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Exact Internal Controllability and Synchronization for a Coupled System of Wave Equations*

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Abstract In this paper the exact internal controllability for a coupled system of wave equations with arbitrarily given coupling matrix is established. Based on this result, the exact internal synchronization and the exact internal synchronization by p-groups are successfully considered.

Keywords Exact internal controllability, Exact internal synchronization by groups, Coupled system of wave equations
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1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary Γ. We consider the following coupled system of wave equations for the variable $U = (u^{(1)}, \cdots, u^{(N)})^{\mathrm{T}}$:

$$\begin{cases} U'' - \Delta U + AU = D\chi_\omega H & \text{in}(0, +\infty) \times \Omega, \\ U = 0 & \text{on}(0, +\infty) \times \Gamma \end{cases} \tag{1.1}$$

with the initial condition

$$t = 0: \ U = \widehat{U}_0, \quad U' = \widehat{U}_1 \quad \text{in}\Omega, \tag{1.2}$$

where " $'$ " stands for the time derivative; $\Delta = \sum\limits_{k=1}^{n} \frac{\partial^2}{\partial x_k^2}$ is the Laplacian operator; the coupling matrix $A = (a_{ij})$ is of order N and the internal control matrix $D = (d_{ij})$ is a full column-rank matrix of order $N \times M (M \leq N)$, both with constant elements; $H = (h^{(1)}, \cdots, h^{(M)})^{\mathrm{T}}$ denotes the internal control; and χ_ω is the characteristic function of the open set $\omega \subset \Omega$.

The exact internal controllability of a single wave equation has been extensively studied. Russell and Lagnese have considered a single 1-d wave equation with locally distributed control in any fixed nonempty subinterval of a bounded interval (see [10, 17]). The exact internal

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controllability for a 1-d semilinear wave equation, under some growth constraints on the nonlinearity, was proved by Zuazua using the Hilbert uniqueness method (HUM for short) and the fixed-point method in [21], and some improved results can be found in [3–4]. Under certain special but reasonable hypotheses, the exact controllability for the 1-d first order quasilinear hyperbolic system can be realized only by internal controls, and the control time can be arbitrarily small (see [20]). Zhuang similarly considered this problem for 1-d quasilinear wave equations (see [19]).

Ammar Khodja and Bader studied the stabilizability of a coupled system of two 1-d wave equations by only one internal control force and gave necessary and sufficient conditions for the exponential stability (see [1]). For a system of two coupled 1-d wave equations with one control, Zhang proved the controllability under certain conditions on the coupling (see [18]).

In higher dimensional case, we consider the following system

$$\begin{cases} y'' - \Delta y = h & \text{in } (0,T)\times\Omega, \\ y = 0 & \text{on } (0,T)\times\Gamma. \end{cases} \tag{1.3}$$

Let $\Omega \subset \mathbb{R}^2$ be a rectangle. Haraux established the exact controllability of system (1.3) by means of a generalized control $h(t,x)$ localized on $(0,T)\times\omega$, where ω is an open subset of Ω (see [8]). Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^2 boundary Γ. The exact controllability of system (1.3) was established by the HUM method in [13] with control distributed in an ε-neighbourhood ω of the boundary satisfying the usual geometrical control condition. Moreover, for a semilinear wave equation, the internal controllability was obtained by Carleman estimates with control supported in a neighbourhood of a portion of the boundary (see [7]).

In this paper, we will consider the exact internal controllability and exact internal synchronization of the coupled system (1.1). We point out that in order to further study the synchronization, the coupling matrix A should be an arbitrarily given matrix.

Let Φ be the solution to the adjoint system

$$\begin{cases} \Phi'' - \Delta\Phi + A^{\mathrm{T}}\Phi = 0 & \text{in } (0,+\infty)\times\Omega, \\ \Phi = 0 & \text{on } (0,+\infty)\times\Gamma \end{cases} \tag{1.4}$$

with the initial data

$$t = 0: \ \Phi = \widehat{\Phi}_0, \quad \Phi' = \widehat{\Phi}_1 \quad \text{in } \Omega. \tag{1.5}$$

Using the HUM method, to obtain the exact internal controllability of system (1.1) at the time $T > 0$, it is sufficient to prove that

$$\|(\widehat{\Phi}_0, \widehat{\Phi}_1)\|^2_{(L^2(\Omega))^N\times(H^{-1}(\Omega))^N} \sim \int_0^{\mathrm{T}}\int_\omega |\Phi|^2 \mathrm{d}x\mathrm{d}t \tag{1.6}$$

for any given coupling matrix A, where " $\sim$ " means there exist positive constants C and C', such that

$$C\int_0^{\mathrm{T}}\int_\omega |\Phi|^2 \mathrm{d}x\mathrm{d}t \le \|(\widehat{\Phi}_0, \widehat{\Phi}_1)\|^2_{(L^2(\Omega))^N\times(H^{-1}(\Omega))^N} \le C'\int_0^{\mathrm{T}}\int_\omega |\Phi|^2 \mathrm{d}x\mathrm{d}t.$$

For any given $x_0 \in \mathbb{R}^n$, we define

$$T(x_0) = 2 \max_{x \in \overline{\Omega}} |x - x_0| \quad \text{and} \quad \Gamma(x_0) = \{x \in \Gamma \mid (x - x_0) \cdot \nu(x) > 0\}, \tag{1.7}$$

where $\nu(x)$ is the unit outer normal vector on Γ (see [13, p.271]).

Before proving (1.6), we recall that for the decoupled system

$$\begin{cases} \widetilde{\Phi}'' - \Delta\widetilde{\Phi} = 0 & \text{in } (0, +\infty) \times \Omega, \\ \widetilde{\Phi} = 0 & \text{on } (0, +\infty) \times \Gamma \end{cases} \tag{1.8}$$

with the initial data

$$t = 0: \ \widetilde{\Phi} = \widetilde{\Phi}_0, \quad \widetilde{\Phi}' = \widetilde{\Phi}_1, \tag{1.9}$$

we have the following proposition.

Proposition 1.1 (see [13]) *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^2 boundary Γ. For any given $x_0 \in \mathbb{R}^n$, assume that ω is a neighbourhood of $\overline{\Gamma}(x_0)$ in Ω and $T > T(x_0)$. Then for any given initial data $(\widetilde{\Phi}_0, \widetilde{\Phi}_1) \in (H_0^1(\Omega))^N \times (L^2(\Omega))^N$, the solution $\widetilde{\Phi}$ to problem* (1.8)–(1.9) *satisfies*

$$\|\widetilde{\Phi}_0\|^2_{(H_0^1(\Omega))^N} + \|\widetilde{\Phi}_1\|^2_{(L^2(\Omega))^N} \sim \int_0^T \int_\omega (|\widetilde{\Phi}|^2 + |\widetilde{\Phi}'|^2) \mathrm{d}x \mathrm{d}t. \tag{1.10}$$

In order to get (1.6) for any given coupling matrix A from Proposition 1.1, we have to overcome a series of difficulties (see Section 2 below), hence the exact internal controllability of the coupled system (1.1) is in fact a delicate problem to be solved.

In Section 2, we will show that $\text{rank}(D) = N$ is necessary and sufficient for the exact internal controllability of the coupled system (1.1) composed of N wave equations. The exact internal synchronization and the exact internal synchronization by p-groups for system (1.1) will be established in Section 3.

2 Exact Internal Controllability and Non-Exact Internal Controllability

2.1 Exact internal controllability

Let us first recall the following standard well-posedness result (see [2, 5, 16]).

Proposition 2.1 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary Γ. For any given initial data $(\widehat{U}_0, \widehat{U}_1) \in (H_0^1(\Omega))^N \times (L^2(\Omega))^N$ and any given $H \in (L^2_{\text{loc}}(\mathbb{R}^+; L^2(\omega)))^M$, system* (1.1) *admits a unique weak solution $U = U(t, x)$ in the space*

$$(C^0_{\text{loc}}(\mathbb{R}^+; H_0^1(\Omega)))^N \cap (C^1_{\text{loc}}(\mathbb{R}^+; L^2(\Omega)))^N. \tag{2.1}$$

Moreover, the mapping

$$(\widehat{U}_0, \widehat{U}_1, H) \to (U, U') \tag{2.2}$$

is linear and continuous for the corresponding topologies.

Definition 2.1 *System* (1.1) *is exactly null controllable at the time* $T > 0$ *in the space* $(H_0^1(\Omega))^N \times (L^2(\Omega))^N$ *if for any given initial data* $(\widehat{U}_0, \widehat{U}_1) \in (H_0^1(\Omega))^N \times (L^2(\Omega))^N$, *there exists an internal control* $H \in (L^2(0, +\infty; L^2(\omega)))^M$ *with compact support in* $[0, T]$, *such that the solution* $U = U(t, x)$ *to problem* (1.1)–(1.2) *satisfies the following condition*:

$$t \geq T: \ U \equiv 0 \quad \text{in } \Omega. \tag{2.3}$$

Remark 2.1 In general, for a linear time-inversible system, the exact internal null controllability is equivalent to the exact internal controllability (see [12–13]).

