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The Soft Point and Its Applications in Body Falling*

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Abstract From the mesoscopic point of view, a definition of soft point is introduced by considering the attributes of geometric profile and mass distribution. After that, this concept is used to develop the soft matching technique to simulate the chaotic behaviors of the equations. Especially, a tennis model with deformation factor $a(t)$ is proposed to derive a generalized Newton-Stokes equation $v'(t) = \lambda(v_T - a(t)v(t))$. Furthermore, a concept of duality of deformation factor $a(t)$ and velocity $v(t)$ with respect to the generalized Newton-Stokes equation is established. To solve this equation, two data-driven models of $a(t)$ are provided, one is based on the concept of soft matching, while the other is by using the amplitude modulation. Finally, the related iterative algorithm is developed to simulate the motion of the falling body via the duality of $a(t)$ and $v(t)$. Numerical examples successfully demonstrate the phenomenon of chaos, which consists of the continual random oscillations and sudden accelerations. Moreover, the algorithm is tested by using larger coefficients corresponding to the terminal velocity and shows more satisfactory results. It may enable us to characterize the total energy of the dynamical system more accurately.

Keywords Soft point, Soft matching, Newton-Stokes equation, Duality, Data-driven model
2000 MR Subject Classification 65Z05, 37M05, 82B03, 70L05, 37A50, 65P20

1 Introduction

On the occasion of the 20th anniversary of the establishment of the ICIAM Su Buchin Prize, we would like to discuss a very elementary concept in geometry, namely, the definition of the point. Another motivation comes from the study of the falling body, especially, a ball is falling in the non-Newtonian fluid. The result of this paper may be generalized to discuss the motion of the high-speed aerocraft.

The establishment of coordinates is a revolutionary milestone to describe a point mathematically (e.g. Euclidean space R^2). This has opened the new epoch to study the mechanical problems rationally and quantitatively. However, any point is actually a mass distribution. We take the point as the mass distribution of a random variable ξ, and propose the concept of soft point.

Based on the definition of soft point, the K-order equivalence is introduced: $\xi \simeq_K \eta$ if $E\xi^k = E\eta^k$ hold for all $k \leq K$ at some mesoscopic level. Then a concept of the soft matching

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$\xi@\eta$ is proposed, more precisely, we can construct ξ to simulate η based on some important features of η, which could be observed and estimated previously.

Furthermore, the concept of soft matching for the difference equation $(u_{j+1} - u_j) = F(u_j)$ will turn to be $(u_{j+1} - u_j)@F(u_j)$. That is, the soft points $\{(u_{j+1} - u_j)\}$ will satisfy the distribution coincident with the important features of soft points $\{F(u_j)\}$. This technique could also be generalized to the dynamical system.

In this paper, the concept of soft point will be introduced more in details, and the idea of the soft matching will be used to describe the motion pattern of a ball falling in the non-Newtonian fluid.

Let us start at the very beginning, the free body falling by Galileo, that is, "the distance is proportional to the square of time". This helps Newton to establish the second mechanical law of $f = mv'$: The force f acting on the object is equal to its mass m multiplying its acceleration v'. Here Newton has taken the object as a mass-point abstractly, that is the point endowed with the attribute of the mass.

These open the new epoch to study physics with mathematics, since the coordinates are the bridge connecting physical locations and mathematical points.

In 1851, Stokes took the drag force as $F_d = 6\pi\mu rv$, which is proportional to the velocity of the ball (see [1]). Here μ is the viscosity of the fluid, r and v are the radius and velocity of the ball, respectively. This derived the Newton-Stokes equation

$$\frac{4\pi}{3} r^3 \rho_b v'(t) = \frac{4\pi}{3} r^3 g(\rho_b - \rho_f) - 6\pi\mu rv(t), \tag{1.1}$$

where ρ_b and ρ_f are the densities of the ball and the fluid, respectively, and g is the gravitational acceleration.

Furthermore, (1.1) can be simplified as

$$v'(t) = \lambda(v_T - v(t)), \tag{1.2}$$

where $\lambda = \frac{9\mu}{2r^2\rho_b}$ and the terminal (final) velocity $v_T = \frac{2(\rho_b-\rho_f)r^2 g}{9\mu}$. The solution of (1.2) can be written as $v(t) = v_T - (v_T - v_0)\mathrm{e}^{-\lambda(t-t_0)}$. When the resultant force or the acceleration $v'(t)$ tends to zero, $v(t)$ will tend to v_T, which is called the equilibrium state macroscopically.

However, in applications, it has been observed that the falling body behaves with random oscillations when it's speed approaches to the terminal velocity (see [2–6]). Such phenomena will happen in physics, economics, biology and many other fields, such as the perturbation of earth orbit, the falling ball in shampoo and the fluctuation of stocks index, etc. The random phenomenon of a falling ball through a non-Newtonian fluid can be observed more clearly as shown in Figure 1 (e.g. [7]).

Lee [8] takes the fluid as spider nets layer by layer, and adds a periodic force caused by the action of the ball which first pushes the spider nets together and breaks through them after. When the ball enters the equilibrium state, the motion pattern is approximatively a harmonic oscillation coupled with the terminal velocity.

