Chin. Ann. Math. Ser. B
44(4), 2023, 615–640
DOI: 10.1007/s11401-023-0035-8

Chinese Annals of Mathematics, Series B

Discrete Differential Calculus on Simplicial Complexes and Constrained Homology*

Shiquan REN[1]

Abstract Let V be a finite set. Let $\mathcal{K}$ be a simplicial complex with its vertices in V. In this paper, the author discusses some differential calculus on V. He constructs some constrained homology groups of $\mathcal{K}$ by using the differential calculus on V. Moreover, he defines an independence hypergraph to be the complement of a simplicial complex in the complete hypergraph on V. Let $\mathcal{L}$ be an independence hypergraph with its vertices in V. He constructs some constrained cohomology groups of $\mathcal{L}$ by using the differential calculus on V.

Keywords Simplicial complexes, Hypergraphs, Chain complexes, Homology, Differential calculus

2000 MR Subject Classification 55U10, 55U15, 53A45, 08A50

1 Introduction

Simplicial complexes play an important and fundamental role in algebraic topology. So far, topologists have developed the homology and cohomology theory for simplicial complexes. We refer to [24, Chapter 1] and [20, Section 2.1] for a systematic introduction to the simplicial homology theory. We also refer to [24, Section 42, Chapter 5] and [20, Sections 3.1–3.2] for an introduction to the simplicial cohomology theory. On the other hand, since 1950's, topologists have developed the simplicial homotopy theory (for example, we may refer to [10–12, 23, 31]), which has been found to have significant applications in various topics in algebraic and geometric topology (for example, we refer to [5, 21, 26] for some of such applications). In simplicial homotopy theory, simplicial complexes are the fundamental models for simplicial sets.

The notion of hypergraphs is a higher dimensional generalization of the notion of graphs (see [1, 25]). In a graph, an edge consists of two vertices while in an oriented hypergraph, a oriented hyperedge is allowed to be consisted of n-vertices for any $n \geq 1$. From a topological point of view, an oriented hypergraph can be obtained by deleting some non-maximal faces in an oriented simplicial complex (see [3, 25]) while an oriented simplicial complex is a special oriented hypergraph with no non-maximal faces missing. The embedded homology of hypergraphs was introduced by Bressan, Li, Ren and Wu [3]. The embedded homology of oriented hypergraphs was proved to be independence on the choice of orientations by Grbić, Wu, Xia and Wei [13, Theorem 2.7].

The complete hypergraph $\Delta[V]$ on a finite set V has its set of hyperedges as all the non-empty subsets of V (see Definition 2.6). A simplicial complex with all of its vertices in V has

Manuscript received July 29, 2021. Revised September 12, 2022.
[1]School of Mathematics and Statistics, Henan University, Kaifeng 475004, Henan, China.
E-mail: renshiquan@henu.edu.cn
*This work was supported by China Postdoctoral Science Foundation (No. 2022M721023).

its set of simplices as a subset of $\Delta[V]$. We call the complement of the set of the simplices in $\Delta[V]$ an independence hypergraph (see Definition 2.9 and Proposition 2.1).

Differential calculus is an important tool in (co)homology theory. In some textbooks in algebraic topology (for example, [2, 22]), the methods of differential calculus have been applied to the (co)homology theory of differentiable manifolds and fibre bundles. During the 1990s, Dimakis and Müller-Hoissen [7–9] initiated the study of discrete differential calculus on discrete sets with a motivation from theoretical physics. During the 2010s, based on the study of [7–9], Grigor'yan, Lin and Yau [14], Grigor'yan, Lin, Muranov and Yau [15–18] and Grigor'yan, Muranov and Yau [19] developed the discrete differential calculus methods on discrete sets and applied the methods to the study of digraphs.

In this paper, we apply the method of the (discrete) differential calculus and give some constrained homology for simplicial complexes as well as constrained cohomology for independence hypergraphs. The constrained cohomology of independence hypergraphs that will be introduced in this paper is in general different from the embedded homology of hypergraphs in [3] and the embedded cohomology of hypergraphs in [13].

Let V be a finite set. Let $\mathcal{K}$ be a simplicial complex whose set of vertices is a subset of V. Let $n \geq 0$. Let $v_0v_1\cdots v_n$ be an n-simplex of $\mathcal{K}$. The usual boundary operator (see [20, p. 105], [24, p. 28]) is given by

$$\partial_n(v_0v_1\cdots v_n) = \sum_{i=0}^{n}(-1)^i v_0\cdots\widehat{v_i}\cdots v_n. \tag{1.1}$$

We generalize the usual boundary operator and define a weighted boundary operator

$$\frac{\partial}{\partial v}(v_0v_1\cdots v_n) = \sum_{i=0}^{n}(-1)^i\delta(v,v_i)v_0\cdots\widehat{v_i}\cdots v_n$$

with respect to any fixed vertex $v \in V$. Note that

$$\partial_n = \sum_{v\in V}\frac{\partial}{\partial v}.$$

We take the exterior algebra $\mathrm{Ext}_*(V)$ generated by the $\frac{\partial}{\partial v}$'s for all $v \in V$. We prove in Subsection 4.2 that for any $t \geq 0$ and any $\alpha \in \mathrm{Ext}_{2t+1}(V)$, there is a constrained homology group of $\mathcal{K}$ with respect to α. Moreover, we prove in Theorem 4.2 that for any $s \geq 0$ and any $\beta \in \mathrm{Ext}_{2s}(V)$, the element β induces a homomorphism between the corresponding constrained homology groups.

We point out that the constrained homology groups which will be investigated in Subsection 4.2 are generalizations of the weighted homology groups investigated by Dawson [6] and Wu, C. Y., Ren, Wu, and Xia [28–30] for weighted simplicial complexes. Let f be a real function on V. We take $t = 1$ and

$$\alpha = \sum_{v\in V} f(v)\frac{\partial}{\partial v}$$

in Definition 4.3, Subsection 4.2. Then the constrained homology groups of the simplicial complex $\mathcal{K}$ with respect to α, which will be investigated in Subsection 4.2, give the weighted homology groups of the weighted simplicial complex $(\mathcal{K}, f)$ which have been investigated in [28–30].

On the other hand, let $\mathcal{L}$ be an independence hypergraph whose set of vertices is a subset of V. For any $v \in V$, we consider the adjoint linear map $\mathrm{d}v$ of the element $\frac{\partial}{\partial v}$ in $\mathrm{Ext}_*(V)$. We define $\mathrm{Ext}^*(V)$ as the exterior algebra generated by the $\mathrm{d}v$'s for all $v \in V$. We prove in Subsection 4.3 that for any $t \geq 0$ and any $\omega \in \mathrm{Ext}^{2t+1}(V)$, there is a constrained cohomology group of $\mathcal{L}$ with respect to ω. Moreover, we prove in Theorem 4.4 that for any $s \geq 0$ and any $\mu \in \mathrm{Ext}^{2s}(V)$, the element μ induces a homomorphism between the constrained cohomology groups.

The remaining part of this paper is organized as follows. In Section 2, we introduce the definitions of hypergraphs, simplicial complexes and independence hypergraphs. In Section 3, as a preparation for Section 4, we discuss some differential calculus for paths on discrete sets. In Section 4, we define the constrained homology groups for simplicial complexes in Definition 4.3 and define the constrained cohomology groups for independence hypergraphs in Definition 4.4. We prove Theorems 4.2 and 4.4. Finally, in Section 5, we give some examples for Section 4.

2 Hypergraphs, Simplicial Complexes and Independence Hypergraphs

Let V be a discrete set whose elements are called vertices. Let $n \geq 0$ be a non-negative integer. Let S_{n+1} be the symmetric group on n-letters. Then S_{n+1} acts on the set of all the sequences $v_0v_1\cdots v_n$, where $v_0, v_1, \cdots, v_n \in V$, by permuting the orders of the vertices.

Definition 2.1 *An oriented n-hyperedge is an equivalent class $[v_0, v_1, \cdots, v_n]$ where the equivalence relation $\sim$ on the set $\{v_0v_1\cdots v_n \mid v_0, v_1, \cdots, v_n \in V\}$ of the sequences is given by $\sigma(v_0v_1\cdots v_n) \sim v_0v_1\cdots v_n$ if and only if $\sigma \in S_{n+1}$ is an even permutation.*

In the remaining part of this paper, suppose V has a total order $\prec$.

Definition 2.2 *An n-hyperedge on V is a sequence*

$$\sigma^{(n)} = v_0v_1\cdots v_n, \tag{2.1}$$

where $v_0 \prec v_1 \prec \cdots \prec v_n$ are vertices in V. For simplicity, an n-hyperedge is also called a hyperedge and $\sigma^{(n)}$ in (2.1) is also denoted as σ.

Remark 2.1 By Definition 2.2, a 0-hyperedge on V is just a single vertex v_0 in V and a 1-hyperedge on V is just an edge v_0v_1 in the complete graph on V.

Definition 2.3 *The complete n-uniform hypergraph $\Delta_n(V)$ on V is the collection of all the possible n-hyperedges on V. In other words, $\Delta_n[V]$ consists of all the subsets of V with $(n+1)$-vertices*:

$$\Delta_n(V) = \{v_0v_1\cdots v_n \mid v_0, v_1, \cdots, v_n \in V \text{ and } v_0 \prec v_1 \prec \cdots \prec v_n\}.$$

Remark 2.2 In particular, let $n = 1$ in Definition 2.3. Then the complete 1-uniform hypergraph $\Delta_1(V)$ is just the complete graph on V.

Definition 2.4 *An n-uniform hypergraph $\mathcal{H}^{(n)}$ on V is a collection of some of the n-hyperedges on V. In other words, $\mathcal{H}^{(n)}$ consists of some of the subsets of V with $(n+1)$-vertices*:

$$\mathcal{H}^{(n)} \subseteq \{v_0v_1\cdots v_n \mid v_0, v_1, \cdots, v_n \in V \textit{ and } v_0 \prec v_1 \prec \cdots \prec v_n\}.$$

Definition 2.5 *A hypergraph on V is a disjoint union*

$$\mathcal{H} = \bigcup_{n\geq 0} \mathcal{H}^{(n)}, \tag{2.2}$$

where $\mathcal{H}^{(n)}$ is an n-uniform hypergraph on V for each $n \geq 0$.

Definition 2.6 *The complete hypergraph $\Delta[V]$ on V is the collection of all the possible hyperedges on V. In other words, $\Delta[V]$ consists of all the non-empty finite subsets of V.*

Remark 2.3 It is direct that we have a disjoint union

$$\Delta[V] = \bigcup_{n\geq 0} \Delta_n(V).$$

Definition 2.7 *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two hypergraphs on V. The complement of $\mathcal{H}_1$ in $\mathcal{H}_2$ is defined to be a hypergraph $\mathcal{H}_2 \setminus \mathcal{H}_1$ on V given by*

$$\mathcal{H}_2 \setminus \mathcal{H}_1 = \{\sigma \text{ is a hyperedge on } V \mid \sigma \in \mathcal{H}_2 \text{ and } \sigma \notin \mathcal{H}_1\}.$$

Definition 2.8 *A simplicial complex (pl. simplicial complexes) $\mathcal{K}$ on V is a hypergraph on V such that for any hyperedge $\sigma \in \mathcal{K}$ and any non-empty subset $\tau \subseteq \sigma$, we always have $\tau \in \mathcal{K}$. An n-hyperedge in a simplicial complex is also called an n-simplex (pl. n-simplices) or simply a simplex (pl. simplices).*

Definition 2.9 *An independence hypergraph $\mathcal{L}$ on V is a hypergraph on V such that for any hyperedge $\sigma \in \mathcal{L}$ and any hyperedge τ on V satisfying $\sigma \subseteq \tau$, we always have $\tau \in \mathcal{L}$.* [1]

Remark 2.4 From Definitions 2.4 and 2.8–2.9, it is direct that

- for any $n \geq 1$, an n-uniform hypergraph is not a simplicial complex;
- for any $n \leq \#V - 1$ where $\#V$ is the cardinality of V (here $\#V$ can be either finite or infinite), an n-uniform hypergraph is not an independence hypergraph.

Remark 2.5 From Definitions 2.6 and 2.8–2.9, it is direct that the complete hypergraph $\Delta[V]$ is a simplicial complex on V and also an independence hypergraph on V.