Theorem 2.1 *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded domain with smooth boundary* Γ. *For any given* $x_0 \in \mathbb{R}^n$, *assume that* ω *is a neighbourhood of* $\overline{\Gamma}(x_0)$ *in* Ω *and* $T > T(x_0)$, *then there exist positive constants* C *and* C' *such that for any given initial data* $(\widehat{\Phi}_0, \widehat{\Phi}_1) \in (L^2(\Omega))^N \times (H^{-1}(\Omega))^N$, *the solution* Φ *to problem* (1.4)–(1.5) *satisfies the following direct and inverse inequalities*:

$$C \int_0^{\mathrm{T}} \int_\omega |\Phi|^2 \mathrm{d}x\mathrm{d}t \leq \|\widehat{\Phi}_0\|^2_{(L^2(\Omega))^N} + \|\widehat{\Phi}_1\|^2_{(H^{-1}(\Omega))^N} \leq C' \int_0^{\mathrm{T}} \int_\omega |\Phi|^2 \mathrm{d}x\mathrm{d}t. \tag{2.4}$$

In order to prove Theorem 2.1, we first consider the special case $A = 0$. Then the proof of the following theorem will be shown in Section 2.2.

Theorem 2.2 *Under the assumptions of Theorem* 2.1, *there exist positive constants* C *and* C' *such that for any given initial data* $(\widetilde{\Phi}_0, \widetilde{\Phi}_1) \in (H_0^1(\Omega))^N \times (L^2(\Omega))^N$, *the solution* $\widetilde{\Phi}$ *to problem* (1.8)–(1.9) *satisfies the following direct and inverse inequalities*:

$$C \int_0^{\mathrm{T}} \int_\omega |\widetilde{\Phi}'|^2 \mathrm{d}x\mathrm{d}t \leq \|\widetilde{\Phi}_0\|^2_{(H_0^1(\Omega))^N} + \|\widetilde{\Phi}_1\|^2_{(L^2(\Omega))^N} \leq C' \int_0^{\mathrm{T}} \int_\omega |\widetilde{\Phi}'|^2 \mathrm{d}x\mathrm{d}t. \tag{2.5}$$

Corollary 2.1 *Under the assumptions of Theorem* 2.1, *there exist positive constants* C *and* C' *such that for any given initial data* $(\widetilde{\Phi}_0, \widetilde{\Phi}_1) \in (L^2(\Omega))^N \times (H^{-1}(\Omega))^N$, *the solution* $\widetilde{\Phi}$ *to problem* (1.8)–(1.9) *satisfies the following direct and inverse inequalities*:

$$C \int_0^{\mathrm{T}} \int_\omega |\widetilde{\Phi}|^2 \mathrm{d}x\mathrm{d}t \leq \|\widetilde{\Phi}_0\|^2_{(L^2(\Omega))^N} + \|\widetilde{\Phi}_1\|^2_{(H^{-1}(\Omega))^N} \leq C' \int_0^{\mathrm{T}} \int_\omega |\widetilde{\Phi}|^2 \mathrm{d}x\mathrm{d}t. \tag{2.6}$$

Proof For any given $(\widetilde{\Phi}_0, \widetilde{\Phi}_1) \in (L^2(\Omega))^N \times (H^{-1}(\Omega))^N$, let

$$\widehat{\Psi}_0 = (\Delta)^{-1}\widetilde{\Phi}_1, \quad \widehat{\Psi}_1 = \widetilde{\Phi}_0. \tag{2.7}$$

Since Δ is a continuous isomorphism from $H_0^1(\Omega)$ to $H^{-1}(\Omega)$, we have

$$\|\widehat{\Psi}_0\|^2_{(H_0^1(\Omega))^N} + \|\widehat{\Psi}_1\|^2_{(L^2(\Omega))^N} \sim \|\widetilde{\Phi}_0\|^2_{(L^2(\Omega))^N} + \|\widetilde{\Phi}_1\|^2_{(H^{-1}(\Omega))^N}. \tag{2.8}$$

Let Ψ be the solution to system (1.8) with the initial data $(\widehat{\Psi}_0, \widehat{\Psi}_1)$ given by (2.7). We have

$$t = 0: \ \Psi' = \widetilde{\Phi}_0, \quad \Psi'' = \Delta\widehat{\Psi}_0 = \widetilde{\Phi}_1.$$

By well-posedness, we get

$$\Psi' = \widetilde{\Phi}. \tag{2.9}$$

Applying Theorem 2.2 to Ψ and noting (2.9), we have (2.6). The proof is then completed.

Finally, using the compact perturbation as in [15], we will give the proof of Theorem 2.1 as follows.

We rewrite system (1.4) as

$$\begin{pmatrix} \Phi \\ \Phi' \end{pmatrix}' = \mathcal{A}\begin{pmatrix} \Phi \\ \Phi' \end{pmatrix} + \mathcal{B}\begin{pmatrix} \Phi \\ \Phi' \end{pmatrix},$$

where

$$\mathcal{A} = \begin{pmatrix} 0 & I_N \\ \Delta & 0 \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} 0 & 0 \\ -A^{\mathrm{T}} & 0 \end{pmatrix},$$

$\Delta : H_0^1(\Omega) \cap H^2(\Omega) \to L^2(\Omega)$ and I_N is the unit matrix of order N. It is easy to see that $\mathcal{A}$ is a skew-adjoint operator with compact resolvent in $(L^2(\Omega))^N \times (H^{-1}(\Omega))^N$, and $\mathcal{B}$ is a compact operator in $(L^2(\Omega))^N \times (H^{-1}(\Omega))^N$. Therefore, operators $\mathcal{A}$ and $\mathcal{A}+\mathcal{B}$ can respectively generate C^0 groups $S_{\mathcal{A}}(t)$ and $S_{\mathcal{A}+\mathcal{B}}(t)$ in space $(L^2(\Omega))^N \times (H^{-1}(\Omega))^N$.

Following a perturbation result in [12, 15], in order to prove the observability inequalities (2.4) for a system of this kind, it is sufficient to check the following assertions:

(i) The direct and inverse inequalities

$$C\int_0^{\mathrm{T}}\int_\omega |\widetilde{\Phi}|^2 \mathrm{d}x\mathrm{d}t \le \|\widetilde{\Phi}_0\|^2_{(L^2(\Omega))^N} + \|\widetilde{\Phi}_1\|^2_{(H^{-1}(\Omega))^N} \le C'\int_0^{\mathrm{T}}\int_\omega |\widetilde{\Phi}|^2 \mathrm{d}x\mathrm{d}t$$

hold for the solution $\widetilde{\Phi} = S_{\mathcal{A}}(t)(\widetilde{\Phi}_0, \widetilde{\Phi}_1)$ to the decoupled problem (1.8)–(1.9).

(ii) The system of root vectors of $\mathcal{A}+\mathcal{B}$ forms a Riesz basis of subspaces in $(L^2(\Omega))^N \times (H^{-1}(\Omega))^N$, i.e., there exists a family of subspaces $\mathcal{V}_m \times \mathcal{H}_m (m \ge 1)$ composed of root vectors of $\mathcal{A}+\mathcal{B}$, such that for any given $x \in (L^2(\Omega))^N \times (H^{-1}(\Omega))^N$, there exists a unique sequence $x_m \in \mathcal{V}_m \times \mathcal{H}_m$ for each $m \ge 1$, such that

$$x = \sum_{m=1}^{+\infty} x_m, \quad C\|x\|^2 \le \sum_{m=1}^{+\infty} \|x_m\|^2 \le C'\|x\|^2,$$

where C, C' are positive constants.

(iii) If $(\Phi, \Psi) \in (L^2(\Omega))^N \times (H^{-1}(\Omega))^N$ and $\lambda \in \mathbb{C}$ such that

$$(\mathcal{A}+\mathcal{B})(\Phi, \Psi) = \lambda(\Phi, \Psi) \quad \text{and} \quad \Phi = 0 \quad \text{on } \omega, \tag{2.10}$$

then $(\Phi, \Psi) \equiv 0$.

Since the assertion (i) was proved in Corollary 2.1, we only have to prove (ii) and (iii).

Proof of (ii). Let e_m be a unit eigenfunction of $-\Delta$ with homogeneous Dirichlet boundary condition:

$$\begin{cases} -\Delta e_m = \lambda_m^2 e_m & \text{in } \Omega, \\ e_m = 0 & \text{on } \Gamma, \end{cases} \tag{2.11}$$

where the sequence $\{\lambda_m\}_{m\geq 1}$ of positive terms is increasing so that $\lambda_m \to +\infty$ as $m \to +\infty$. Clearly, $\{e_m\}_{m\geq 1}$ is a Hilbert basis in $L^2(\Omega)$.

