ros, A. and Urbano, F., Lagrangian submanifolds of $\mathbb{C}^n$ with conformal Maslov form and the Whitney sphere, *J. Math. Soc. Japan*, **50**(1), 1998, 203–226.

[25] Savas-Halilaj, A. and Smoczyk, K., Lagrangian mean curvature flow of Whitney spheres, *Geom. Topol.*, **23**(2), 2019, 1057–1084.

[26] Shen, C. L., A global pinching theorem for minimal hypersurfaces in sphere, *Proc. Amer. Math. J.*, **105**, 1989, 192–198.

[27] Wang, H., Some global pingching theorems for minimal submanifolds in a sphere, *Acta Math. Sin.*, **31**, 1988, 503–509.

[28] Weinstein, A., Lectures on symplectic manifolds, *Regional Conference Series in Mathematics,*, **29**, American Mathematical Society, R.I., 1977.

[29] Xu, H. W., $L_{\frac{n}{2}}$-pinching theorems for submanifolds with parallel mean curvature in a sphere, *J. Math. Soc. Japan*, **46**(3), 1994, 503–515.

[30] Xu, H. W. and Gu, J. R., A general gap theorem for submanifolds with parallel mean curvature in $\mathbb{R}^{n+p}$, *Comm. Anal. Geom.*, **15**(1), 2007, 175–194.

[31] Yun, G., Total scalar curvature and L^2 harmonic 1-forms on a minimal hypersurface in Euclidean space, *Geom. Dedicata*, **89**, 2002, 135–141.

[32] Zhang, L. Y., An energy gap phenomenon for the Whitney sphere, *Math. Z.*, **297**(3–4), 2021, 1601–1611.