Proposition 2.1 *Let $\Delta[V]$ be the complete hypergraph on V. Let $\mathcal{K}$ be a simplicial complex on V. Let $\mathcal{L}$ be an independence hypergraph on V. Then both the followings are satisfied:*

(i) $\Delta[V] \setminus \mathcal{K}$ *is an independence hypergraph on V;*

(ii) $\Delta[V] \setminus \mathcal{L}$ *is a simplicial complex on V.*

Proof (i) Let $\sigma \in \Delta[V] \setminus \mathcal{K}$. Let τ be a hyperedge on V such that $\sigma \subseteq \tau$. In order to prove that $\Delta[V] \setminus \mathcal{K}$ is an independence hypergraph, it suffices to prove $\tau \in \Delta[V] \setminus \mathcal{K}$. Suppose to the contrary, $\tau \notin \Delta[V] \setminus \mathcal{K}$. Then $\tau \in \mathcal{K}$. Since $\mathcal{K}$ is a simplicial complex and $\sigma \subseteq \tau$, we have $\sigma \in \mathcal{K}$. This contradicts $\sigma \in \Delta[V] \setminus \mathcal{K}$. Therefore, $\tau \in \Delta[V] \setminus \mathcal{K}$, which implies that $\Delta[V] \setminus \mathcal{K}$ is an independence hypergraph.

[1] The reason that we use the term "independence hypergraph" is as follows. By Proposition 2.1, the set of hyperedges of an independence hypergraph $\mathcal{L}$ on V is the complement of the set of simplices of a simplicial complex $\mathcal{K}$ on V in the set of hyperedges of the complete hypergraph $\Delta[V]$. That is, $\mathcal{K} = \Delta[V]\setminus\mathcal{L}$, or equivalently, $\mathcal{L} = \Delta[V] \setminus \mathcal{K}$. If we regard each simplex $\sigma \in \mathcal{K}$ as a relation on V, then the vertices of each hyperedge $\sigma \in \mathcal{L}$ are independent from the relations in $\mathcal{K}$. Since $\mathcal{L}$ consists of all the hyperedges $\sigma \in \Delta[V]$ such that the vertices of each hyperedge are independent from the relations in $\mathcal{K}$, we call $\mathcal{L}$ an independence hypergraph.

(ii) Let $\sigma \in \Delta[V] \setminus \mathcal{L}$. Let τ be a hyperedge on V such that $\tau \subseteq \sigma$. In order to prove that $\Delta[V] \setminus \mathcal{L}$ is a simplicial complex, it suffices to prove $\tau \in \Delta[V] \setminus \mathcal{L}$. Suppose to the contrary, $\tau \notin \Delta[V] \setminus \mathcal{L}$. Then $\tau \in \mathcal{L}$. Since $\mathcal{L}$ is an independence hypergraph and $\tau \subseteq \sigma$, we have $\sigma \in \mathcal{L}$. This contradicts $\sigma \in \Delta[V] \setminus \mathcal{L}$. Therefore, $\tau \in \Delta[V] \setminus \mathcal{L}$, which implies that $\Delta[V] \setminus \mathcal{L}$ is a simplicial complex.

Example 2.1 Consider the set $V = \{v_0, v_1, v_2, v_3, v_4, v_5\}$. Then

(i) $\sigma^{(3)} = v_0v_2v_4v_5$ is 3-hyperedge on V;

(ii) $\mathcal{H}^{(2)} = \{v_0v_2v_3, v_1v_2v_3, v_1v_3v_5, v_2v_4v_5\}$ is a 2-uniform hypergraph on V;

(iii) $\mathcal{H} = \{v_0, v_0v_1, v_4v_5, v_0v_1v_2, v_2v_3v_4v_5\}$ is a hypergraph on V;

(iv) $\Delta[V] = \{v_i \mid 0 \le i \le 5\} \cup \{v_iv_j \mid 0 \le i < j \le 5\} \cup \{v_iv_jv_k \mid 0 \le i < j < k \le 5\} \cup \{v_iv_jv_kv_l \mid 0 \le i < j < k < l \le 5\} \cup \{v_iv_jv_kv_lv_s \mid 0 \le i < j < k < l < s \le 5\} \cup \{v_0v_1v_2v_3v_4v_5\}$;

(v) $\mathcal{K} = \{v_0, v_0v_1, v_0v_2, v_1v_2, v_0v_1v_2\}$ is a simplicial complex on V;

(vi) $\mathcal{L} = \{v_0v_1v_2v_4, v_0v_1v_2v_3v_5, v_0v_1v_2v_3v_4, v_0v_1v_2v_4v_5, v_0v_1v_2v_3v_4v_5\}$ is an independence hypergraph on V.

Example 2.2 Consider the set $V = \mathbb{Z}$ of all the integers. Then

(i) for any $p \in \mathbb{Z}$ and any $q \ge 0$, the sequence $p(p+1)\cdots(p+q)$ of subsequent integers is a q-hyperedge on V;

(ii) for any $q \ge 0$, the collection $\mathcal{H}^{(q)} = \{p(p+1)\cdots(p+q) \mid p \equiv 1 \pmod 3\}$ of sequences of subsequent integers is a q-uniform hypergraph on V;

(iii) the collection $\mathcal{H} = \{p(p+1)\cdots(p+q) \mid p \equiv 1 \pmod 3 \text{ and } 2 \le q \le 5\}$ of sequences of subsequent integers is a hypergraph on V;

(iv) $\Delta[V] = \{i_0 \in \mathbb{Z}\} \cup \{i_0i_1 \in \mathbb{Z} \times \mathbb{Z} \mid i_0 < i_1\} \cup \{i_0i_1i_2 \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \mid i_0 < i_1 < i_2\} \cup \cdots$;

(v) the collection $\mathcal{K} = \{p(p+1)\cdots(p+q) \mid p \in \mathbb{Z} \text{ and } 0 \le q \le 5\}$ of sequences of subsequent integers is a simplicial complex on V;

(vi) the collection $\mathcal{L} = \{p(p+1)\cdots(p+q) \mid p \in \mathbb{Z} \text{ and } q > 5\}$ of sequences is an independence hypergraph on V.

3 Differential Calculus for Paths on Discrete Sets

In this section, we review the definitions of paths and elementary paths on a discrete set (see [15]). By applying some discrete differential calculus, we construct certain chain complexes and co-chain complexes for the space of paths on a discrete set.

3.1 Paths on discrete sets

Throughout this section, we let V be a discrete set. Let n be a non-negative integer.

Definition 3.1 (see [15, Definition 2.1]) *An elementary n-path on V is an ordered sequence $v_0v_1\cdots v_n$ of $n+1$ vertices in V. Here for any integers $0 \le i < j \le n$, we do not require $v_i \prec v_j$, $v_j \prec v_i$ or $v_i \ne v_j$.*

Definition 3.2 (see [15, Definition 2.2]) *A formal linear combination of elementary n-paths on V with coefficients in the real numbers $\mathbb{R}$ is called an n-path on V.*

Notation 3.1 (see [15, Subsection 2.1]) Denote by $\Lambda_n(V)$ the vector space of all n-paths on V. Then any element in $\Lambda_n(V)$ is of the form

$$\sum_{v_0,v_1,\cdots,v_n\in V} r_{v_0v_1\cdots v_n} v_0v_1\cdots v_n, \quad r_{v_0v_1\cdots v_n}\in\mathbb{R}.$$

Notation 3.2 Letting n run over all non-negative integers, we have a graded vector space

$$\Lambda_*(V)=\bigoplus_{n=0}^{\infty}\Lambda_n(V).$$

Notation 3.3 For each $n\geq 0$, we have a canonical inner product

$$\langle\ ,\ \rangle:\ \Lambda_n(V)\times\Lambda_n(V)\to\mathbb{R}$$

on $\Lambda_n(V)$ given by

$$\langle u_0u_1\cdots u_n, v_0v_1\cdots v_n\rangle=\prod_{i=0}^{n}\delta(u_i,v_i). \tag{3.1}$$

Remark 3.1 It follows from (3.1) that

• if $u_0u_1\cdots u_n$ and $v_0v_1\cdots v_n$ are identically the same elementary n-path, then

$$\langle u_0u_1\cdots u_n, v_0v_1\cdots v_n\rangle=1;$$

• if $u_0u_1\cdots u_n$ and $v_0v_1\cdots v_n$ are not the same elementary n-path, then

$$\langle u_0u_1\cdots u_n, v_0v_1\cdots v_n\rangle=0.$$

3.2 Partial derivatives on path spaces

Definition 3.3 *For any $v\in V$, we define the partial derivative of $\Lambda_*(V)$ with respect to v to be a sequence of linear maps*

$$\frac{\partial}{\partial v}:\ \Lambda_n(V)\to\Lambda_{n-1}(V),\quad n\geq 0$$

by letting

$$\frac{\partial}{\partial v}(v_0v_1\cdots v_n)=\sum_{i=0}^{n}(-1)^i\delta(v,v_i)v_0\cdots\widehat{v_i}\cdots v_n. \tag{3.2}$$

Here in (3.2)*, for any vertices $u,v\in V$, we use the notation $\delta(u,v)=1$ if $u=v$ and $\delta(u,v)=0$ if $u\neq v$. We extend* (3.2) *linearly over $\mathbb{R}$.*

Remark 3.2 By Definition 3.3, for any distinct vertices $v_0,v_1,\cdots,v_n$ in V we have the followings:

• If $v_i=v$ for some $0\leq i\leq n$, then

$$\frac{\partial}{\partial v}(v_0v_1\cdots v_n)=(-1)^i v_0\cdots\widehat{v_i}\cdots v_n;$$

• if $v_i\neq v$ for any $0\leq i\leq n$, then

$$\frac{\partial}{\partial v}(v_0v_1\cdots v_n)=0.$$

Lemma 3.1 (see [27, Lemma 2.7]) *For any $u, v \in V$, we have*

$$\frac{\partial}{\partial u} \circ \frac{\partial}{\partial v} = -\frac{\partial}{\partial v} \circ \frac{\partial}{\partial u}. \tag{3.3}$$

Proof Since both $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial v}$ are linear, it follows that both $\frac{\partial}{\partial u} \circ \frac{\partial}{\partial v}$ and $\frac{\partial}{\partial v} \circ \frac{\partial}{\partial u}$ are linear as well. Hence in order to prove the identity (3.3) as linear maps from $\Lambda_n(V; R)$ to $\Lambda_{n-1}(V; R)$, we only need to verify the identity (3.3) on an elementary n-path $v_0 v_1 \cdots v_n$. By the definition (3.2), we have

$$\begin{aligned}
\frac{\partial}{\partial u} \circ \frac{\partial}{\partial v}(v_0 v_1 \cdots v_n) &= \frac{\partial}{\partial u}\Big(\sum_{j=0}^{n} (-1)^j \delta(v, v_j) v_0 \cdots \widehat{v_j} \cdots v_n\Big) \\
&= \sum_{j=0}^{n} (-1)^j \delta(v, v_j) \frac{\partial}{\partial u}(v_0 \cdots \widehat{v_j} \cdots v_n) \\
&= \sum_{j=0}^{n} (-1)^j \delta(v, v_j) \sum_{i=0}^{j-1} (-1)^i \delta(u, v_i)(v_0 \cdots \widehat{v_i} \cdots \widehat{v_j} \cdots v_n) \\
&\quad + \sum_{j=0}^{n} (-1)^j \delta(v, v_j) \sum_{i=j+1}^{n} (-1)^{i-1} \delta(u, v_i)(v_0 \cdots \widehat{v_i} \cdots \widehat{v_j} \cdots v_n) \\
&= \sum_{0 \le i < j \le n} (-1)^{i+j} \delta(u, v_i) \delta(v, v_j)(v_0 \cdots \widehat{v_i} \cdots \widehat{v_j} \cdots v_n) \\
&\quad + \sum_{0 \le j < i \le n} (-1)^{i+j-1} \delta(u, v_i) \delta(v, v_j)(v_0 \cdots \widehat{v_j} \cdots \widehat{v_i} \cdots v_n).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{\partial}{\partial v} \circ \frac{\partial}{\partial u}(v_0 v_1 \cdots v_n) &= \sum_{0 \le j < i \le n} (-1)^{i+j} \delta(u, v_i) \delta(v, v_j)(v_0 \cdots \widehat{v_i} \cdots \widehat{v_j} \cdots v_n) \\
&\quad + \sum_{0 \le i < j \le n} (-1)^{i+j-1} \delta(u, v_i) \delta(v, v_j)(v_0 \cdots \widehat{v_j} \cdots \widehat{v_i} \cdots v_n).
\end{aligned}$$

Therefore, for any elementary n-path $v_0 v_1 \cdots v_n$ on V, we have

$$\frac{\partial}{\partial u} \circ \frac{\partial}{\partial v}(v_0 v_1 \cdots v_n) + \frac{\partial}{\partial v} \circ \frac{\partial}{\partial u}(v_0 v_1 \cdots v_n) = 0.$$

Consequently, by the linear property of $\frac{\partial}{\partial u} \circ \frac{\partial}{\partial v}$ and $\frac{\partial}{\partial v} \circ \frac{\partial}{\partial u}$, we obtain (3.3).