Let

$$\mathcal{V}_m \times \mathcal{H}_m = \{(\alpha e_m, \beta e_m) : \alpha, \beta \in \mathbb{C}^N\}. \tag{2.12}$$

Obviously, the subspaces $\mathcal{V}_m \times \mathcal{H}_m (m = 1, 2, \cdots)$ are mutually orthogonal and

$$(L^2(\Omega))^N \times (H^{-1}(\Omega))^N = \bigoplus_{m\geq 1} \mathcal{V}_m \times \mathcal{H}_m, \tag{2.13}$$

where $\bigoplus$ stands for the direct sum of subspaces. In particular, for any given $x \in (L^2(\Omega))^N \times (H^{-1}(\Omega))^N$, there exists a sequence $x_m \in \mathcal{V}_m \times \mathcal{H}_m$, such that

$$x = \sum_{m=1}^{+\infty} x_m, \quad \|x\|^2 = \sum_{m=1}^{+\infty} \|x_m\|^2. \tag{2.14}$$

On the other hand, since $\mathcal{V}_m \times \mathcal{H}_m$ is an invariant subspace of $\mathcal{A}+\mathcal{B}$ and with finite dimension, the restriction of $\mathcal{A}+\mathcal{B}$ in the subspace $\mathcal{V}_m \times \mathcal{H}_m$ is a linear bounded operator, hence its root vectors constitute a basis in the finite dimensional complex space $\mathcal{V}_m \times \mathcal{H}_m$. This together with (2.13)–(2.14) implies that the system of root vectors of $\mathcal{A}+\mathcal{B}$ forms a Riesz basis of subspaces in $(L^2(\Omega))^N \times (H^{-1}(\Omega))^N$.

Proof of (iii). Let $(\Phi, \Psi) \in (L^2(\Omega))^N \times (H^{-1}(\Omega))^N$ and $\lambda \in \mathbb{C}$ satisfy (2.10). We have

$$\Psi = \lambda\Phi, \quad \Delta\Phi - A^{\mathrm{T}}\Phi = \lambda\Psi,$$

namely,

$$\begin{cases} \Delta\Phi = (\lambda^2 I + A^{\mathrm{T}})\Phi & \text{in } \Omega, \\ \Phi = 0 & \text{on } \Gamma. \end{cases}$$

It follows from the classic elliptic theory that $\Phi \in H^2(\Omega)$. Moreover, noting that Φ satisfies

$$\Phi = 0 \ \text{ on } \omega,$$

by [12, Proposition 3.4], we get $\Phi \equiv 0$, then $\Psi \equiv 0$. The proof is complete.

We now show that system (1.1) is exactly controllable under N internal controls by a standard application of the HUM method of J.-L. Lions (see [13]).

Theorem 2.3 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary Γ. Assume that ω is a neighbourhood of $\overline{\Gamma}(x_0)$ in Ω. Assume furthermore that D is invertible. Then system* (1.1) *is exactly controllable at the time $T > T(x_0)$ in the space $(H^1_0(\Omega))^N \times (L^2(\Omega))^N$.*

Proof Let Φ be the solution to the adjoint problem (1.4)–(1.5) with $(\widehat{\Phi}_0, \widehat{\Phi}_1) \in (L^2(\Omega))^N \times (H^{-1}(\Omega))^N$. Let

$$H = \chi_\omega D^{-1}\Phi. \tag{2.15}$$

By direct inequality in (2.4), $H \in L^2(0,T;L^2(\omega))^N$. By Proposition 2.1, the corresponding backward problem

$$\begin{cases} V'' - \Delta V + AV = \chi_\omega \Phi & \text{in } (0,T)\times\Omega, \\ V = 0 & \text{on}(0,T)\times\Gamma, \\ V(T) = V'(T) = 0 & \text{in } \Omega \end{cases} \tag{2.16}$$

admits a unique weak solution $V \in (C^0([0,T];H_0^1(\Omega)))^N \cap (C^1([0,T];L^2(\Omega)))^N$. Then the linear map

$$\Lambda(\widehat{\Phi}_0, \widehat{\Phi}_1) = (-V'(0), V(0)) \tag{2.17}$$

is well defined and continuous from $(L^2(\Omega))^N \times (H^{-1}(\Omega))^N$ into $(L^2(\Omega))^N \times (H_0^1(\Omega))^N$. We define the Hilbert norm

$$\|(\widehat{\Phi}_0, \widehat{\Phi}_1)\|_F = \Big(\int_0^{\mathrm{T}} \int_\omega |\Phi|^2 \mathrm{d}x\mathrm{d}t\Big)^{\frac{1}{2}}. \tag{2.18}$$

Denote by F the completion of $\mathcal{D}(\Omega)\times\mathcal{D}(\Omega)$ with respect to the $\|\cdot\|_F$ norm. By Theorem 2.1, we have $F = (L^2(\Omega))^N \times (H^{-1}(\Omega))^N$.

Let Ψ be the solution to system (1.4) with

$$t = 0: \ \Psi = \widehat{\Psi}_0, \quad \Psi' = \widehat{\Psi}_1.$$

Multiplying the backward problem (2.16) by Ψ and intergrating by parts, we have

$$-\int_\Omega (V'(0), \widehat{\Psi}_0)\mathrm{d}x + \int_\Omega (V(0), \widehat{\Psi}_1)\mathrm{d}x = \int_0^{\mathrm{T}} \int_\omega (\Phi(t), \Psi(t))\mathrm{d}x\mathrm{d}t,$$

namely,

$$\langle \Lambda(\widehat{\Phi}_0, \widehat{\Phi}_1), (\widehat{\Psi}_0, \widehat{\Psi}_1)\rangle_{F'\times F} = \int_0^{\mathrm{T}} \int_\omega (\Phi(t), \Psi(t))\mathrm{d}x\mathrm{d}t.$$

Then

$$|\langle \Lambda(\widehat{\Phi}_0, \widehat{\Phi}_1), (\widehat{\Psi}_0, \widehat{\Psi}_1)\rangle_{F'\times F}| \le \|(\widehat{\Phi}_0, \widehat{\Phi}_1)\|_F \|(\widehat{\Psi}_0, \widehat{\Psi}_1)\|_F$$

and

$$\langle \Lambda(\widehat{\Phi}_0, \widehat{\Phi}_1), (\widehat{\Phi}_0, \widehat{\Phi}_1)\rangle_{F'\times F} = \|(\widehat{\Phi}_0, \widehat{\Phi}_1)\|_F^2.$$

Therefore, $\langle \Lambda(\widehat{\Phi}_0, \widehat{\Phi}_1), (\widehat{\Phi}_0, \widehat{\Phi}_1)\rangle_{F'\times F}$ is a bilinear, symmetric, continuous and coercive form on $F\times F$. By Lax and Milgram's lemma, Λ is an isomorphism from $(L^2(\Omega))^N \times (H^{-1}(\Omega))^N$ onto $(L^2(\Omega))^N \times (H_0^1(\Omega))^N$. Then for any given $(\widehat{U}_0, \widehat{U}_1) \in (H_0^1(\Omega))^N \times (L^2(\Omega))^N$, there exists a unique $(\widehat{\Phi}_0, \widehat{\Phi}_1) \in (L^2(\Omega))^N \times (H^{-1}(\Omega))^N$, such that

$$\Lambda(\widehat{\Phi}_0, \widehat{\Phi}_1) = (-\widehat{U}_1, \widehat{U}_0) = (-V'(0), V(0)).$$

By well-posedness, the solution U to problem (1.1)–(1.2) with the control given by (2.15) satisfies the final condition (2.3). The proof is completed.

2.2 The proof of Theorem 2.2

Now, we pass to the proof of Theorem 2.2. By Proposition 1.1, we try to get rid of the lower term $\int_0^T\int_\omega|\widetilde{\Phi}|^2\mathrm{d}x\mathrm{d}t$ in (1.10). By (1.10), there exists a positive constant C' such that

$$\|\widetilde{\Phi}_0\|^2_{(H_0^1(\Omega))^N}+\|\widetilde{\Phi}_1\|^2_{(L^2(\Omega))^N}\le C'\int_0^T\int_\omega(|\widetilde{\Phi}|^2+|\widetilde{\Phi}'|^2)\mathrm{d}x\mathrm{d}t,$$

where C' depends on the size of ω. However, since C' could be very large like $\frac{1}{\varepsilon^3}$, where ε is the thickness of ω $\big($i.e., $\varepsilon=\min\limits_{x\in\overline{\Gamma}(x_0)}d(x,\omega^c)$, where ω^c denotes the complementary set of $\omega\big)$, we can not absorb it directly from the left-hand side.

Inspired by [9, 11], we can first show the observability inequality (2.5) for the initial data with higher frequencies, and then extend it to the whole space $(H_0^1(\Omega))^N\times(L^2(\Omega))^N$ based on the following lemma.

Lemma 2.1 (see [11]) *Let $\mathcal{F}$ be a Hilbert space endowed with the p-norm $\|\cdot\|_p$. Assume that*

$$\mathcal{F}=\mathcal{N}\oplus\mathcal{L}, \tag{2.19}$$

where $\bigoplus$ denotes the direct sum and $\mathcal{L}$ is a finite co-dimensional closed subspace in $\mathcal{F}$. Assume that q is another norm in $\mathcal{F}$, such that the projection from $\mathcal{F}$ into $\mathcal{N}$ is continuous with respect to the q-norm $\|\cdot\|_q$. Assume furthermore that

$$q(y)\le p(y),\quad \forall y\in\mathcal{L}. \tag{2.20}$$

Then there exists a positive constant C such that

$$q(z)\le Cp(z),\quad \forall z\in\mathcal{F}. \tag{2.21}$$

In order to extend (2.20) to the whole space $\mathcal{F}$ (i.e., $(H_0^1(\Omega))^N\times(L^2(\Omega))^N$), it is sufficient to verify the continuity of the projection from $\mathcal{F}$ into $\mathcal{N}$ for the q-norm. In many situations, it often occurs that the subspaces $\mathcal{N}$ and $\mathcal{L}$ are mutually orthogonal with respect to the q-inner product.

We now give the proof of Theorem 2.2.

Step 1. We define the linear unbounded operator $-\Delta$ in $(L^2(\Omega))^N$ by

$$D(-\Delta)=\{\Phi\in(H^2(\Omega))^N:\Phi|_\Gamma=0\}.$$

Clearly, $-\Delta$ is a densely defined self-adjoint and coercive operator with a compact resolvent in $(L^2(\Omega))^N$. Then we can define the power operator $(-\Delta)^{\frac{s}{2}}$ for any given $s\in\mathbb{R}$. Moreover, the domain $\mathcal{H}_s=D((-\Delta)^{\frac{s}{2}})$ endowed with the norm $\|\Phi\|_{\mathcal{H}_s}=\|(-\Delta)^{\frac{s}{2}}\Phi\|_{(L^2(\Omega))^N}$ is a Hilbert space, and its dual space with respect to the pivot space $(L^2(\Omega))^N$ is $\mathcal{H}_s'=\mathcal{H}_{-s}$. In particular, we have

$$\mathcal{H}_1=D(\sqrt{-\Delta})=(H_0^1(\Omega))^N.$$

Then we formulate (1.8)–(1.9) into an abstract evolution problem:

$$\begin{cases} \widetilde{\Phi}'' - \Delta\widetilde{\Phi} = 0 \\ t = 0: \ \widetilde{\Phi} = \widetilde{\Phi}_0, \quad \widetilde{\Phi}' = \widetilde{\Phi}_1. \end{cases} \tag{2.22}$$

Clearly, problem (2.22) generates a C^0-semigroup in the space $\mathcal{H}_s \times \mathcal{H}_{s-1}$.

For each $m \geq 1$, we define the subspace Z_m by

$$Z_m = \{\alpha e_m : \alpha \in \mathbb{R}^N\}. \tag{2.23}$$

For any given integers $m \neq n$ and any given vectors $\alpha, \beta \in \mathbb{R}^N$, we have

$$\begin{aligned} (\alpha e_m, \beta e_n)_{\mathcal{H}_s} &= (\alpha, \beta)((-\Delta)^{\frac{s}{2}} e_m, (-\Delta)^{\frac{s}{2}} e_n)_{L^2(\Omega)} \\ &= (\alpha, \beta)\lambda_m^s \lambda_n^s (e_m, e_n)_{L^2(\Omega)} \\ &= (\alpha, \beta)\lambda_m^s \lambda_n^s \delta_{mn}. \end{aligned} \tag{2.24}$$

Then the subspaces $Z_m (m \geq 1)$ are mutually orthogonal in the Hilbert space $\mathcal{H}_s$ for any given $s \in \mathbb{R}$ and in particular, we have

$$\|\widetilde{\Phi}\|_{\mathcal{H}_s} = \frac{1}{\lambda_m} \|\widetilde{\Phi}\|_{\mathcal{H}_{s+1}}, \quad \forall \widetilde{\Phi} \in Z_m. \tag{2.25}$$

Noting that Z_m is invariant for $-\Delta$, the solution $\widetilde{\Phi}$ to problem (2.22) with $(\widetilde{\Phi}_0, \widetilde{\Phi}_1) \in Z_m \times Z_m$ is in the space Z_m.

Let $m_0 \geq 1$ be an integer. We denote by $\bigoplus\limits_{m \geq m_0} (Z_m \times Z_m)$ the linear hull of the subspaces $(Z_m \times Z_m)$ for $m \geq m_0$. In other words, $\bigoplus\limits_{m \geq m_0} (Z_m \times Z_m)$ is composed of all finite linear combinations of elements of $(Z_m \times Z_m)$ for $m \geq m_0$.

In particular, by (2.25), we have

$$\|\widetilde{\Phi}_0\|^2_{(L^2(\Omega))^N} + \|\widetilde{\Phi}_1\|^2_{(H^{-1}(\Omega))^N} \leq \frac{1}{\lambda_{m_0}^2} (\|\widetilde{\Phi}_0\|^2_{(H_0^1(\Omega))^N} + \|\widetilde{\Phi}_1\|^2_{(L^2(\Omega))^N}) \tag{2.26}$$

for any given $(\widetilde{\Phi}_0, \widetilde{\Phi}_1) \in \bigoplus\limits_{m \geq m_0} (Z_m \times Z_m)$.

Step 2. Recalling (1.10), by well-posedness of problem (2.22) in $(L^2(\Omega))^N \times (H^{-1}(\Omega))^N$ and (2.26), there exists a positive constant C such that

$$\begin{aligned} \int_0^{\mathrm{T}} \int_\omega |\widetilde{\Phi}|^2 \mathrm{d}x\mathrm{d}t &\leq C(\|\widetilde{\Phi}_0\|^2_{(L^2(\Omega))^N} + \|\widetilde{\Phi}_1\|^2_{(H^{-1}(\Omega))^N}) \\ &\leq \frac{C}{\lambda_{m_0}^2} (\|\widetilde{\Phi}_0\|^2_{(H_0^1(\Omega))^N} + \|\widetilde{\Phi}_1\|^2_{(L^2(\Omega))^N}). \end{aligned}$$

Taking $m_0 \geq 1$ so large that $\frac{C}{\lambda_{m_0}^2} < 1$, the lower term $\int_0^{\mathrm{T}} \int_\omega |\widetilde{\Phi}|^2 \mathrm{d}x\mathrm{d}t$ in (1.10) can be absorbed by the left-hand side, namely, there exists a positive constant C' such that

$$\|\widetilde{\Phi}_0\|^2_{(H_0^1(\Omega))^N} + \|\widetilde{\Phi}_1\|^2_{(L^2(\Omega))^N} \leq C' \int_0^{\mathrm{T}} \int_\omega |\widetilde{\Phi}'|^2 \mathrm{d}x\mathrm{d}t \tag{2.27}$$

for any given $(\widetilde{\Phi}_0, \widetilde{\Phi}_1) \in \bigoplus_{m \geq m_0} (Z_m \times Z_m)$.

Step 3. For any given $(\widetilde{\Phi}_0, \widetilde{\Phi}_1) \in \bigoplus_{m \geq 1} (Z_m \times Z_m)$, define

$$p(\widetilde{\Phi}_0, \widetilde{\Phi}_1) = \Big(\int_0^{\mathrm{T}} \int_\omega |\widetilde{\Phi}'|^2 \mathrm{d}x \mathrm{d}t \Big)^{\frac{1}{2}}, \tag{2.28}$$

where $\widetilde{\Phi}$ is the solution to the corresponding adjoint problem (2.22). By Holmgren's uniqueness Theorem (see [13, Theorem 8.1]), for $T > 0$ large enough, $p(\cdot)$ defines well a Hilbert norm in $\bigoplus_{m \geq 1} (Z_m \times Z_m)$. Then, we denote by $\mathcal{F}$ the completion of $\bigoplus_{m \geq 1} (Z_m \times Z_m)$ with respect to the p-norm. Clearly, $\mathcal{F}$ is a Hilbert space.

We next take

$$\mathcal{N} = \bigoplus_{1 \leq m < m_0} (Z_m \times Z_m), \quad \mathcal{L} = \overline{\{ \bigoplus_{m \geq m_0} (Z_m \times Z_m)\}}^p \tag{2.29}$$

in (2.19).

Clearly, $\mathcal{N}$ is a finite-dimensional subspace and $\mathcal{L}$ is a closed subspace in $\mathcal{F}$. In particular, (2.27) can be extended by continuity to all initial data $(\widetilde{\Phi}_0, \widetilde{\Phi}_1)$ in the whole subspace $\mathcal{L}$.