Notation 3.4 We denote $\frac{\partial}{\partial v} \circ \frac{\partial}{\partial u}$ as $\frac{\partial}{\partial v} \wedge \frac{\partial}{\partial u}$ for any $u, v \in V$.

Definition 3.4 *We consider the exterior algebra*

$$\mathrm{Ext}_*(V) = \bigwedge \Big(\frac{\partial}{\partial v} \Big| v \in V\Big)$$

and call it the differential algebra on V.

We have the following observations:

• The differential algebra $\mathrm{Ext}_*(V)$ is a direct sum

$$\mathrm{Ext}_*(V) = \bigoplus_{k=0}^{\infty} \mathrm{Ext}_k(V);$$

• $\mathrm{Ext}_0(V)=\mathbb{R}$ while for each $k\geq 1$, the space $\mathrm{Ext}_k(V)$ is the vector space spanned by all the following elements

$$\frac{\partial}{\partial v_1}\wedge\frac{\partial}{\partial v_2}\wedge\cdots\wedge\frac{\partial}{\partial v_k},\quad v_1,v_2,\cdots,v_k\in V$$

modulo the relation

$$\frac{\partial}{\partial v_1}\wedge\cdots\wedge\frac{\partial}{\partial v_i}\wedge\frac{\partial}{\partial v_{i+1}}\wedge\cdots\wedge\frac{\partial}{\partial v_k}=-\frac{\partial}{\partial v_1}\wedge\cdots\wedge\frac{\partial}{\partial v_{i+1}}\wedge\frac{\partial}{\partial v_i}\wedge\cdots\wedge\frac{\partial}{\partial v_k}$$

for any $1\leq i\leq k-1$;

• the exterior product

$$\wedge:\ \mathrm{Ext}_k(V)\times\mathrm{Ext}_l(V)\to\mathrm{Ext}_{k+l}(V),\quad k,l\geq 1$$

is the composition of linear maps. It is given by

$$\Big(\frac{\partial}{\partial v_1}\wedge\cdots\wedge\frac{\partial}{\partial v_k}\Big)\wedge\Big(\frac{\partial}{\partial u_1}\wedge\cdots\wedge\frac{\partial}{\partial u_l}\Big)=\frac{\partial}{\partial v_1}\wedge\cdots\wedge\frac{\partial}{\partial v_k}\wedge\frac{\partial}{\partial u_1}\wedge\cdots\wedge\frac{\partial}{\partial u_l}$$

which extends bilinearly over $\mathbb{R}$.

• For any $k\geq 1$ and any $\alpha\in\mathrm{Ext}_k(V)$, we have that α gives a sequence of linear maps

$$\alpha_n:\ \Lambda_n(V)\to\Lambda_{n-k}(V),\quad n\geq 0. \tag{3.4}$$

Here we adopt the notation that $\Lambda_{-n}(V)=0$ for any $n\geq 0$. Precisely, if we write

$$\alpha=\sum_{v_1,v_2,\cdots,v_k\in V} r_{v_1,v_2,\cdots,v_k}\frac{\partial}{\partial v_1}\wedge\frac{\partial}{\partial v_2}\wedge\cdots\wedge\frac{\partial}{\partial v_k},\quad r_{v_1,v_2,\cdots,v_k}\in\mathbb{R},$$

then for any elementary n-path $u_0u_1\cdots u_n$ on V with $n\geq k$, we have [2]

$$\begin{aligned}\alpha(u_0u_1\cdots u_n)&=\sum_{v_1,v_2,\cdots,v_k\in V} r_{v_1,v_2,\cdots,v_k}\frac{\partial}{\partial v_1}\wedge\frac{\partial}{\partial v_2}\wedge\cdots\wedge\frac{\partial}{\partial v_k}(u_0u_1\cdots u_n)\\&=\sum_{0\leq i_1<i_2<\cdots<i_k\leq n}\ \sum_{\sigma\in S_k} r_{u_{i_{\sigma(1)}},u_{i_{\sigma(2)}},\cdots,u_{i_{\sigma(k)}}}\mathrm{sgn}(\sigma)\\&\quad(-1)^{i_1+i_2+\cdots+i_k}u_0\cdots\widehat{u_{i_1}}\cdots\widehat{u_{i_2}}\cdots\widehat{u_{i_k}}\cdots u_n.\end{aligned}$$

Here S_k is the permutation group on k-letters and for any permutation $\sigma\in S_k$, we use $\mathrm{sgn}(\sigma)$ to denote the signature of σ.

[2] The expression of $\alpha(u_0u_1\cdots u_n)$ follows from the following two observations:
(i) For any $0\leq i_1<i_2<\cdots<i_k\leq n$, by applying (3.2) for k-times, we have

$$\frac{\partial}{\partial u_{i_1}}\wedge\frac{\partial}{\partial u_{i_2}}\wedge\cdots\wedge\frac{\partial}{\partial u_{i_k}}(u_0u_1\cdots u_n)=(-1)^{i_1+i_2+\cdots+i_k}u_0\cdots\widehat{u_{i_1}}\cdots\widehat{u_{i_2}}\cdots\widehat{u_{i_k}}\cdots u_n;$$

(ii) for any $\sigma\in S_k$, by applying (3.3) iteratively, we have

$$\frac{\partial}{\partial u_{i_{\sigma(1)}}}\wedge\frac{\partial}{\partial u_{i_{\sigma(2)}}}\wedge\cdots\wedge\frac{\partial}{\partial u_{i_{\sigma(k)}}}=\mathrm{sgn}(\sigma)\frac{\partial}{\partial u_{i_1}}\wedge\frac{\partial}{\partial u_{i_2}}\wedge\cdots\wedge\frac{\partial}{\partial u_{i_k}}.$$

3.3 Partial differentiations on path spaces

Definition 3.5 *For any $v \in V$, we define the partial differentiation* $\mathrm{d}v$ *with respect to v to be a sequence of linear maps*

$$\mathrm{d}v: \ \Lambda_n(V) \to \Lambda_{n+1}(V), \quad n \geq 0,$$

such that $\mathrm{d}v$ *is the adjoint linear map of $\frac{\partial}{\partial v}$ for each $n \geq 0$. Precisely, for any $n \geq 0$, any $\xi \in \Lambda_n(V)$, and any $\eta \in \Lambda_{n+1}(V)$, we have*

$$\Big\langle \frac{\partial}{\partial v}(\eta), \xi \Big\rangle = \langle \eta, \mathrm{d}v(\xi) \rangle. \tag{3.5}$$

The next lemma gives an explicit formula for $\mathrm{d}v$.

Lemma 3.2 (see [27, Lemma 2.10]) *For any $n \geq 1$, any $v \in V$, and any elementary $(n-1)$-path $u_0 u_1 \cdots u_{n-1}$ on V, we have*

$$\mathrm{d}v(u_0 u_1 \cdots u_{n-1}) = \sum_{i=0}^{n} (-1)^i u_0 u_1 \cdots u_{i-1} v u_i u_{i+1} \cdots u_{n-1}. \tag{3.6}$$

Proof In (3.5), we take η to be an elementary n-path $v_0 v_1 \cdots v_n \in \Lambda_n(V)$ and take ξ to be an elementary $(n-1)$-path $u_0 u_1 \cdots u_{n-1} \in \Lambda_{n-1}(V)$. Then

$$\begin{aligned}
\langle v_0 v_1 \cdots v_n, \mathrm{d}v(u_0 u_1 \cdots u_{n-1}) \rangle &= \Big\langle \frac{\partial}{\partial v}(v_0 v_1 \cdots v_n), u_0 u_1 \cdots u_{n-1} \Big\rangle \\
&= \Big\langle \sum_{i=0}^{n} (-1)^i \delta(v, v_i) v_0 \cdots \widehat{v_i} \cdots v_n, u_0 u_1 \cdots u_{n-1} \Big\rangle \\
&= \sum_{i=0}^{n} (-1)^i \delta(v, v_i) \prod_{j=0}^{i-1} \delta(v_j, u_j) \prod_{j=i}^{n-1} \delta(v_{j+1}, u_j).
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
\mathrm{d}v(u_0 u_1 \cdots u_{n-1}) &= \sum_{v_0, v_1, \cdots, v_n \in V} \langle v_0 v_1 \cdots v_n, \mathrm{d}v(u_0 u_1 \cdots u_{n-1}) \rangle v_0 v_1 \cdots v_n \\
&= \sum_{v_0, v_1, \cdots, v_n \in V} \Big(\sum_{i=0}^{n} (-1)^i \delta(v, v_i) \prod_{j=0}^{i-1} \delta(v_j, u_j) \prod_{j=i}^{n-1} \delta(v_{j+1}, u_j) \Big) v_0 v_1 \cdots v_n \\
&= \sum_{i=0}^{n} (-1)^i \Big(\sum_{v_0, v_1, \cdots, v_n \in V} \delta(v, v_i) \prod_{j=0}^{i-1} \delta(v_j, u_j) \prod_{j=i}^{n-1} \delta(v_{j+1}, u_j) \Big) v_0 v_1 \cdots v_n \\
&= \sum_{i=0}^{n} (-1)^i u_0 u_1 \cdots u_{i-1} v u_i u_{i+1} \cdots u_{n-1}.
\end{aligned} \tag{3.7}$$

We obtain (3.6).

The next corollary gives the case $n = 1$ in Lemma 3.2.

Corollary 3.1 *For any $u, v \in V$, we have* $\mathrm{d}v(u) = vu - uv$.

Similarly to the proof of Lemma 3.1, it is direct to verify the next lemma.

Lemma 3.3 (see [27, Lemma 2.7]) *For any $u, v \in V$, we have*

$$\mathrm{d}u \circ \mathrm{d}v = -\mathrm{d}v \circ \mathrm{d}u. \tag{3.8}$$

Proof [3] For any $n \geq 0$ and any elementary n-path $v_0v_1 \cdots v_n \in \Lambda_n(V)$, by (3.6), we have

$$\begin{aligned}
\mathrm{d}u \circ \mathrm{d}v(v_0v_1 \cdots v_n) &= \mathrm{d}u\Big(\sum_{i=0}^{n+1}(-1)^i v_0 \cdots v_{i-1}vv_i \cdots v_n\Big) \\
&= \sum_{i=0}^{n+1}(-1)^i \mathrm{d}u(v_0 \cdots v_{i-1}vv_i \cdots v_n) \\
&= \sum_{i=0}^{n+1}(-1)^i \Big(\sum_{j=0}^{i-1}(-1)^j v_0 \cdots v_{j-1}uv_j \cdots v_{i-1}vv_i \cdots v_n \\
&\quad + (-1)^i v_0 \cdots v_{i-1}uvv_i \cdots v_n + (-1)^{i+1} v_0 \cdots v_{i-1}vuv_i \cdots v_n \\
&\quad + \sum_{j=i+1}^{n+1}(-1)^{j+1} v_0 \cdots v_{i-1}vv_i \cdots v_{j-1}uv_j \cdots v_n\Big)
\end{aligned}$$

while

$$\begin{aligned}
\mathrm{d}v \circ \mathrm{d}u(v_0v_1 \cdots v_n) &= \sum_{i=0}^{n+1}(-1)^i \Big(\sum_{j=0}^{i-1}(-1)^j v_0 \cdots v_{j-1}vv_j \cdots v_{i-1}uv_i \cdots v_n \\
&\quad + (-1)^i v_0 \cdots v_{i-1}vuv_i \cdots v_n + (-1)^{i+1} v_0 \cdots v_{i-1}uvv_i \cdots v_n \\
&\quad + \sum_{j=i+1}^{n+1}(-1)^{j+1} v_0 \cdots v_{i-1}uv_i \cdots v_{j-1}vv_j \cdots v_n\Big).
\end{aligned}$$

Thus

$$\mathrm{d}u \circ \mathrm{d}v(v_0v_1 \cdots v_n) = -\mathrm{d}v \circ \mathrm{d}u(v_0v_1 \cdots v_n)$$

for any $n \geq 0$ and any elementary n-path $v_0v_1 \cdots v_n \in \Lambda_n(V)$. Consequently, sine both $\mathrm{d}u \circ \mathrm{d}v$ and $\mathrm{d}v \circ \mathrm{d}u$ are linear, we obtain (3.8).