We introduce the second norm

$$q(\widetilde{\Phi}_0, \widetilde{\Phi}_1) = \|(\widetilde{\Phi}_0, \widetilde{\Phi}_1)\|_{(H_0^1(\Omega))^N \times (L^2(\Omega))^N}, \quad \forall (\widetilde{\Phi}_0, \widetilde{\Phi}_1) \in \mathcal{F}. \tag{2.30}$$

Since

$$(\widetilde{\Phi}_0, \widetilde{\Phi}_1) = (\widetilde{\Phi}_0^{(\mathcal{N})}, \widetilde{\Phi}_1^{(\mathcal{N})}) + (\widetilde{\Phi}_0^{(\mathcal{L})}, \widetilde{\Phi}_1^{(\mathcal{L})}),$$

where

$$(\widetilde{\Phi}_0^{(\mathcal{N})}, \widetilde{\Phi}_1^{(\mathcal{N})}) \in \mathcal{N}, \quad (\widetilde{\Phi}_0^{(\mathcal{L})}, \widetilde{\Phi}_1^{(\mathcal{L})}) \in \mathcal{L}$$

by (2.27), for all $(\widetilde{\Phi}_0, \widetilde{\Phi}_1) \in \mathcal{F}$, we have

$$\begin{aligned} \|(\widetilde{\Phi}_0, \widetilde{\Phi}_1)\|_{(H_0^1(\Omega))^N \times (L^2(\Omega))^N} &= \|(\widetilde{\Phi}_0^{(\mathcal{N})}, \widetilde{\Phi}_1^{(\mathcal{N})})\|_{(H_0^1(\Omega))^N \times (L^2(\Omega))^N} \\ &\quad + \|(\widetilde{\Phi}_0^{(\mathcal{L})}, \widetilde{\Phi}_1^{(\mathcal{L})})\|_{(H_0^1(\Omega))^N \times (L^2(\Omega))^N} \\ &\leq \|(\widetilde{\Phi}_0^{(\mathcal{N})}, \widetilde{\Phi}_1^{(\mathcal{N})})\|_{(H_0^1(\Omega))^N \times (L^2(\Omega))^N} + C' \Big(\int_0^{\mathrm{T}} \int_\omega |\widetilde{\Phi}'|^2 \mathrm{d}x \mathrm{d}t \Big)^{\frac{1}{2}} \\ &< +\infty. \end{aligned}$$

Moreover, (2.27) means that

$$q(\widetilde{\Phi}_0, \widetilde{\Phi}_1) \leq C' p(\widetilde{\Phi}_0, \widetilde{\Phi}_1), \quad \forall (\widetilde{\Phi}_0, \widetilde{\Phi}_1) \in \mathcal{L}. \tag{2.31}$$

Since $\mathcal{N}$ is an orthogonal complement of $\mathcal{L}$ for the q-inner product, the projection from $\mathcal{F}$ into $\mathcal{L}$ is continuous for the q-norm. By Lemma 2.1, we can extend (2.27) to the whole space $\mathcal{F}$:

$$C p(\widetilde{\Phi}_0, \widetilde{\Phi}_1) \leq q(\widetilde{\Phi}_0, \widetilde{\Phi}_1) \leq C' p(\widetilde{\Phi}_0, \widetilde{\Phi}_1), \quad \forall (\widetilde{\Phi}_0, \widetilde{\Phi}_1) \in \mathcal{F},$$

which implies

$$\mathcal{F} = (H_0^1(\Omega))^N \times (L^2(\Omega))^N.$$

The proof is then completed.

Remark 2.2 In the special case that $\omega = \Omega$, (2.6) holds for any given $T > 0$ (see [13, Chapter 7]). Then, system (1.1) is exactly controllable at any given time $T > 0$ under N internal controls.

2.3 Non-exact internal controllability

For any given initial data $(\widehat{U}_0, \widehat{U}_1) \in (H_0^1(\Omega))^N \times (L^2(\Omega))^N$, let $\mathcal{U}_{ad}(\widehat{U}_0, \widehat{U}_1)$ denote the set of all the controls H which realize the exact internal controllability for system (1.1) at the time T.

Similarly to [12, Theorem 3.8], we have the following theorem.

Theorem 2.4 *Assume that system* (1.1) *is exactly controllable in the space* $(H_0^1(\Omega))^N \times (L^2(\Omega))^N$. *Then for* $\varepsilon > 0$ *small enough, the values of* $H \in \mathcal{U}_{ad}(\widehat{U}_0, \widehat{U}_1)$ *on* $(T-\varepsilon, T) \times \omega$ *can be arbitrarily chosen.*

It is easy to see that $\mathcal{U}_{ad}$ is a convex, closed and non-empty set. By Hilbert projection theorem, there exists a unique control $H_0 \in \mathcal{U}_{ad}(\widehat{U}_0, \widehat{U}_1)$, such that

$$\|H_0\|_{(L^2(0,T;L^2(\omega)))^M} = \inf_{H \in \mathcal{U}_{ad}} \|H\|_{(L^2(0,T;L^2(\omega)))^M}. \tag{2.32}$$

Proposition 2.2 *Assume that system* (1.1) *is exactly controllable at the time* T *in* $(H_0^1(\Omega))^N \times (L^2(\Omega))^N$. *There exists a positive constant* $C > 0$, *such that for any given initial data* $(\widehat{U}_0, \widehat{U}_1) \in (H_0^1(\Omega))^N \times (L^2(\Omega))^N$, *the optimal internal control* H_0 *given by* (2.32) *satisfies the following estimate*:

$$\|H_0\|_{(L^2(0,T;L^2(\omega)))^M} \le C\|(\widehat{U}_0, \widehat{U}_1)\|_{(H_0^1(\Omega))^N \times (L^2(\Omega))^N}. \tag{2.33}$$

Proof For any given $H \in (L^2(0,T;L^2(\omega)))^M$, we solve the following backward problem:

$$\begin{cases} V'' - \Delta V + AV = D\chi_\omega H & \text{in } (0,T) \times \Omega, \\ V = 0 & \text{on } (0,T) \times \Gamma, \\ t = T: \ V = V' = 0 & \text{in } \Omega. \end{cases} \tag{2.34}$$

By Proposition 2.1, the map

$$\mathcal{T}: \ H \to (V(0), V'(0))$$

is linear and continuous from $(L^2(0,T;L^2(\omega)))^M$ into $(H_0^1(\Omega))^N \times (L^2(\Omega))^N$.

Let $\mathcal{N}$ denote the kernel of $\mathcal{T}$, which is a closed subspace in $(L^2(0,T;L^2(\omega)))^M$. The quotient space $(L^2(0,T;L^2(\omega)))^M/\mathcal{N}$ is a Hilbert space and $\mathcal{T}$ is still continuous from $(L^2(0,T;L^2(\omega)))^M/\mathcal{N}$ into $(H_0^1(\Omega))^N \times (L^2(\Omega))^N$.

By definition, $\mathcal{T}$ is injective. On the other hand, the exact internal controllability of system (1.1) implies that $\mathcal{T}$ is surjective, therefore, it is a bijection from $(L^2(0,T;L^2(\omega)))^M/\mathcal{N}$ into $(H_0^1(\Omega))^N \times (L^2(\Omega))^N$. By Banach-Schauder's open mapping theorem, $\mathcal{T}^{-1}$ is bounded from $(H_0^1(\Omega))^N \times (L^2(\Omega))^N$ into $(L^2(0,T;L^2(\omega)))^M/\mathcal{N}$. Then there exists a constant $C>0$, such that

$$\inf_{H\in\mathcal{U}_{ad}} \|H\|_{(L^2(0,T;L^2(\omega)))^M} = \|\dot{H}\|_{(L^2(0,T;L^2(\omega)))^M} \le C\|(\widehat{U}_0,\widehat{U}_1)\|_{(H_0^1(\Omega))^N\times(L^2(\Omega))^N},$$

where $\dot{H}$ belongs to the equivalence class $H+\mathcal{N}$ of H. Then, noting (2.32), we get (2.33).

In the case of fewer internal controls, we have the following negative result.

Theorem 2.5 *Assume that* $\operatorname{rank}(D) < N$. *Then, no matter how large the time* $T>0$ *is, system* (1.1) *is not exactly controllable in the space* $(H_0^1(\Omega))^N \times (L^2(\Omega))^N$.

Proof Let $E\in\mathbb{R}^N$ be a unit vector such that $D^{\mathrm{T}}E=0$. For any given $\theta\in\mathcal{D}(\Omega)$, we choose the special initial data as

$$t=0:\ U=0,\quad U'=\theta E. \tag{2.35}$$

If system (1.1) is exactly controllable, by Proposition 2.2, there exists a positive constant C such that the optimal control H_0 satisfies the following estimate:

$$\|H_0\|_{(L^2(0,T;L^2(\omega)))^M} \le C\|\theta\|_{L^2(\Omega)}.$$

By Proposition 2.1, the corresponding solution U possesses the regularity

$$U\in(C^0(0,T;H_0^1(\Omega)))^N \tag{2.36}$$

with the continuous dependence

$$\|U\|_{(C^0(0,T;H_0^1(\Omega)))^N} \le C'\|\theta\|_{L^2(\Omega)}, \tag{2.37}$$

where C' is a positive constant independent of θ, and then the mapping $\theta\to U$ is compact.

Now, applying E to (1.1) and noting $w=(E,U)$, we get the following backward problem:

$$\begin{cases} w''-\Delta w = -(E,AU) & \text{in } (0,T)\times\Omega,\\ w=0 & \text{on}(0,T)\times\Gamma,\\ t=T:\ w=w'=0 & \text{in } \Omega. \end{cases} \tag{2.38}$$

Noting (2.36), the right-hand side in (2.38) belongs to $C^0(0,T;H_0^1(\Omega))$. By Proposition 2.1 and the compactness of the mapping $\theta\to U$, the mapping

$$\begin{aligned} T:\quad \theta \quad &\to \quad w\\ L^2(\Omega) &\to C^1(0,T;H_0^1(\Omega)) \end{aligned}$$

is also compact, which contradicts the fact that

$$\|\theta\|_{H_0^1(\Omega)} = \|w'(0)\|_{H_0^1(\Omega)} \le \|w\|_{C^1(0,T;H_0^1(\Omega))}. \tag{2.39}$$

The proof is complete.

Combining Theorems 2.3 and 2.5, we have the following theorem.

Theorem 2.6 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary Γ. Assume that ω is a neighbourhood of $\overline{\Gamma}(x_0)$ in Ω. Then the coupled system* (1.1) *composed of N wave equations is exactly controllable at the time $T > T(x_0)$ in the space $(H_0^1(\Omega))^N \times (L^2(\Omega))^N$ if and only if the internal control matrix D has the rank N.*

3 Exact Internal Synchronization by p-Groups

According to Theorem 2.6, system (1.1) is exactly controllable if and only if $\mathrm{rank}(D) = N$, namely, $M = N$ and the internal control matrix D is invertible. When the number of internal controls $M = \mathrm{rank}(D) < N$, the exact internal controllability fails. In order to consider the situation that the number of internal controls is reduced, we will investigate the exact internal synchronization by groups for system (1.1).

Since the consideration on the exact internal synchronization by p-groups is quite similar to that on the exact boundary synchronization by p-groups (see [12]), in what follows we only give the main results and some brief explanations.

3.1 Exact internal synchronization by p-groups

Let $p \geq 1$ be an integer and

$$0 = n_0 < n_1 < n_2 < \cdots < n_p = N$$

with $n_r - n_{r-1} \geq 2$ for all $1 \leq r \leq p$. We re-arrange the components of the state variable U into p groups

$$(u^{(1)}, \cdots, u^{(n_1)}), (u^{(n_1+1)}, \cdots, u^{(n_2)}), \cdots, (u^{(n_{p-1}+1)}, \cdots, u^{(N)}).$$

Definition 3.1 *System* (1.1) *is exactly synchronizable by p-groups at the time $T > 0$ if for any given initial data $(\widehat{U}_0, \widehat{U}_1) \in (H_0^1(\Omega))^N \times (L^2(\Omega))^N$, there exists an internal control $H \in (L^2(0, +\infty; L^2(\omega)))^{N-p}$ with compact support in $[0, T]$, such that the corresponding solution $U = U(t, x)$ to problem* (1.1)–(1.2) *satisfies*

$$t \geq T : \begin{cases} u^{(1)} \equiv \cdots \equiv u^{(n_1)} := u_1, \\ u^{(n_1+1)} \equiv \cdots \equiv u^{(n_2)} := u_2, \\ \cdots \\ u^{(n_{p-1}+1)} \equiv \cdots \equiv u^{(N)} := u_p, \end{cases} \tag{3.1}$$

where $u = (u_1, \cdots, u_p)^{\mathrm{T}}$ is called the exactly synchronizable state by p-groups, which is a priori unknown.

In the special case $p = 1$, the exact internal synchronization by p-groups becomes the exact internal synchronization, and the exactly synchronizable state by p-groups $u = (u_1, \cdots, u_p)^{\mathrm{T}}$ becomes the exactly synchronizable state u.

Let S_r be the matrix of order $(n_r - n_{r-1} - 1) \times (n_r - n_{r-1})$, defined by

$$S_r = \begin{pmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & 1 & -1 \end{pmatrix}, \quad 1 \le r \le p.$$

Define the $(N-p) \times N$ matrix C_p of synchronization by p-groups as

$$C_p = \begin{pmatrix} S_1 & & & \\ & S_2 & & \\ & & \ddots & \\ & & & S_p \end{pmatrix}.$$

Defining the orthogonal vectors by

$$(e_r)_j = \begin{cases} 1, & n_{r-1} + 1 \le j \le n_r, \\ 0, & \text{otherwise}, \end{cases} \tag{3.2}$$

where $1 \le r \le p$, we have

$$\mathrm{Ker}(C_p) = \mathrm{Span}\{e_1, \cdots, e_p\}. \tag{3.3}$$

It is easy to see that the exact internal synchronization by p-groups (3.1) can be written as

$$t \ge T: \quad C_p U \equiv 0, \tag{3.4}$$

or, equivalently,

$$t \ge T: \quad U = \sum_{r=1}^{p} u_r e_r. \tag{3.5}$$

Definition 3.2 *The matrix A satisfies the condition of C_p-compatibility if there exists a matrix $\overline{A_p}$ of order $(N-p)$, such that*

$$C_p A = \overline{A_p} C_p, \tag{3.6}$$

where the matrix $\overline{A_p}$ is called the reduced matrix of A by C_p.

Lemma 3.1 (see [12]) *The matrix A satisfies the condition of C_p-compatibility* (3.6) *if and only if the kernel of C_p is an invariant subspace of A:*

$$A\mathrm{Ker}(C_p) \subseteq \mathrm{Ker}(C_p). \tag{3.7}$$

Moreover, the reduced matrix $\overline{A_p}$ is given by

$$\overline{A_p} = C_p A C_p^{+},$$

where

$$C_p^{+} = C_p^{\mathrm{T}} (C_p C_p^{\mathrm{T}})^{-1} \tag{3.8}$$

is the Moore-Penrose inverse of C_p.

Remark 3.1 When $p = 1$, condition (3.7) is equivalent to the following row-sum condition:

$$\sum_{l=1}^{N} a_{kl} = a, \quad k = 1, \cdots, N, \tag{3.9}$$

where a is a constant independent of $k = 1, \cdots, N$.

By (3.6), applying C_p to problem (1.1)–(1.2) and setting $W_p = C_pU$, we get the following reduced system:

$$\begin{cases} W_p'' - \Delta W_p + \overline{A_p} W_p = C_p D \chi_\omega H & \text{in } (0, +\infty) \times \Omega, \\ W_p = 0 & \text{on } (0, +\infty) \times \Gamma \end{cases} \tag{3.10}$$

with the initial condition

$$t = 0: \ W_p = C_p \widehat{U}_0, \quad W_p' = C_p \widehat{U}_1 \quad \text{in} \Omega. \tag{3.11}$$

Under the condition of C_p-compatibility (3.6), the exact internal synchronization by p-groups of the original system (1.1) is equivalent to the exact internal null controllability of the reduced system (3.10). Moreover, by Theorem 2.6, we immediately get the following theorem.

Theorem 3.1 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary Γ. Assume that ω is a neighbourhood of $\overline{\Gamma}(x_0)$ in Ω. Under the condition of C_p-compatibility* (3.6)*, system* (1.1) *is exactly synchronizable by p-groups in the space $(H_0^1(\Omega))^N \times (L^2(\Omega))^N$ if and only if*

$$\text{rank}(C_p D) = N - p. \tag{3.12}$$

We next give the necessary condition for the exact internal synchronization by p-groups of system (1.1).

Theorem 3.2 *Assume that system* (1.1) *is exactly synchronizable by p-groups. Then we necessarily have*

$$\text{rank}(C_p D) = N - p. \tag{3.13}$$

Moreover, if

$$\text{rank}(D) = N - p, \tag{3.14}$$

then, we necessarily have the condition of C_p-compatibility (3.6).

Proof We define an $(N - \widetilde{p}) \times N$ full row-rank matrix $\widetilde{C}_{\widetilde{p}} (0 \leq \widetilde{p} \leq p)$ by

$$\text{Im}(\widetilde{C}_{\widetilde{p}}^{\mathrm{T}}) = \text{Span}(C_p^{\mathrm{T}}, A^{\mathrm{T}} C_p^{\mathrm{T}}, \cdots, (A^{\mathrm{T}})^{N-1} C_p^{\mathrm{T}}). \tag{3.15}$$

By Cayley-Hamilton's theorem, we have $A^{\mathrm{T}} \text{Im}(\widetilde{C}_{\widetilde{p}}^{\mathrm{T}}) \subseteq \text{Im}(\widetilde{C}_{\widetilde{p}}^{\mathrm{T}})$, namely,

$$A \text{Ker}(\widetilde{C}_{\widetilde{p}}) \subseteq \text{Ker}(\widetilde{C}_{\widetilde{p}}). \tag{3.16}$$

By Lemma 3.1, there exists a matrix $\widetilde{A}_{\widetilde{p}}$ of order $(N-\widetilde{p})$, such that

$$\widetilde{C}_{\widetilde{p}}A = \widetilde{A}_{\widetilde{p}}\widetilde{C}_{\widetilde{p}}. \tag{3.17}$$