Notation 3.5 We denote $\mathrm{d}u \circ \mathrm{d}v$ as $\mathrm{d}u \wedge \mathrm{d}v$ for any $u, v \in V$.

Definition 3.6 *We consider the exterior algebra*

$$\mathrm{Ext}^*(V) = \wedge(\mathrm{d}v \mid v \in V)$$

and call it the co-differential algebra on V.

[3] An alternative proof for Lemma 3.3 follows from Lemma 3.1 directly: Let $u, v \in V$. For any $n \geq 0$, any $\xi \in \Lambda_n(V)$ and any $\eta \in \Lambda_{n+2}(V)$, we have

$$\begin{aligned}
\langle \eta, \mathrm{d}u \wedge \mathrm{d}v(\xi)\rangle &= \Big\langle \frac{\partial}{\partial u}(\eta), \mathrm{d}v(\xi)\Big\rangle = \Big\langle \frac{\partial}{\partial v} \wedge \frac{\partial}{\partial u}(\eta), \xi\Big\rangle \\
&= -\Big\langle \frac{\partial}{\partial u} \wedge \frac{\partial}{\partial v}(\eta), \xi\Big\rangle = -\Big\langle \frac{\partial}{\partial v}(\eta), \mathrm{d}u(\xi)\Big\rangle = -\langle \eta, \mathrm{d}v \wedge \mathrm{d}u(\xi)\rangle.
\end{aligned}$$

This implies (3.8). Nevertheless, the proof for Lemma 3.3 in the main-body consolidates (3.6) in Lemma 3.2.

We have the following observations:

• $\mathrm{Ext}^*(V)$ is a direct sum

$$\mathrm{Ext}^*(V)=\bigoplus_{k=0}^{\infty}\mathrm{Ext}^k(V).$$

• $\mathrm{Ext}^0(V)=\mathbb{R}$ while for each $k\geq 1$, the space $\mathrm{Ext}^k(V)$ is spanned by

$$\mathrm{d}v_1\wedge\mathrm{d}v_2\wedge\cdots\wedge\mathrm{d}v_k,\quad v_1,v_2,\cdots,v_k\in V$$

modulo the relation

$$\mathrm{d}v_1\wedge\cdots\wedge\mathrm{d}v_i\wedge\mathrm{d}v_{i+1}\wedge\cdots\wedge\mathrm{d}v_k=-\mathrm{d}v_1\wedge\cdots\wedge\mathrm{d}v_{i+1}\wedge\mathrm{d}v_i\wedge\cdots\wedge\mathrm{d}v_k$$

for any $1\leq i\leq k-1$.

• For any $k\geq 1$ and any $\omega\in\mathrm{Ext}^k(V)$, we have that ω gives a sequence of linear maps

$$\omega_n:\ \Lambda_n(V)\to\Lambda_{n+k}(V),\quad n\geq 0. \tag{3.9}$$

Definition 3.7 *Let $k\geq 1$, $\alpha\in\mathrm{Ext}_k(V)$ and $\omega\in\mathrm{Ext}^k(V)$. We say that α and ω are adjoint to each other if for any $n\geq 0$, any $\xi\in\Lambda_n(V)$ and any $\eta\in\Lambda_{n+k}(V)$, the identity*

$$\langle\alpha(\eta),\xi\rangle=\langle\eta,\omega(\xi)\rangle$$

is satisfied.

Proposition 3.1 *Let $k\geq 1$ be any positive integer. Let $\alpha\in\mathrm{Ext}_k(V)$ be given by*

$$\alpha=\sum_{v_1,v_2,\cdots,v_k\in V}r_{v_1,v_2,\cdots,v_k}\frac{\partial}{\partial v_1}\wedge\frac{\partial}{\partial v_2}\wedge\cdots\wedge\frac{\partial}{\partial v_k},\quad r_{v_1,v_2,\cdots,v_k}\in\mathbb{R}.$$

Suppose $\omega\in\mathrm{Ext}^k(V)$ is adjoint to α. Then ω is given by

$$\omega=\mathrm{sgn}(k)\sum_{v_1,v_2,\cdots,v_k\in V}r_{v_1,v_2,\cdots,v_k}\mathrm{d}v_1\wedge\mathrm{d}v_2\wedge\cdots\wedge\mathrm{d}v_k,\quad r_{v_1,v_2,\cdots,v_k}\in\mathbb{R}, \tag{3.10}$$

where $\mathrm{sgn}(k)=1$ *if* $k\equiv 0,1\ (\mathrm{mod}\,4)$ *and* $\mathrm{sgn}(k)=-1$ *if* $k\equiv 2,3\ (\mathrm{mod}\,4)$.

Proof Let $n\geq 0$, $\xi\in\Lambda_n(V)$ and $\eta\in\Lambda_{n+k}(V)$. Then we have

$$\begin{aligned}\langle\alpha(\eta),\xi\rangle&=\sum_{v_1,v_2,\cdots,v_k\in V}r_{v_1,v_2,\cdots,v_k}\Big\langle\frac{\partial}{\partial v_1}\wedge\frac{\partial}{\partial v_2}\wedge\cdots\wedge\frac{\partial}{\partial v_k}(\eta),\xi\Big\rangle\\&=\sum_{v_1,v_2,\cdots,v_k\in V}r_{v_1,v_2,\cdots,v_k}\Big\langle\frac{\partial}{\partial v_2}\wedge\cdots\wedge\frac{\partial}{\partial v_k}(\eta),\mathrm{d}v_1(\xi)\Big\rangle\\&=\sum_{v_1,v_2,\cdots,v_k\in V}r_{v_1,v_2,\cdots,v_k}\Big\langle\frac{\partial}{\partial v_3}\wedge\cdots\wedge\frac{\partial}{\partial v_k}(\eta),\mathrm{d}v_2\wedge\mathrm{d}v_1(\xi)\Big\rangle\\&=\cdots\end{aligned}$$

$$= \sum_{v_1,v_2,\cdots,v_k\in V} r_{v_1,v_2,\cdots,v_k}\langle \eta, \mathrm{d}v_k \wedge \mathrm{d}v_{k-1} \wedge \cdots \wedge \mathrm{d}v_1(\xi)\rangle$$
$$= \sum_{v_1,v_2,\cdots,v_k\in V} r_{v_1,v_2,\cdots,v_k}\mathrm{sgn}(k)\langle \eta, \mathrm{d}v_1 \wedge \mathrm{d}v_2 \wedge \cdots \wedge \mathrm{d}v_k(\xi)\rangle.$$

The last equality follows from the fact that the permutation $(k, k-1, \cdots, 1)$ of $(1, 2, \cdots, k)$ has the signature

$$\mathrm{sgn}\begin{pmatrix} 1, 2, \cdots, k \\ k, k-1, \cdots, 1 \end{pmatrix} = (-1)^{(k-1)+(k-2)+\cdots+1} = (-1)^{\frac{k(k-1)}{2}}$$

for any $k \geq 2$ and the permutation $(k, k-1, \cdots, 1)$ of $(1, 2, \cdots, k)$ has the signature 1 for $k = 1$. In other words, the permutation $(k, k-1, \cdots, 1)$ of $(1, 2, \cdots, k)$ has the signature 1 for $k \equiv 0, 1 \pmod 4$ and has the signature -1 for $k \equiv 2, 3 \pmod 4$. Therefore, we have that ω given by (3.10) is adjoint to α. The proposition follows.

3.4 Some chain complexes and co-chain complexes on path spaces

Proposition 3.2 *Let t be a non-negative integer. Let $\alpha \in \mathrm{Ext}_{2t+1}(V)$ and $\omega \in \mathrm{Ext}^{2t+1}(V)$. Then for any $0 \leq q \leq 2t$, we have a chain complex*

$$\cdots \xrightarrow{\alpha} \Lambda_{n(2t+1)+q}(V) \xrightarrow{\alpha} \Lambda_{(n-1)(2t+1)+q}(V) \xrightarrow{\alpha}$$

$$\cdots \xrightarrow{\alpha} \Lambda_{(2t+1)+q}(V) \xrightarrow{\alpha} \Lambda_q(V) \xrightarrow{\alpha} 0$$

and a co-chain complex

$$\cdots \xleftarrow{\omega} \Lambda_{n(2t+1)+q}(V) \xleftarrow{\omega} \Lambda_{(n-1)(2t+1)+q}(V) \xleftarrow{\omega}$$

$$\cdots \xleftarrow{\omega} \Lambda_{(2t+1)+q}(V) \xleftarrow{\omega} \Lambda_q(V) \xleftarrow{\omega} 0.$$

Proof Let $t \geq 0$. Let $\alpha \in \mathrm{Ext}_{2t+1}(V)$ and $\omega \in \mathrm{Ext}^{2t+1}(V)$. Let $0 \leq q \leq 2t$. Note that for each $n \geq 0$, the maps

$$\alpha: \ \Lambda_{n(2t+1)+q}(V) \to \Lambda_{(n-1)(2t+1)+q}(V)$$

and

$$\omega: \ \Lambda_{n(2t+1)+q}(V) \to \Lambda_{(n+1)(2t+1)+q}(V)$$

are well-defined. By the anti-symmetric property of exterior algebras,

$$\alpha \wedge \alpha = (-1)^{(2t+1)^2}\alpha \wedge \alpha, \quad \omega \wedge \omega = (-1)^{(2t+1)^2}\omega \wedge \omega.$$

Since $(2t+1)^2$ is odd,

$$\alpha \circ \alpha = \alpha \wedge \alpha = 0, \quad \omega \circ \omega = \omega \wedge \omega = 0.$$

Thus for any $0 \leq q \leq 2t$, we have the chain complex as well as the co-chain complex given in the proposition.

Notation 3.6 Let $0 \leq q \leq 2t$. Let $\alpha \in \mathrm{Ext}_{2t+1}(V)$ and $\omega \in \mathrm{Ext}^{2t+1}(V)$. We adopt the following notations:

(i) Denote the chain complex in Proposition 3.2 as

$$\Lambda_*(V, \alpha, q) = \{\Lambda_{n(2t+1)+q}(V), \alpha\}_{n \geq 0};$$

(ii) denote the co-chain complex in Proposition 3.2 as

$$\Lambda^*(V, \omega, q) = \{\Lambda_{n(2t+1)+q}(V), \omega\}_{n \geq 0}.$$

Notation 3.7 For any integer m, there is a unique integer λ (not necessarily non-negative) and a unique integer $0 \leq q \leq 2t$ such that $m = \lambda(2t+1) + q$. We adopt the following notations:

(i) Denote the chain complex

$$\cdots \xrightarrow{\alpha} \Lambda_{(n+\lambda)(2t+1)+q}(V) \xrightarrow{\alpha} \Lambda_{(n-1+\lambda)(2t+1)+q}(V) \xrightarrow{\alpha}$$

$$\cdots \xrightarrow{\alpha} \Lambda_{(1+\lambda)(2t+1)+q}(V) \xrightarrow{\alpha} \Lambda_{\lambda(2t+1)+q}(V) \xrightarrow{\alpha} 0$$

as

$$\Lambda_*(V, \alpha, m) = \{\Lambda_{(n+\lambda)(2t+1)+q}(V), \alpha\}_{n \geq 0};$$

(ii) denote the co-chain complex

$$\cdots \xleftarrow{\omega} \Lambda_{(n+\lambda)(2t+1)+q}(V) \xleftarrow{\omega} \Lambda_{(n-1+\lambda)(2t+1)+q}(V) \xleftarrow{\omega}$$

$$\cdots \xleftarrow{\omega} \Lambda_{(1+\lambda)(2t+1)+q}(V) \xleftarrow{\omega} \Lambda_{\lambda(2t+1)+q}(V) \xleftarrow{\omega} 0$$

as

$$\Lambda^*(V, \omega, m) = \{\Lambda_{(n+\lambda)(2t+1)+q}(V), \omega\}_{n \geq 0}.$$

Here in both (i) and (ii), we use the notation $\Lambda_k(V) = 0$ for $k < 0$.

Proposition 3.3 *Let t, s be non-negative integers. Let $m \in \mathbb{Z}$. Let $\alpha \in \mathrm{Ext}_{2t+1}(V)$ and $\omega \in \mathrm{Ext}^{2t+1}(V)$. Let $\beta \in \mathrm{Ext}_{2s}(V)$ and $\mu \in \mathrm{Ext}^{2s}(V)$. Then β gives a chain map*

$$\beta: \ \Lambda_*(V, \alpha, m) \to \Lambda_*(V, \alpha, m - 2s)$$

and μ gives a co-chain map

$$\mu: \ \Lambda^*(V, \omega, m) \to \Lambda^*(V, \omega, m + 2s).$$

Proof Note that as linear maps,

$$\beta: \ \Lambda_{(n+\lambda)(2t+1)+q}(V) \to \Lambda_{(n+\lambda)(2t+1)+q-2s}(V)$$

and

$$\mu: \ \Lambda_{(n+\lambda)(2t+1)+q}(V) \to \Lambda_{(n+\lambda)(2t+1)+q+2s}(V)$$

are well-defined. By the anti-symmetric property of exterior algebras, we have (see [4, p, 53, Anticommutative Law])

$$\alpha \wedge \beta = (-1)^{2s(2t+1)} \beta \wedge \alpha = \beta \wedge \alpha.$$

That is,

$$\alpha \circ \beta = \beta \circ \alpha.$$

Thus β is a chain map from $\Lambda_*(V, \alpha, m)$ to $\Lambda_*(V, \alpha, m-2s)$. Moreover, we also have (see [4, p, 53, Anticommutative Law])

$$\omega \wedge \mu = (-1)^{2s(2t+1)} \mu \wedge \omega = \mu \wedge \omega.$$

That is,

$$\omega \circ \mu = \mu \circ \omega.$$

Thus μ is a co-chain map from $\Lambda^*(V, \omega, m)$ to $\Lambda^*(V, \omega, m+2s)$. The proposition follows.

4 Constrained Homology for Simplicial Complexes and Constrained Cohomology for Independence Hypergraphs

In this section, we define the constrained homology groups for simplicial complexes and the constrained cohomology groups for independence hypergraphs. We prove that any element $\beta \in \mathrm{Ext}_{2s}(V)$, where $s \geq 0$, induces homomorphisms between the constrained homology groups for the simplicial complexes on V. We also prove that any element $\mu \in \mathrm{Ext}^{2s}(V)$, where $s \geq 0$, induces homomorphisms between the constrained cohomology groups for the independence hypergraphs on V.

4.1 Some auxiliaries

Throughout this section, we let V be a finite set. Let $\Delta[V]$ be the complete hypergraph on V. For each integer $n \geq 0$, let

$$C_n(\Delta[V]; \mathbb{R}) = \mathrm{Span}_{\mathbb{R}}\{\sigma^{(n)} \in \Delta[V]\}$$

be the vector space consisting of all the linear combinations of the n-hyperedges on V. Consider the direct sum

$$C_*(\Delta[V]; \mathbb{R}) = \bigoplus_{n \geq 0} C_n(\Delta[V]; \mathbb{R}). \tag{4.1}$$

Note that since V is assumed to be a finite set, the direct sum in the right-hand side of (4.1) is a finite sum.

Lemma 4.1 *Let t be a non-negative integer. Let $m \in \mathbb{Z}$. Suppose $m = \lambda(2t+1) + q$ where $\lambda \in \mathbb{Z}$ and the integer $0 \leq q \leq 2t$. Then for any $\alpha \in \mathrm{Ext}_{2t+1}(V)$, the graded vector space*

$$C_{(n+\lambda)(2t+1)+q}(\Delta[V]; \mathbb{R}), \quad n \geq 0 \tag{4.2}$$

equipped with the boundary map α gives a sub-chain complex of $\Lambda_*(V, \alpha, m)$, which will be denoted as $C_*(\Delta[V], \alpha, m)$.

Proof For each $n \geq 0$, the vector space $C_{(n+\lambda)(2t+1)+q}(\Delta[V]; \mathbb{R})$ is a subspace of the vector space $\Lambda_{(n+\lambda)(2t+1)+q}(V)$. Hence in order to prove that (4.2) equipped with α is a sub-chain complex of $\Lambda_*(V, \alpha, m)$, it suffices to prove that the map

$$\alpha: \ C_{(n+\lambda)(2t+1)+q}(\Delta[V]; \mathbb{R}) \to C_{(n-1+\lambda)(2t+1)+q}(\Delta[V]; \mathbb{R}) \tag{4.3}$$

is well-defined for each $n \geq 0$. This follows from the observation that for any $[(n+\lambda)(2t+1)+q]$-simplex

$$v_0 v_1 \cdots v_{(n+\lambda)(2t+1)+q} \in C_{(n+\lambda)(2t+1)+q}(\Delta[V]; R)$$

and any

$$\alpha = \frac{\partial}{\partial u_1} \wedge \frac{\partial}{\partial u_2} \wedge \cdots \wedge \frac{\partial}{\partial u_{2t+1}}$$

where $u_1, u_2, \cdots, u_{2t+1} \in V$ and $u_1 \prec u_2 \prec \cdots \prec u_{2t+1}$, we have

$$\alpha(v_0 v_1 \cdots v_{(n+\lambda)(2t+1)+q}) \in C_{(n-1+\lambda)(2t+1)+q}(\Delta[V]; R).$$

By a calculation of linear combinations, it follows that the map (4.3) is well-defined. Therefore, the graded vector space (4.2) equipped with α is a sub-chain complex of $\Lambda_*(V, \alpha, m)$.

Definition 4.1 *For any $n \geq 1$ and any elementary n-path $v_0 v_1 \cdots v_n$ on V, we call $v_0 v_1 \cdots v_n$ a non-simplicial elementary n-path if there exist integers $0 \leq i < j \leq n$ such that either $v_j \prec v_i$ or $v_j = v_i$.*

Definition 4.2 *Let $\mathcal{O}_n(V)$ be the vector space spanned by all the non-simplicial elementary n-paths on V. Then $\mathcal{O}_n(V)$ consists of all the linear combinations of the non-simplicial elementary n-paths on V. We call an element in $\mathcal{O}_n(V)$ a non-simplicial n-path on V.*

Lemma 4.2 *Let $t \geq 0$ be an integer. Let $m \in \mathbb{Z}$. Suppose $m = \lambda(2t+1) + q$ where $\lambda \in \mathbb{Z}$ and the integer $0 \leq q \leq 2t$. Then for any $\omega \in \mathrm{Ext}^{2t+1}(V)$, the graded vector space*

$$\mathcal{O}_{(n+\lambda)(2t+1)+q}(V), \quad n \geq 0 \tag{4.4}$$

equipped with the co-boundary map ω gives a sub-co-chain complex of $\Lambda^*(V, \omega, m)$, which will be denoted as $\mathcal{O}^*(V, \omega, m)$.

Proof It suffices to verify that the map

$$\omega: \ \mathcal{O}_{(n+\lambda)(2t+1)+q}(V) \to \mathcal{O}_{(n+1+\lambda)(2t+1)+q}(V) \tag{4.5}$$

is well-defined for each $n \geq 0$. This follows from the observation that after adding some vertices to any non-simplicial elementary path, we still get a non-simplicial elementary path. Hence for any non-simplicial elementary $[(n+\lambda)(2t+1)+q]$-path

$$v_0 v_1 \cdots v_{(n+\lambda)(2t+1)+q} \in \mathcal{O}_{(n+\lambda)(2t+1)+q}(V)$$

and any

$$\omega = \mathrm{d}u_1 \wedge \mathrm{d}u_2 \wedge \cdots \wedge \mathrm{d}u_{2t+1}$$

where $u_1, u_2, \cdots, u_{2t+1} \in V$ and $u_1 \prec u_2 \prec \cdots \prec u_{2t+1}$, we have

$$\omega(v_0 v_1 \cdots v_{(n+\lambda)(2t+1)+q}) \in \mathcal{O}_{(n+1+\lambda)(2t+1)+q}(V).$$

By a calculation of linear combinations, it follows that the map (4.5) is well-defined. Therefore, the graded vector space (4.4) equipped with ω is a sub-co-chain complex of $\Lambda^*(V, \omega, m)$.

Lemma 4.3 *Let $t \geq 0$ be an integer. Let $m \in \mathbb{Z}$. Suppose $m = \lambda(2t+1) + q$ where $\lambda \in \mathbb{Z}$ and the integer $0 \leq q \leq 2t$. Then for any $\omega \in \mathrm{Ext}^{2t+1}(V)$, the graded vector space*

$$C_{(n+\lambda)(2t+1)+q}(\Delta[V]; \mathbb{R}), \quad n \geq 0 \tag{4.6}$$

equipped with the co-boundary map ω gives a quotient co-chain complex

$$\Lambda^*(V, \omega, m)/\mathcal{O}^*(V, \omega, m),$$

which will be denoted as $C^*(\Delta[V], \omega, m)$.

Proof Note that the canonical inclusion of the sub-co-chain complex $\mathcal{O}^*(V, \omega, m)$ into the co-chain complex $\Lambda^*(V, \omega, m)$ gives a quotient co-chain complex $\Lambda^*(V, \omega, m)/\mathcal{O}^*(V, \omega, m)$. On the other hand, for each $n \geq 0$, the quotient vector space

$$\Lambda_{(n+\lambda)(2t+1)+q}(V)/\mathcal{O}_{(n+\lambda)(2t+1)+q}(V)$$

is canonically isomorphic to the vector space $C_{(n+\lambda)(2t+1)+q}(\Delta[V]; \mathbb{R})$. Therefore, the quotient co-chain complex $\Lambda^*(V, \omega, m)/\mathcal{O}^*(V, \omega, m)$ is given by the graded vector space (4.6) equipped with the co-boundary map ω. The lemma follows.

With the help of Proposition 3.3, the next proposition follows.

Proposition 4.1 *Let t, s be non-negative integers. Let $m \in \mathbb{Z}$. Let $\alpha \in \mathrm{Ext}_{2t+1}(V)$ and $\omega \in \mathrm{Ext}^{2t+1}(V)$. Let $\beta \in \mathrm{Ext}_{2s}(V)$ and $\mu \in \mathrm{Ext}^{2s}(V)$. Then β is a chain map*

$$\beta: \ C_*(\Delta[V], \alpha, m) \to C_*(\Delta[V], \alpha, m - 2s) \tag{4.7}$$

and μ is a co-chain map

$$\mu: \ C^*(\Delta[V], \omega, m) \to C^*(\Delta[V], \omega, m + 2s). \tag{4.8}$$

Proof By a similar argument in the proof of Lemma 4.1, it can be verified that as a linear map, β in (4.7) is well-defined. Thus by Proposition 3.3 and Lemma 4.1, it follows that β in (4.7) is a chain map. On the other hand, by a similar argument in the proof of Lemma 4.3, it can be verified that as a linear map, μ in (4.8) is well-defined. Thus by Proposition 3.3 and Lemma 4.3, it follows that ω in (4.8) is a co-chain map.

4.2 Constrained homology for simplicial complexes

Let $\mathcal{K}$ be a simplicial complex with its vertices in V.

Notation 4.1 For each non-negative integer n, let $C_n(\mathcal{K}; \mathbb{R})$ be the (real) vector space consisting of all the linear combinations of the n-simplices in $\mathcal{K}$.

Theorem 4.1 *Let t, s be non-negative integers. Let $m \in \mathbb{Z}$. Suppose $m = \lambda(2t+1) + q$ where $\lambda \in \mathbb{Z}$ and $0 \le q \le 2t$. Then*

(i) *for any $\alpha \in \mathrm{Ext}_{2t+1}(V)$, the graded vector space*

$$C_{(n+\lambda)(2t+1)+q}(\mathcal{K};\mathbb{R}), \quad n \ge 0 \tag{4.9}$$

equipped with the chain map α gives a sub-chain complex of $C_*(\Delta[V], \alpha, m)$, which will be denoted as $C_*(\mathcal{K}, \alpha, m)$;

(ii) *for any $\beta \in \mathrm{Ext}_{2s}(V)$, there is an induced chain map*

$$\beta: \ C_*(\mathcal{K}, \alpha, m) \to C_*(\mathcal{K}, \alpha, m - 2s). \tag{4.10}$$

Proof We prove (i) and (ii) subsequently.