On the other hand, applying C_p to the equations in system (1.1), by (3.4) we can successively get

$$t \geq T: \ C_pAU = 0, \quad C_pA^2U = 0, \cdots,$$

then, noting (3.15), we have

$$t \geq T: \ \widetilde{C}_{\widetilde{p}}U = 0. \tag{3.18}$$

Noting (3.17) and setting $\widetilde{W}_{\widetilde{p}} = \widetilde{C}_{\widetilde{p}}U$, it follows from (3.18) that the reduced system

$$\begin{cases} \widetilde{W}_{\widetilde{p}}'' - \Delta\widetilde{W}_{\widetilde{p}} + \widetilde{A}_{\widetilde{p}}\widetilde{W}_{\widetilde{p}} = \widetilde{C}_{\widetilde{p}}D\chi_\omega H & \text{in}(0,+\infty)\times\Omega, \\ \widetilde{W}_{\widetilde{p}} = 0 & \text{on}(0,+\infty)\times\Gamma \end{cases} \tag{3.19}$$

is exactly controllable in the space $(H_0^1(\Omega))^{N-\widetilde{p}} \times (L^2(\Omega))^{N-\widetilde{p}}$. By Theorem 2.6, we get

$$\operatorname{rank}(\widetilde{C}_{\widetilde{p}}D) = N - \widetilde{p}. \tag{3.20}$$

Noting that $\operatorname{rank}(\widetilde{C}_{\widetilde{p}}) = N - \widetilde{p}$, by [11, Proposition 2.11], we have

$$\operatorname{Ker}(D^{\mathrm{T}}) \cap \operatorname{Im}(\widetilde{C}_{\widetilde{p}}^{\mathrm{T}}) = \{0\}.$$

Since $\operatorname{Im}(C_p^{\mathrm{T}}) \subseteq \operatorname{Im}(\widetilde{C}_{\widetilde{p}}^{\mathrm{T}})$, we have

$$\operatorname{Ker}(D^{\mathrm{T}}) \cap \operatorname{Im}(C_p^{\mathrm{T}}) = \{0\}.$$

Applying again by [12, Proposition 2.11], we have (3.13).

On the other hand, conditions (3.14) and (3.20) imply that $p = \widetilde{p}$. It follows from the definition (3.15) that $A^{\mathrm{T}}\operatorname{Im}(C_p^{\mathrm{T}}) \subseteq \operatorname{Im}(C_p^{\mathrm{T}})$, namely, the condition of C_p-compatibility (3.6) holds true. The proof is completed.

Remark 3.2 The rank M of the internal control matrix D presents the number of internal controls applied to the original system (1.1), while, the rank of the matrix C_pD presents the number of internal controls effectively applied to the reduced system (3.10). Let us write

$$D = D_0 + D_1 \tag{3.21}$$

with $D_0 \in \operatorname{Ker}(C_p)$ and $D_1 \in \operatorname{Im}(C_p^{\mathrm{T}})$. The part D_0 will disappear in the reduced system (3.10), and is useless for the exact internal synchronization by p-groups of the original system (1.1). So, in order to minimize the number of internal controls, we are interested in the control matrix D such that

$$\operatorname{Im}(D) \cap \operatorname{Ker}(C_p) = \{0\} \tag{3.22}$$

or, by [12, Proposition 2.11], such that

$$\operatorname{rank}(C_pD) = \operatorname{rank}(D) = N - p. \tag{3.23}$$

3.2 Exact synchronizable states by p-groups

Under the condition of C_p-compatibility (3.6), it is easy to see that for $t \geq T$, the exactly synchronizable state by p-groups $u = (u_1, \cdots, u_p)^{\mathrm{T}}$ satisfies the following coupled system of homogenous wave equations:

$$\begin{cases} u'' - \Delta u + \widetilde{A} u = 0 & \text{in } (T, +\infty) \times \Omega, \\ u = 0 & \text{on } (T, +\infty) \times \Gamma, \end{cases} \tag{3.24}$$

where $\widetilde{A} = (\alpha_{rs})$ is given by

$$\alpha_{rs} = \sum_{j=n_{s-1}+1}^{n_s} a_{ij}, \quad n_{r-1} + 1 \leq i \leq n_r, \quad 1 \leq r, s \leq p. \tag{3.25}$$

Hence, the evolution of the exactly synchronizable state by p-groups $u = (u_1, \cdots, u_p)^{\mathrm{T}}$ with respect to t is completely determined by the values of (u, u') at the time $t = T$ as the initial condition

$$t = T: \ u = \widehat{u}_0, \quad u' = \widehat{u}_1 \quad \text{in}\Omega, \tag{3.26}$$

where $\widehat{u}_0 = (\widehat{u}_0^{(1)}, \cdots, \widehat{u}_0^{(p)})^{\mathrm{T}}$ and $\widehat{u}_1 = (\widehat{u}_1^{(1)}, \cdots, \widehat{u}_1^{(p)})^{\mathrm{T}}$.

Theorem 3.3 *Assume that the coupling matrix A satisfies the condition of C_p-compatibility* (3.6). *Then the attainable set* (*see* [6, 12, 14]) *of the values* $(\widehat{u}_0, \widehat{u}_1)$ *at the time* $t = T$ *of the exactly synchronizable state by p-groups* $u = (u_1, \cdots, u_p)^{\mathrm{T}}$ *is actually the whole space* $(H_0^1(\Omega))^p \times (L^2(\Omega))^p$ *as the initial data* $(\widehat{U}_0, \widehat{U}_1)$ *vary in the space* $(H_0^1(\Omega))^N \times (L^2(\Omega))^N$.

Proof Let

$$V = \Big\{ \Big(\sum_{r=1}^{p} \widehat{u}_0^{(r)} e_r, \sum_{r=1}^{p} \widehat{u}_1^{(r)} e_r \Big) \Big| (\widehat{u}_0^{(r)}, \widehat{u}_1^{(r)}) \in H_0^1(\Omega) \times L^2(\Omega), \ 1 \leq r \leq p \Big\},$$

where $e_r (1 \leq r \leq p)$ are given by (3.2). Since A satisfies the condition of C_p-compatibility (3.6), V is invariant for

$$\begin{pmatrix} 0 & I_N \\ \Delta - A & 0 \end{pmatrix}.$$

Then system (1.1) generates a C^0 semi-group $S(t)$ on V. By time invertibility, $\mathrm{Im}\, S(T)$ fulfils V, accordingly, the state $(u(T), u'(T))$ fulfils $(H_0^1(\Omega))^p \times (L^2(\Omega))^p$. The proof is complete.

Generally speaking, the exactly synchronizable state by p-groups $u = (u_1, \cdots, u_p)^{\mathrm{T}}$ depends on applied internal control H. We next estimate the difference between the exactly synchronizable state by p-groups $u = (u_1, \cdots, u_p)^{\mathrm{T}}$ and the solution to a problem which is independent of H.

We first consider the special case that A^{T} admits an invariant subspace $\mathrm{Span}\{E_1, \cdots, E_p\}$ which is bi-orthonormal to $\mathrm{Ker}(C_p) = \mathrm{Span}\{e_1, \cdots, e_p\}$, namely, we have

$$(e_i, E_j) = \delta_{ij}, \quad 1 \leq i, j \leq p, \tag{3.27}$$

where $e_1, \cdots, e_p$ are given by (3.2) and δ_{ij} is the Kronecker symbol. Let

$$\mathcal{D}_{N-p} = \{D \in \mathbb{M}^{N\times(N-p)} \mid \operatorname{rank}(C_pD) = \operatorname{rank}(D) = N-p\}. \tag{3.28}$$

Similarly to [12, Theorems 7.1–7.2], the following results can be easily established.

Theorem 3.4 *Assume that the matrix A satisfies the condition of C_p-compatibility* (3.6). *Assume furthermore that A^{T} admits an invariant subspace* $\operatorname{Span}\{E_1, \cdots, E_p\}$ *which is bi-orthonormal to* $\operatorname{Ker}(C_p) = \operatorname{Span}\{e_1, \cdots, e_p\}$. *Then there exists an internal control matrix $D \in \mathcal{D}_{N-p}$, such that the exactly synchronizable state by p-groups $u = (u_1, \cdots, u_p)^{\mathrm{T}}$ is uniquely determined by*

$$t \geq T: \ u = \phi, \tag{3.29}$$

where $\phi = (\phi_1, \cdots, \phi_p)^{\mathrm{T}}$ is the solution to the following problem independent of applied internal controls H : For $s = 1, \cdots, p$,

$$\begin{cases} \phi_s'' - \Delta\phi_s + \sum\limits_{r=1}^{p} \alpha_{sr}\phi_s = 0 & \text{in}(0,+\infty)\times\Omega, \\ \phi_s = 0 & \text{on}(0,+\infty)\times\Gamma, \\ t=0: \ \phi_s = (E_s, \widehat{U}_0), \quad \phi_s' = (E_s, \widehat{U}_1) & \text{in}\Omega, \end{cases} \tag{3.30}$$

where $\alpha_{sr} (s, r = 1, \cdots, p)$ are given by (3.25).

Theorem 3.5 *Assume that the condition of C_p-compatibility* (3.6) *holds. Then for any given control matrix $D \in \mathcal{D}_{N-p}$, there exists a positive constant c_T independent of initial data, but depending on T, such that each exactly synchronizable state by p-groups $u = (u_1, \cdots, u_p)^{\mathrm{T}}$ satisfies the following estimate*:

$$\|(u,u')(T) - (\phi,\phi')(T)\|^2_{(H^2(\Omega))^p\times(H^1(\Omega))^p} \leq c_T\|C_p(\widehat{U}_0, \widehat{U}_1)\|^2_{(H_0^1(\Omega))^{N-p}\times(L^2(\Omega))^{N-p}}, \tag{3.31}$$

where $\phi = (\phi_1, \cdots, \phi_p)^{\mathrm{T}}$ is the solution to problem (3.30) *in which* $\operatorname{Span}\{E_1, \cdots, E_p\}$ *is bi-orthonormal to* $\operatorname{Span}\{e_1, \cdots, e_p\}$.