(i) For each $n \ge 0$, the vector space $C_{(n+\lambda)(2t+1)+q}(\mathcal{K};\mathbb{R})$ is a subspace of the vector space $C_{(n+\lambda)(2t+1)+q}(\Delta[V];\mathbb{R})$. Hence in order to prove that the graded vector space (4.9) equipped with the chain map α is a sub-chain complex of $C_*(\Delta[V], \alpha, m)$, it suffices to prove that the map

$$\alpha: \ C_{(n+\lambda)(2t+1)+q}(\mathcal{K};\mathbb{R}) \to C_{(n-1+\lambda)(2t+1)+q}(\mathcal{K};\mathbb{R}) \tag{4.11}$$

is well-defined for each $n \ge 0$. This follows from the observation that for any $[(n+\lambda)(2t+1)+q]$-simplex

$$v_0 v_1 \cdots v_{(n+\lambda)(2t+1)+q} \in C_{(n+\lambda)(2t+1)+q}(\mathcal{K}; R)$$

and any

$$\alpha = \frac{\partial}{\partial u_1} \wedge \frac{\partial}{\partial u_2} \wedge \cdots \wedge \frac{\partial}{\partial u_{2t+1}},$$

where $u_1, u_2, \cdots, u_{2t+1} \in V$ and $u_1 \prec u_2 \prec \cdots \prec u_{2t+1}$, we have

$$\alpha(v_0 v_1 \cdots v_{(n+\lambda)(2t+1)+q}) \in C_{(n-1+\lambda)(2t+1)+q}(\mathcal{K}; R).$$

By a calculation of linear combinations, it follows that the map (4.11) is well-defined. Therefore, the graded vector space (4.9) equipped with the chain map α is a sub-chain complex of $C_*(\Delta[V], \alpha, m)$.

(ii) Similar with the verification that the map α in (4.11) is well-defined for each $n \ge 0$, we can prove that the map

$$\beta: \ C_{(n+\lambda)(2t+1)+q}(\mathcal{K};\mathbb{R}) \to C_{(n+\lambda)(2t+1)+q-2s}(\mathcal{K};\mathbb{R})$$

is well-defined for each $n \ge 0$. Therefore, with the help of (4.7) in Proposition 4.1, we have that β gives a chain map in (4.10).

Definition 4.3 *Let t be a non-negative integer. Let $\alpha \in \mathrm{Ext}_{2t+1}(V)$. Let $m \in \mathbb{Z}$. Suppose $m = \lambda(2t+1) + q$ where $\lambda \in \mathbb{Z}$ and $0 \le q \le 2t$. Let $\mathcal{K}$ be a simplicial complex with its vertices in V. For each $n \ge 0$, we define the n-th constrained homology group $H_n(\mathcal{K}, \alpha, m)$ of $\mathcal{K}$ with respect to α and m to be the n-th homology group*

$$\begin{aligned} H_n(\mathcal{K}, \alpha, m) &:= H_n(C_*(\mathcal{K}, \alpha, m)) \\ &= \frac{\mathrm{Ker}(\alpha : C_{(n+\lambda)(2t+1)+q}(\mathcal{K};\mathbb{R}) \to C_{(n-1+\lambda)(2t+1)+q}(\mathcal{K};\mathbb{R}))}{\mathrm{Im}(\alpha : C_{(n+1+\lambda)(2t+1)+q}(\mathcal{K};\mathbb{R}) \to C_{(n+\lambda)(2t+1)+q}(\mathcal{K};\mathbb{R}))} \end{aligned}$$

of the chain complex $C_*(\mathcal{K}, \alpha, m)$.

The next theorem follows from Theorem 4.1 and Definition 4.3 immediately.

Theorem 4.2 (Main Result I) *Let t, s be non-negative integers. Let $m \in \mathbb{Z}$. Suppose $m = \lambda(2t+1)+q$ where $\lambda \in \mathbb{Z}$ and $0 \le q \le 2t$. Then for any $\alpha \in \mathrm{Ext}_{2t+1}(V)$ and $\beta \in \mathrm{Ext}_{2s}(V)$, there is an induced homomorphism*

$$\beta_* : \ H_n(\mathcal{K}, \alpha, m) \to H_n(\mathcal{K}, \alpha, m-2s), \quad n \ge 0 \tag{4.12}$$

of the constrained homology groups.

Proof Apply the homology functor to the chain complex in Theorem 4.1(i) and the chain map in Theorem 4.1(ii). We obtain the homomorphism β_* of the constrained homology groups in (4.12).

4.3 Constrained cohomology for independence hypergraphs

Let $\mathcal{L}$ be an independence hypergraph with its vertices in V.

Notation 4.2 For each non-negative integer n, let $C_n(\mathcal{L}; \mathbb{R})$ be the (real) vector space consisting of all the linear combinations of the n-hyperedges in $\mathcal{L}$.

Theorem 4.3 *Let t, s be non-negative integers. Let $m \in \mathbb{Z}$. Suppose $m = \lambda(2t+1) + q$ where $\lambda \in \mathbb{Z}$ and $0 \le q \le 2t$. Then*

(i) *for any $\omega \in \mathrm{Ext}^{2t+1}(V)$, the graded vector space*

$$C_{(n+\lambda)(2t+1)+q}(\mathcal{L}; \mathbb{R}), \quad n \ge 0 \tag{4.13}$$

equipped with the co-boundary map ω gives a sub-co-chain complex of $C^*(\Delta[V], \omega, m)$, which will be denoted as $C^*(\mathcal{L}, \omega, m)$;

(ii) *for any $\mu \in \mathrm{Ext}^{2s}(V)$, there is an induced co-chain map*

$$\mu : \ C^*(\mathcal{L}, \omega, m) \to C^*(\mathcal{L}, \omega, m+2s). \tag{4.14}$$

Proof We prove (i) and (ii) subsequently.

(i) For each $n \ge 0$, the vector space $C_{(n+\lambda)(2t+1)+q}(\mathcal{L}; \mathbb{R})$ is a subspace of the vector space $C_{(n+\lambda)(2t+1)+q}(\Delta[V]; \mathbb{R})$. Hence in order to prove that the graded vector space (4.13) equipped with the co-boundary map ω is a sub-co-chain complex of $C^*(\Delta[V], \omega, m)$, it suffices to prove that the map

$$\omega : \ C_{(n+\lambda)(2t+1)+q}(\mathcal{L}; \mathbb{R}) \to C_{(n+1+\lambda)(2t+1)+q}(\mathcal{L}; \mathbb{R}) \tag{4.15}$$

is well-defined for each $n \ge 0$. This follows from the observation that for any $[(n+\lambda)(2t+1)+q]$-hyperedge

$$v_0 v_1 \cdots v_{(n+\lambda)(2t+1)+q} \in C_{(n+\lambda)(2t+1)+q}(\mathcal{L}; R)$$

and any

$$\omega = \mathrm{d}u_1 \wedge \mathrm{d}u_2 \wedge \cdots \wedge \mathrm{d}u_{2t+1}$$

where $u_1, u_2, \cdots, u_{2t+1} \in V$ and $u_1 \prec u_2 \prec \cdots \prec u_{2t+1}$, we have

$$\omega(v_0 v_1 \cdots v_{(n+\lambda)(2t+1)+q}) \in C_{(n+1+\lambda)(2t+1)+q}(\mathcal{L}; R).$$

By a calculation of linear combinations, it follows that the map (4.15) is well-defined. Therefore, the graded vector space (4.13) equipped with the co-boundary map ω is a sub-co-chain complex of $C^*(\Delta[V], \omega, m)$.

(ii) Similar with the verification that the map ω in (4.15) is well-defined for each $n \geq 0$, we can prove that the map

$$\mu : \ C_{(n+\lambda)(2t+1)+q}(\mathcal{L}; \mathbb{R}) \to C_{(n+\lambda)(2t+1)+q+2s}(\mathcal{L}; \mathbb{R})$$

is well-defined for each $n \geq 0$. Therefore, with the help of (4.8) in Proposition 4.1, we have that μ gives a co-chain map in (4.14).

Definition 4.4 *Let t be a non-negative integer. Let $\omega \in \mathrm{Ext}^{2t+1}(V)$. Let $m \in \mathbb{Z}$. Suppose $m = \lambda(2t+1) + q$ where $\lambda \in \mathbb{Z}$ and $0 \leq q \leq 2t$. Let $\mathcal{L}$ be an independence hypergraph with its vertices in V. For each $n \geq 0$, we define the n-th constrained cohomology group $H^n(\mathcal{L}, \omega, m)$ of $\mathcal{L}$ with respect to ω and m to be the cohomology group*

$$\begin{aligned} H^n(\mathcal{L}, \omega, m) &:= H^n(C^*(\mathcal{L}, \omega, m)) \\ &= \frac{\mathrm{Ker}(\omega : C_{(n+\lambda)(2t+1)+q}(\mathcal{L}; \mathbb{R}) \to C_{(n+1+\lambda)(2t+1)+q}(\mathcal{L}; \mathbb{R}))}{\mathrm{Im}(\omega : C_{(n-1+\lambda)(2t+1)+q}(\mathcal{L}; \mathbb{R}) \to C_{(n+\lambda)(2t+1)+q}(\mathcal{L}; \mathbb{R}))} \end{aligned}$$

of the co-chain complex $C^*(\mathcal{L}, \omega, m)$.

The next theorem follows from Theorem 4.3 and Definition 4.4 immediately.

Theorem 4.4 (Main Result II) *Let t, s be non-negative integers. Let $m \in \mathbb{Z}$. Suppose $m = \lambda(2t+1)+q$ where $\lambda \in \mathbb{Z}$ and $0 \leq q \leq 2t$. Then for any $\omega \in \mathrm{Ext}^{2t+1}(V)$ and $\mu \in \mathrm{Ext}^{2s}(V)$, there is an induced homomorphism*

$$\mu_* : \ H^n(\mathcal{L}, \omega, m) \to H^n(\mathcal{L}, \omega, m + 2s), \quad n \geq 0 \tag{4.16}$$

of the constrained cohomology groups.

Proof Apply the cohomology functor to the co-chain complex in Theorem 4.3(i) and the co-chain map in Theorem 4.3(ii). We obtain the homomorphism μ_* of the constrained cohomology groups in (4.16).

5 Examples

We give some examples for Theorems 4.1–4.4.

Example 5.1 Let V be any finite set. Then we have the followings.

(i) Any element $\alpha \in \mathrm{Ext}_1(V)$ can be expressed as

$$\alpha = \sum_{v \in V} f(v) \frac{\partial}{\partial v} \tag{5.1}$$

for some function $f : V \to \mathbb{R}$. Let $\mathcal{K}$ be a simplicial complex with its vertices in V. Then for any $n \geq 0$ and any n-simplex $v_0 v_1 \cdots v_n$ in $\mathcal{K}$, we have

$$\alpha(v_0 v_1 \cdots v_n) = \sum_{v \in V} f(v) \frac{\partial}{\partial v}(v_0 v_1 \cdots v_n)$$

$$
\begin{aligned}
&= \sum_{v\in V} f(v) \sum_{i=0}^{n} (-1)^i \delta(v, v_i) v_0 \cdots \widehat{v_i} \cdots v_n \\
&= \sum_{i=0}^{n} (-1)^i \Big(\sum_{v\in V} \delta(v, v_i) f(v) \Big) v_0 \cdots \widehat{v_i} \cdots v_n \\
&= \sum_{i=0}^{n} (-1)^i f(v_i) v_0 \cdots \widehat{v_i} \cdots v_n.
\end{aligned}
$$

In [28–30], α given in (5.1) is called the f-weighted boundary operator on $\mathcal{K}$ and the $(\alpha, 0)$-homology of $\mathcal{K}$ is denoted as the weighted homology $H_*(\mathcal{K}, f)$ of the weighted simplicial complex $(\mathcal{K}, f)$. Particularly, if f takes the constant value 1 for all $v \in V$, then α is the usual boundary operator ∂_* given in (1.1) and $H_*(\mathcal{K}, f)$ is the usual homology $H_*(\mathcal{K})$ (see [24, Chapter 1] and [20, Section 2.1]) of $\mathcal{K}$.

(ii) Any element $\omega \in \mathrm{Ext}^1(V)$ can be expressed as

$$
\omega = \sum_{v\in V} f(v)\mathrm{d}v \tag{5.2}
$$

for some function $f : V \to \mathbb{R}$. Let $\mathcal{L}$ be an independence hypergraph with its vertices in V. Then for any $n \geq 0$ and any n-hyperedge $v_0 v_1 \cdots v_n$ in $\mathcal{L}$, we have

$$
\begin{aligned}
\omega(v_0 v_1 \cdots v_n) &= \sum_{v\in V} f(v) \mathrm{d}v(v_0 v_1 \cdots v_n) \\
&= \sum_{v\in V} f(v) \sum_{i=0}^{n+1} (-1)^i v_0 v_1 \cdots v_{i-1} v v_i v_{i+1} \cdots v_n \\
&= \sum_{i=0}^{n+1} (-1)^i \Big(\sum_{v\in V} f(v) v_0 \cdots\cdots v_{i-1} v v_i \cdots v_n \Big).
\end{aligned}
$$

Similar with (i), we call ω given in (5.2) the f-weighted co-boundary operator on $\mathcal{L}$ and denote the $(\omega, 0)$-cohomology of $\mathcal{L}$ as $H^*(\mathcal{L}, f)$. Particularly, if f takes the constant value 1 for all $v \in V$, then we denote the ω as d_* denote the $H^*(\mathcal{L}, f)$ as $H^*(\mathcal{L})$.

Example 5.2 Let $V = \{v_0, v_1, v_2\}$. Let $f : V \to \mathbb{R}$ be a function on V.

(i) Let

$$
\mathcal{K} = \{v_0, v_1, v_2, v_0v_1, v_0v_2, v_1v_2\}
$$

be a simplicial complex with its vertices in V. Then we have

$$
\begin{aligned}
C_0(\mathcal{K}; \mathbb{R}) &= \mathrm{Span}_{\mathbb{R}}\{v_0, v_1, v_2\}, \\
C_1(\mathcal{K}; \mathbb{R}) &= \mathrm{Span}_{\mathbb{R}}\{v_0v_1, v_0v_2, v_1v_2\}, \\
C_n(\mathcal{K}; \mathbb{R}) &= 0 \quad \text{for all } n \geq 2.
\end{aligned}
$$

- Let $t = 1$. Let

$$
\alpha = f(v_0)\frac{\partial}{\partial v_0} + f(v_1)\frac{\partial}{\partial v_1} + f(v_2)\frac{\partial}{\partial v_2}.
$$

With the help of Example 5.1(i), we have

$$
\alpha(v_0) = \alpha(v_1) = \alpha(v_2) = 0,
$$

$$\begin{aligned}
\alpha(v_0v_1) &= f(v_0)v_1 - f(v_1)v_0,\\
\alpha(v_0v_2) &= f(v_0)v_2 - f(v_2)v_0,\\
\alpha(v_1v_2) &= f(v_1)v_2 - f(v_2)v_1,\\
\alpha(v_0v_1v_2) &= f(v_0)v_1v_2 - f(v_1)v_0v_2 + f(v_2)v_0v_1.
\end{aligned}$$

Note that

$$\dim \mathrm{Ker}(\alpha : C_0(\mathcal{K};\mathbb{R}) \to 0) = 3$$

and

$$\dim \mathrm{Im}(\alpha : C_1(\mathcal{K};\mathbb{R}) \to C_0(\mathcal{K};\mathbb{R})) = \begin{cases} 2, & \text{if } f(v_i) \neq 0 \text{ for some } i = 0,1,2;\\ 0, & \text{if } f(v_0) = f(v_1) = f(v_2) = 0. \end{cases}$$

Thus

$$H_0(\mathcal{K}, f) = H_0(\mathcal{K}, \alpha, 0) = \begin{cases} \mathbb{R}, & \text{if } f(v_i) \neq 0 \text{ for some } i = 0,1,2;\\ \mathbb{R}^3, & \text{if } f(v_0) = f(v_1) = f(v_2) = 0. \end{cases}$$

Note that

$$\dim \mathrm{Ker}(\alpha : C_1(\mathcal{K};\mathbb{R}) \to C_0(\mathcal{K};\mathbb{R})) = \begin{cases} 1, & \text{if } f(v_i) \neq 0 \text{ for some } i = 0,1,2;\\ 3, & \text{if } f(v_0) = f(v_1) = f(v_2) = 0 \end{cases}$$

and

$$\dim \mathrm{Im}(\alpha : C_2(\mathcal{K};\mathbb{R}) \to C_1(\mathcal{K};\mathbb{R})) = 0.$$

Thus

$$H_1(\mathcal{K}, f) = H_1(\mathcal{K}, \alpha, 0) = \begin{cases} \mathbb{R}, & \text{if } f(v_i) \neq 0 \text{ for some } i = 0,1,2;\\ \mathbb{R}^3, & \text{if } f(v_0) = f(v_1) = f(v_2) = 0. \end{cases}$$

- Let $s = 1$. Let

$$\beta = b_{01}\frac{\partial}{\partial v_0} \wedge \frac{\partial}{\partial v_1} + b_{02}\frac{\partial}{\partial v_0} \wedge \frac{\partial}{\partial v_2} + b_{12}\frac{\partial}{\partial v_1} \wedge \frac{\partial}{\partial v_2}.$$

Then

$$\beta(v_i) = 0 \quad \text{for } 0 \leq i \leq 2$$

and

$$\beta(v_iv_j) = 0 \quad \text{for } 0 \leq i < j \leq 2.$$

Thus the induced homomorphism β_* between the homology groups is identically zero.

(ii) Let

$$\mathcal{L} = \{v_0v_1, v_0v_2, v_0v_1v_2\}$$

be an independence hypergraph with its vertices in V. Then we have

$$C_0(\mathcal{L};\mathbb{R}) = 0,$$

$$
\begin{aligned}
C_1(\mathcal{L};\mathbb{R}) &= \mathrm{Span}_{\mathbb{R}}\{v_0v_1, v_0v_2\},\\
C_2(\mathcal{L};\mathbb{R}) &= \mathrm{Span}_{\mathbb{R}}\{v_0v_1v_2\},\\
C_n(\mathcal{L};\mathbb{R}) &= 0 \quad \text{for all } n \geq 3.
\end{aligned}
$$

• Let $t = 1$. Let

$$\omega = f(v_0)\mathrm{d}v_0 + f(v_1)\mathrm{d}v_1 + f(v_2)\mathrm{d}v_2 + f(v_3)\mathrm{d}v_3.$$

With the help of Example 5.1(ii), we have

$$
\begin{aligned}
\omega(v_0v_1) &= f(v_2)v_0v_1v_2,\\
\omega(v_0v_2) &= -f(v_1)v_0v_1v_2,\\
\omega(v_0v_1v_2) &= 0.
\end{aligned}
$$

Note that

$$\dim \mathrm{Ker}(\omega : C_1(\mathcal{L};\mathbb{R}) \to C_2(\mathcal{L};\mathbb{R})) = \begin{cases} 1, & \text{if } f(v_i) \neq 0 \text{ for some } i = 1,2;\\ 2, & \text{if } f(v_1) = f(v_2) = 0 \end{cases}$$

or equivalently,

$$\dim \mathrm{Im}(\omega : C_1(\mathcal{L};\mathbb{R}) \to C_2(\mathcal{L};\mathbb{R})) = \begin{cases} 1, & \text{if } f(v_i) \neq 0 \text{ for some } i = 1,2;\\ 0, & \text{if } f(v_1) = f(v_2) = 0. \end{cases}$$

Thus

$$H^1(\mathcal{L}, f) = H^1(\mathcal{L}, \omega, 0) = \begin{cases} \mathbb{R}, & \text{if } f(v_i) \neq 0 \text{ for some } i = 1,2;\\ \mathbb{R}^2, & \text{if } f(v_1) = f(v_2) = 0 \end{cases}$$

and

$$H^2(\mathcal{L}, f) = H^2(\mathcal{L}, \omega, 0) = \begin{cases} 0, & \text{if } f(v_i) \neq 0 \text{ for some } i = 1,2;\\ \mathbb{R}, & \text{if } f(v_1) = f(v_2) = 0. \end{cases}$$

Moreover,

$$H^n(\mathcal{L}, f) = H^n(\mathcal{L}, \omega, 0) = 0$$

for any $n \neq 1, 2$.

• Let $s = 1$. Let

$$\mu = u_{01}\mathrm{d}v_0 \wedge \mathrm{d}v_1 + u_{02}\mathrm{d}v_0 \wedge \mathrm{d}v_2 + u_{12}\mathrm{d}v_1 \wedge \mathrm{d}v_2.$$

Then

$$\mu(v_0v_1v_2) = \mu(v_0v_1) = \mu(v_0v_2) = 0.$$

Thus the induced homomorphism μ_* between the cohomology groups is identically zero.

Example 5.3 Let $V = \{v_0, v_1, v_2, v_3\}$. Let $t = 1$. Then any $\alpha \in \mathrm{Ext}_3(V)$ can be expressed as

$$\alpha = f(v_0, v_1, v_2)\frac{\partial}{\partial v_0} \wedge \frac{\partial}{\partial v_1} \wedge \frac{\partial}{\partial v_2} + f(v_0, v_1, v_3)\frac{\partial}{\partial v_0} \wedge \frac{\partial}{\partial v_1} \wedge \frac{\partial}{\partial v_3}$$

$$+ f(v_0, v_2, v_3)\frac{\partial}{\partial v_0} \wedge \frac{\partial}{\partial v_2} \wedge \frac{\partial}{\partial v_3} + f(v_1, v_2, v_3)\frac{\partial}{\partial v_1} \wedge \frac{\partial}{\partial v_2} \wedge \frac{\partial}{\partial v_3},$$

where

$$f: \ V \times V \times V \to \mathbb{R}$$

is a real function on the 3-fold Cartesian product of V. By Proposition 3.1, the adjoint $\omega \in \mathrm{Ext}^3(V)$ of α is given by

$$\begin{aligned}\omega = &-f(v_0, v_1, v_2)\mathrm{d}v_0 \wedge \mathrm{d}v_1 \wedge \mathrm{d}v_2 - f(v_0, v_1, v_3)\mathrm{d}v_0 \wedge \mathrm{d}v_1 \wedge \mathrm{d}v_3 \\ &- f(v_0, v_2, v_3)\mathrm{d}v_0 \wedge \mathrm{d}v_2 \wedge \mathrm{d}v_3 - f(v_1, v_2, v_3)\mathrm{d}v_1 \wedge \mathrm{d}v_2 \wedge \mathrm{d}v_3.\end{aligned}$$

Let $s = 1$. Then any $\beta \in \mathrm{Ext}_2(V)$ can be expressed as

$$\begin{aligned}\beta = &\, g(v_0, v_1)\frac{\partial}{\partial v_0} \wedge \frac{\partial}{\partial v_1} + g(v_0, v_2)\frac{\partial}{\partial v_0} \wedge \frac{\partial}{\partial v_2} + g(v_0, v_3)\frac{\partial}{\partial v_0} \wedge \frac{\partial}{\partial v_3} \\ &+ g(v_1, v_2)\frac{\partial}{\partial v_1} \wedge \frac{\partial}{\partial v_2} + g(v_1, v_3)\frac{\partial}{\partial v_1} \wedge \frac{\partial}{\partial v_3} + g(v_2, v_3)\frac{\partial}{\partial v_2} \wedge \frac{\partial}{\partial v_3},\end{aligned}$$

where

$$g: \ V \times V \to \mathbb{R}$$

is a real function on the 2-fold Cartesian product of V. By Proposition 3.1, the adjoint $\mu \in \mathrm{Ext}^2(V)$ of β is given by

$$\begin{aligned}\mu = &-g(v_0, v_1)\mathrm{d}v_0 \wedge \mathrm{d}v_1 - g(v_0, v_2)\mathrm{d}v_0 \wedge \mathrm{d}v_2 - g(v_0, v_3)\mathrm{d}v_0 \wedge \mathrm{d}v_3 \\ &- g(v_1, v_2)\mathrm{d}v_1 \wedge \mathrm{d}v_2 - g(v_1, v_3)\mathrm{d}v_1 \wedge \mathrm{d}v_3 - g(v_2, v_3)\mathrm{d}v_2 \wedge \mathrm{d}v_3.\end{aligned}$$

Consider the complete hypergraph

$$\begin{aligned}\Delta[V] = \{&v_0, v_1, v_2, v_3, v_0v_1, v_0v_2, v_0v_3, v_1v_2, v_1v_3, v_2v_3, \\ &v_0v_1v_2, v_0v_1v_3, v_0v_2v_3, v_1v_2v_3, v_0v_1v_2v_3\}.\end{aligned}$$

Then $\Delta[V]$ is a simplicial complex and is also an independence hypergraph.