3.3 Exact internal synchronization

For the special case $p = 1$, we have some further results on the exact internal synchronization.

Let $\varepsilon_1, \cdots, \varepsilon_q$ (resp. $\mathcal{E}_1, \cdots, \mathcal{E}_q$) be a Jordan chain of length q of A (resp. A^{T}), such that

$$\begin{cases} A\varepsilon_l = a\varepsilon_l + \varepsilon_{l+1}, & 1 \leq l \leq q, \\ A^{\mathrm{T}}\mathcal{E}_k = a\mathcal{E}_k + \mathcal{E}_{k-1}, & 1 \leq k \leq q, \\ (\mathcal{E}_k, \varepsilon_l) = \delta_{kl}, & 1 \leq k, l \leq q, \end{cases} \tag{3.32}$$

where

$$\varepsilon_q = (1, \cdots, 1)^{\mathrm{T}}, \quad \varepsilon_{q+1} = 0, \quad \mathcal{E}_0 = 0. \tag{3.33}$$

Clearly $\varepsilon_q = e_1$ is an eigenvector of A, respectively, $\mathcal{E}_1 = E_1$ is an eigenvector of A^{T} associated with the same eigenvalue a.

Consider the projection P on the subspace $\text{Span}\{\varepsilon_1, \cdots, \varepsilon_q\}$ as follows:

$$P = \sum_{k=1}^{q} \varepsilon_k \otimes \mathcal{E}_k, \tag{3.34}$$

where $\otimes$ stands for the tensor product such that

$$(\varepsilon_k \otimes \mathcal{E}_k)U = (\mathcal{E}_k, U)\varepsilon_k, \quad \forall U \in \mathbb{R}^N, \quad 1 \le k \le q.$$

The projection P can be represented by a matrix of order N. We can then decompose

$$\mathbb{R}^N = \text{Im}(P) \oplus \text{Ker}(P),$$

where $\oplus$ stands for the direct sum of subspaces. Moreover, we have

$$\text{Im}(P) = \text{Span}\{\varepsilon_1, \cdots, \varepsilon_q\}, \quad \text{Ker}(P) = (\text{Span}\{\mathcal{E}_1, \cdots, \mathcal{E}_q\})^{\perp} \tag{3.35}$$

and $PA = AP$.

Now let $U = U(t, x)$ be the solution to problem (1.1)–(1.2). We define

$$\begin{cases} U_c = (I - P)U, \\ U_s = PU. \end{cases} \tag{3.36}$$

If system (1.1) is exactly synchronizable, we have

$$t \ge T: \ U = u\varepsilon_q, \tag{3.37}$$

where $u = u(t, x)$ is the exactly synchronizable state. Then, noting (3.36)–(3.37), we have

$$t \ge T: \ \begin{cases} U_c = u(I - P)\varepsilon_q = 0, \\ U_s = uP\varepsilon_q = u\varepsilon_q. \end{cases} \tag{3.38}$$

Thus, U_c and U_s can be called the controllable part and the synchronizable part of U, respectively.

Recalling $PA = AP$ and applying the projection P on problem (1.1)–(1.2), we immediately get the following proposition.

Proposition 3.1 *The controllable part U_c is the solution to the following problem:*

$$\begin{cases} U_c'' - \Delta U_c + AU_c = (I - P)D\chi_\omega H & \text{in}(0, +\infty) \times \Omega, \\ U_c = 0 & \text{on}(0, +\infty) \times \Gamma, \\ t = 0: \ U_c = (I - P)\widehat{U}_0, \quad U_c' = (I - P)\widehat{U}_1 & \text{in}\Omega, \end{cases} \tag{3.39}$$

while, the synchronizable part U_s is the solution to the following problem:

$$\begin{cases} U_s'' - \Delta U_s + AU_s = PD\chi_\omega H & \text{in}(0, +\infty) \times \Omega, \\ U_s = 0 & \text{on}(0, +\infty) \times \Gamma, \\ t = 0: \ U_s = P\widehat{U}_0, \quad U_s' = P\widehat{U}_1 & \textit{in } \Omega. \end{cases} \tag{3.40}$$

Proposition 3.2 *Let a matrix D of order $N \times (N-1)$ be defined by*

$$\mathrm{Im}(D) = (\mathrm{Span}\{\mathcal{E}_q\})^{\perp}. \tag{3.41}$$

We have $D \in \mathcal{D}_{N-1}$.

Similarly to [12, Theorem 5.5], we have the following theorem.

Theorem 3.6 *When $q = 1$, we can take an internal control matrix $D \in \mathcal{D}_{N-1}$, such that the synchronizable part U_s is independent of internal controls H. Inversely, if the synchronizable part U_s is independent of internal controls H, then we necessarily have $q = 1$.*

We next discuss the general case $q \geq 1$. Let us denote

$$\phi_k = (\mathcal{E}_k, U), \quad 1 \leq k \leq q \tag{3.42}$$

and write

$$U_s = \sum_{k=1}^{q} (\mathcal{E}_k, U)\varepsilon_k = \sum_{k=1}^{q} \phi_k \varepsilon_k.$$

Then, $(\phi_1, \cdots, \phi_q)$ are the coordinates of U_s on the bi-orthonormal basis $\{\varepsilon_1, \cdots, \varepsilon_q\}$ and $\{\mathcal{E}_1, \cdots, \mathcal{E}_q\}$.

Noting (3.32), we easily get the following theorem.

Theorem 3.7 *Let $\varepsilon_1, \cdots, \varepsilon_q$ (resp. $\mathcal{E}_1, \cdots, \mathcal{E}_q$) be a Jordan chain of A (resp. A^{T}) corresponding to the eigenvalue a and $\varepsilon_q = (1, \cdots, 1)^{\mathrm{T}}$. The synchronizable part $U_s = (\phi_1, \cdots, \phi_q)$ is determined by the solution of the following problem: For $1 \leq k \leq q$,*

$$\begin{cases} \phi_k'' - \Delta\phi_k + a\phi_k + \phi_{k-1} = \chi_\omega h_k & \mathrm{in}(0, +\infty) \times \Omega, \\ \phi_k = 0 & \mathrm{on}(0, +\infty) \times \Gamma, \\ t = 0: \quad \phi_k = (\mathcal{E}_k, \widehat{U}_0), \quad \phi_k' = (\mathcal{E}_k, \widehat{U}_1) & \mathrm{in}\Omega, \end{cases} \tag{3.43}$$

where

$$\phi_0 = 0 \quad \text{and} \quad h_k = (\mathcal{E}_k, DH), \quad 1 \leq k \leq q. \tag{3.44}$$

Moreover, the exactly synchronizable state is given by $u = \phi_q$ for $t \geq T$.

Noting (3.37) and (3.42), we have

$$t \geq T: \ \phi_k = (\mathcal{E}_k, U) = (\mathcal{E}_k, u\varepsilon_q) = u\delta_{kq}, \quad 1 \leq k \leq q. \tag{3.45}$$

The relation (3.45) shows that only the last component ϕ_q is synchronized, while, the others are steered to zero.

In the special case $q = 1$, by Theorems 3.6–3.7, we have the following corollary.

Corollary 3.1 *When $q = 1$, we can take an internal control matrix $D \in \mathcal{D}_{N-1}$, such that $D^{\mathrm{T}}\mathcal{E}_1 = 0$. Then the exactly synchronizable state u is determined by $u = \phi$ for $t \geq T$, where ϕ*

is the solution to the following problem:

$$\begin{cases}\phi'' - \Delta\phi + a\phi = 0 & \text{in}(0,+\infty)\times\Omega,\\ \phi = 0 & \text{on}(0,+\infty)\times\Gamma,\\ t=0:\ \ \phi = (\mathcal{E}_1, \widehat{U}_0), \quad \phi' = (\mathcal{E}_1, \widehat{U}_1) & \text{in}\Omega.\end{cases} \tag{3.46}$$

Inversely, if the synchronizable part U_s *is independent of internal controls* H, *then we necessarily have*

$$q = 1 \quad \text{and} \quad D^{\mathrm{T}}\mathcal{E}_1 = 0.$$

Consequently, the exactly synchronizable state u *is given by* $u = \phi$ *for* $t \geq T$, *where* ϕ *is the solution to problem* (3.46). *In particular, if*

$$(\mathcal{E}_1, \widehat{U}_0) = (\mathcal{E}_1, \widehat{U}_1) = 0, \tag{3.47}$$

then system (1.1) *is exactly controllable for such initial data* $(\widehat{U}_0, \widehat{U}_1)$.

Declarations

Conflicts of interest Tatsien LI is an editor-in-chief for Chinese Annals of Mathematics Series B and was not involved in the editorial review or the decision to publish this article. All authors declare that there are no conflicts of interest.

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