- By a direct calculation,

$$\begin{aligned}\alpha(v_i) &= 0, \quad i = 0, 1, 2, 3, \\ \alpha(v_iv_j) &= 0, \quad 0 \le i < j \le 3, \\ \alpha(v_iv_jv_k) &= 0, \quad 0 \le i < j < k \le 3, \\ \alpha(v_0v_1v_2v_3) &= (-1)^{0+1+2}f(v_0, v_1, v_2)v_3 + (-1)^{0+1+3}f(v_0, v_1, v_3)v_2 \\ &\quad + (-1)^{0+2+3}f(v_0, v_2, v_3)v_1 + (-1)^{1+2+3}f(v_1, v_2, v_3)v_0 \\ &= -f(v_0, v_1, v_2)v_3 + f(v_0, v_1, v_3)v_2 - f(v_0, v_2, v_3)v_1 \\ &\quad + f(v_1, v_2, v_3)v_0.\end{aligned}$$

It follows that

$$\dim \mathrm{Im}(\alpha : C_3(\Delta[V]; \mathbb{R}) \to C_0(\Delta[V]; \mathbb{R}))$$

$$= \begin{cases} 1, & \text{if } f(v_i, v_j, v_k), 0 \leq i < j < k \leq 3, \text{ are not all zero;} \\ 0, & \text{if } f(v_i, v_j, v_k) = 0 \text{ for any } 0 \leq i < j < k \leq 3, \end{cases}$$

or equivalently,

$$\begin{aligned} &\dim \operatorname{Ker}(\alpha : C_3(\Delta[V]; \mathbb{R}) \to C_0(\Delta[V]; \mathbb{R})) \\ &= \begin{cases} 0, & \text{if } f(v_i, v_j, v_k), 0 \leq i < j < k \leq 3, \text{ are not all zero;} \\ 1, & \text{if } f(v_i, v_j, v_k) = 0 \text{ for any } 0 \leq i < j < k \leq 3. \end{cases} \end{aligned}$$

Consequently,

$$H_0(\Delta[V], \alpha, 0) = \begin{cases} \mathbb{R}^3, & \text{if } f(v_i, v_j, v_k), 0 \leq i < j < k \leq 3, \text{ are not all zero;} \\ \mathbb{R}^4, & \text{if } f(v_i, v_j, v_k) = 0 \text{ for any } 0 \leq i < j < k \leq 3 \end{cases}$$

and

$$H_3(\Delta[V], \alpha, 0) = \begin{cases} 0, & \text{if } f(v_i, v_j, v_k), 0 \leq i < j < k \leq 3, \text{ are not all zero;} \\ \mathbb{R}, & \text{if } f(v_i, v_j, v_k) = 0 \text{ for any } 0 \leq i < j < k \leq 3. \end{cases}$$

By a similar calculation, we have

$$H_1(\Delta[V], \alpha, 0) = \mathbb{R}^6, \quad H_2(\Delta[V], \alpha, 0) = \mathbb{R}^4.$$

Moreover,

$$H_n(\Delta[V], \alpha, 0) = 0$$

for any $n \neq 0, 1, 2, 3$.

• It is direct that $\beta \circ \alpha(v_i) = 0$ for any $0 \leq i \leq 3$, $\beta \circ \alpha(v_i v_j) = 0$ for any $0 \leq i < j \leq 3$, $\beta \circ \alpha(v_i v_j v_k) = 0$ for any $0 \leq i < j < k \leq 3$, and $\beta \circ \alpha(v_0 v_1 v_2 v_3) = 0$. Therefore, the induced homomorphism β_* between the homology groups is the zero map.

• By a direct calculation,

$$\begin{aligned} \omega(v_0) &= -f(v_1, v_2, v_3) \mathrm{d}v_1 \wedge \mathrm{d}v_2 \wedge \mathrm{d}v_3(v_0) \\ &= f(v_1, v_2, v_3) v_0 v_1 v_2 v_3, \\ \omega(v_1) &= -f(v_0, v_2, v_3) \mathrm{d}v_0 \wedge \mathrm{d}v_2 \wedge \mathrm{d}v_3(v_1) \\ &= -f(v_0, v_2, v_3) v_0 v_1 v_2 v_3, \\ \omega(v_2) &= -f(v_0, v_1, v_3) \mathrm{d}v_0 \wedge \mathrm{d}v_1 \wedge \mathrm{d}v_3(v_2) \\ &= f(v_0, v_1, v_3) v_0 v_1 v_2 v_3, \\ \omega(v_3) &= -f(v_0, v_1, v_2) \mathrm{d}v_0 \wedge \mathrm{d}v_1 \wedge \mathrm{d}v_2(v_3) \\ &= -f(v_0, v_1, v_2) v_0 v_1 v_2 v_3, \\ \omega(v_i v_j) &= 0, \quad 0 \leq i < j \leq 3, \\ \omega(v_i v_j v_k) &= 0, \quad 0 \leq i < j < k \leq 3, \\ \omega(v_0 v_1 v_2 v_3) &= 0. \end{aligned}$$

It follows that

$$\dim \operatorname{Im}(\omega : C_0(\Delta[V]; \mathbb{R}) \to C_3(\Delta[V]; \mathbb{R}))$$

$$= \begin{cases} 1, & \text{if } f(v_i, v_j, v_k), 0 \le i < j < k \le 3, \text{ are not all zero;} \\ 0, & \text{if } f(v_i, v_j, v_k) = 0 \text{ for any } 0 \le i < j < k \le 3, \end{cases}$$

or equivalently,

$$\begin{aligned} &\dim \mathrm{Ker}(\omega : C_0(\Delta[V]; \mathbb{R}) \to C_3(\Delta[V]; \mathbb{R})) \\ &= \begin{cases} 3, & \text{if } f(v_i, v_j, v_k), 0 \le i < j < k \le 3, \text{ are not all zero;} \\ 4, & \text{if } f(v_i, v_j, v_k) = 0 \text{ for any } 0 \le i < j < k \le 3. \end{cases} \end{aligned}$$

Consequently,

$$H^0(\Delta[V], \omega, 0) = \begin{cases} \mathbb{R}^3, & \text{if } f(v_i, v_j, v_k), 0 \le i < j < k \le 3, \text{ are not all zero;} \\ \mathbb{R}^4, & \text{if } f(v_i, v_j, v_k) = 0 \text{ for any } 0 \le i < j < k \le 3 \end{cases}$$

and

$$H^3(\Delta[V], \omega, 0) = \begin{cases} 0, & \text{if } f(v_i, v_j, v_k), 0 \le i < j < k \le 3, \text{ are not all zero;} \\ \mathbb{R}, & \text{if } f(v_i, v_j, v_k) = 0 \text{ for any } 0 \le i < j < k \le 3. \end{cases}$$

By a similar calculation, we have

$$H^1(\Delta[V], \omega, 0) = \mathbb{R}^6, \quad H^2(\Delta[V], \omega, 0) = \mathbb{R}^4.$$

Moreover,

$$H^n(\Delta[V], \omega, 0) = 0$$

for any $n \neq 0, 1, 2, 3$.

• It is direct to see that the induced homomorphism μ_* between the cohomology groups is the zero map.

Acknowledgement The author would like to express his deep gratitude to the referee for the helpful comments and valuable suggestions for the improvement of the paper.

References

[1] Berge, C., Graphs and Hypergraphs, North-Holland Mathematical Library, Amsterdam, 1973.

[2] Bott, R. and Tu, L. W., Differential Forms in Algebraic Topology, Springer-Verlag, Berlin and Heidelberg, 1982.

[3] Bressan, S., Li, J. Y., Ren, S. Q. and Wu, J., The embedded homology of hypergraphs and applications, *Asian Journal of Mathematics*, **23**(3), 2019, 479–500.

[4] Chern, S. S., Chen, W. H. and Lam, K. S., Lectures on Differential Geometry, Series on University Mathematics, **1**, World Scientific, River Edge, NJ, 2000.

[5] Cohen, F. R. and Wu, J., On braid groups and homotopy groups, *Geometry and Topology Monographs*, **13**, 2008, 169–193.

[6] Dawson, R. J. MacG., Homology of weighted simplicial complexes, *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, **31**(3), 1990, 229–243.

[7] Dimakis, A. and Müller-Hoissen, F., Differential calculus and gauge theory on finite sets, *Journal of Physics A: Mathematical and General*, **27**(9), 1994, 3159–3178.

[8] Dimakis, A. and Müller-Hoissen, F., Discrete differential calculus: Graphs, topologies, and gauge theory, *Journal of Mathematical Physics*, **35**(12), 1994, 6703–6735.

[9] Dimakis, A. and Müller-Hoissen, F., Discrete Riemannian geometry, *Journal of Mathematical Physics*, **40**(3), 1999, 1518–1548.

[10] Curtis, E. B., Simplicial homotopy theory, *Advance in mathematics*, **6**, 1971, 107–209.

[11] Eilenberg, S. and Zilber, J. A., Semi-simplicial complexes and singular homology, *Annals of Mathematics*, **51**, 1950, 499–513.

[12] Goerss, Paul G. and Jardine John, Simplicial Homotopy Theory, Birkhäuser, Basel, 1999.

[13] Grbić, J., Wu, J., Xia, K. L. and Wei, G.-W., Aspects of topological approaches for data science, *Foundations of Data Science*, **4**, 2022, 165–216.

[14] Grigor'yan, A., Lin, Y. and Yau, S.-T., Torsion of digraphs and path complexes, 2020, arXiv: 2012.07302v1.

[15] Grigor'yan, A., Lin, Y., Muranov, Y. and Yau, S.-T., Homologies of path complexes and digraphs, 2013, arXiv: 1207.2834.

[16] Grigor'yan, A., Lin, Y., Muranov, Y. and Yau, S.-T., Homotopy theory for digraphs, *Pure and Applied Mathematics Quarterly*, **10**(4), 2014, 619–674.

[17] Grigor'yan, A., Lin, Y., Muranov, Y. and Yau, S.-T., Cohomology of digraphs and (undirected) graphs, *Asian Journal of Mathematics*, **15**(5), 2015, 887–932.

[18] Grigor'yan, A., Lin, Y., Muranov, Y. and Yau, S.-T., Path complexes and their homologies, *Journal of Mathematical Sciences*, **248**(5), 2020, 564–599.

[19] Grigor'yan, A., Muranov, Y. and Yau, S.-T., Homologies of digraphs and Künneth formulas, *Communications in Analysis and Geometry*, **25**, 2017, 969–1018.

[20] Hatcher, A., Algebraic Topology, Cambridge University Press, Cambridge, 2002.

[21] Lei, F. C., Li, F. L. and Wu, J., On simplicial resolutions of framed links, *Transactions of The American Mathematical Society*, **366**(6), 2014, 3075–3093.

[22] Madsen, Ib H. and Tornehave, J., From Calculus to Cohomology: De Rham Cohomology and Characteristic Classes, Cambridge University Press, Cambridge, 1997.

[23] Milnor, J., The geometric realization of a semi-simplicial complex, *Annals of Mathematics, 2nd Ser.*, **65**(2), 1957, 357–362.

[24] Munkres, J. R., Elements of Algebraic Topology, Addison-Wesley Publishing Company, California, 1984.

[25] Parks, A. D. and Lipscomb, S. L., Homology and Hypergraph Acyclicity: A Combinatorial Invariant for Hypergraphs, Naval Surface Warfare Center, 1991, http://www.dtic.mil/docs/citations/ADA241584.

[26] Pavutnitskiy, F. and Wu, J., A simplicial James-Hopf map and decompositions of the unstable Adams spectral sequence for suspensions, *Algebraic and Geometric Topology*, **19**(1), 2019, 77–108.

[27] Ren, S. Q., Simplicial-like identities for the paths and the regular paths on discrete sets (unpublished manuscript), 2021, arXiv 2107.09868.

[28] Ren, S. Q., Wu, C. Y. and Wu, J., Weighted persistent homology, *Rocky Mountain Journal of Mathematics*, **48**(8), 2018, 2661–2687.

[29] Ren, S. Q., Wu, C. Y. and Wu, J., Computational tools in weighted persistent homology, *Chinese Annals of Mathematics, Ser. B*, **42**(2), 2021, 237–258.

[30] Wu, C. Y., Ren, S. Q., Wu, J. and Xia, K. L., Discrete Morse theory for weighted simplicial complexes, *Topology and its Applications*, **270**, 2020, Article 107038.

[31] Wu, J., Simplicial objects and homotopy groups, Braids, 31–181, Lecture Notes Series, Institute of Mathematical Sciences, National University of Singapore, **19**, World Scientific Publishing, Hackensack, NJ, 2010.