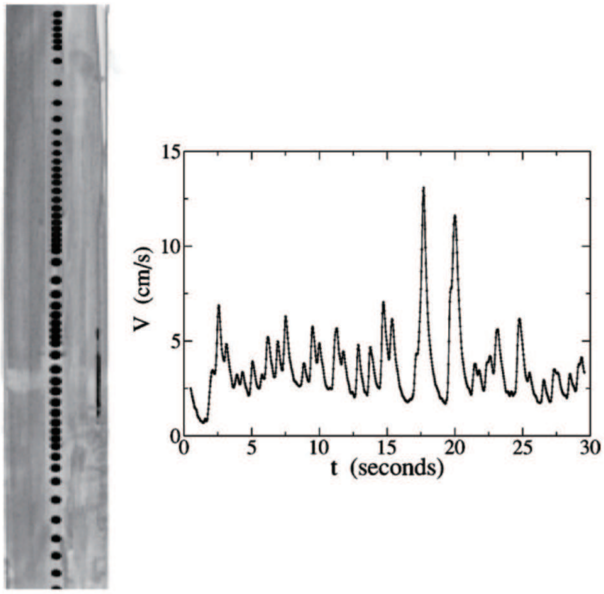


Figure 1 Oscillations of a solid ball falling through a wormlike micellar fluid.

The models above are all under the framework of Newtonian mechanics and can be described by some second order ODEs (ordinary differential equations). Discretize these ODEs, the related iterative algorithms can be obtained to simulate the motion of the falling ball in the category of Newton.

However, people have already observed a lot of non-Newtonian fluids such as shampoo, toothpaste, cream, slurry and so on. In fact, all fluids will exhibit it's non-Newtonian properties, if the falling ball approaches to the terminal velocity and is deformed easily, and the fluid possesses a strong viscosity.

The researches of [9–10] developed a data-driven model of the falling ball by using the LASSO (Least Absolute Shrinkage and Selection Operator) approach. Moreover, we can develop the soft LASSO approach based on the concept of soft matching of soft points.

This paper generalizes the concept of soft point by considering not only the geometric profile but also the mass distribution and consists of four sections. In Section 2, a generalized concept of soft point of order $(N+M)$ is introduced. Based on this concept, the soft matching technique is developed in order to simulate the chaotic behaviors of the equations and some simple cases are discussed. In Section 3, taking the deformation of the ball into account, a tennis model is proposed to derive a generalized Newton-Stokes equation to describe the motion of the falling ball. Meanwhile, a dual concept of deformation factor $a(t)$ and velocity $v(t)$ with respect to the generalized Newton-Stokes equation is suggested. Two data-driven models of $a(t)$ are derived at some mesoscopic level, one is based on the concept of soft matching, while the other is by using the amplitude modulation. Furthermore, the related discrete iterative algorithm is developed to simulate the trajectory of the velocity. The algorithm is tested by using different values of coefficient d corresponding to the terminal velocity v_T. The results are more satisfactory when we use dv_T instead of v_T in the algorithm with a lager d. It may enable us to characterize the total energy of the dynamical system more accurately. Numerical examples are provided and show the chaotic phenomena with the behavior of the continual random oscillations coupled with sudden accelerations. Some conclusions are given in the last Section.

2 A New Concept: Soft Point

In this Section, we will generalize the concepts of point in mathematics and mass-point in physics, which will be called "soft point" from a more mesoscopic perspective.

A point P in mathematics is abstractly represented only by using its position with some coordinates, but having no size, no shape and no mass. A mass-point (P, m) in physics contains not only the position in mathematics but also the mass. However, in the real world, a point would coincide with two attributes, the geometric attribute (size and shape) and the physical attribute (mass and its distribution). Roughly speaking, a point P is a distribution of mass in a certain geometry. Therefore, there could be at least two kinds of way to observe a point mesoscopically. The key feature of difference of the concepts in geometry and in physics is that, the mathematician takes the uniform distribution $\chi_\Omega(x)$ on some geometry region Ω, while the physicist takes the mass distribution $m(x)$ to describe the concept of the point.

The concept of the generating function is used to generalize the definition of the point mesoscopically. We define the "soft point" of order $(N+M)$, where N and M are the orders of the generating function with respect to geometry and physics, respectively.

Definition 2.1 *Assume that a point whose mass distribution is $m(x)$ defined on some region $\Omega \in R^d$ and $u(x) = \chi_\Omega(x)$ is the uniform distribution on Ω. We define*

$$N_j := \int_\Omega x^j u(x)\mathrm{d}x \quad \text{and} \quad M_k := \int_\Omega x^k m(x)\mathrm{d}x, \tag{2.1}$$

where $x^j = x_1^{j_1} \cdots x_d^{j_d}$ is the notation of multiple variable, with $x = (x_1, \cdots, x_d) \in R^d$ and $j = (j_1, \cdots, j_d)$, $k = (k_1, \cdots, k_d) \in Z^d$. Furthermore, the values of $\{N_j, M_k\}$ characterize a soft point in a mesoscopic level of order $(N+M)$, where $|j| := j_1 + \cdots + j_d \leq N$ and $|k| := k_1 + \cdots + k_d \leq M$.

For example, when $d = 1$, $N = 1$ and $M = 1$, N_0 is the area of the region Ω, M_0 is the total amount of mass, N_1/N_0 is the geometric center P_g, and M_1/M_0 is the mass barycentre P_m, respectively. P_g and P_m are most used in the application of mathematics and physics to define the position of a point.

Remark 2.1 The concept of soft point can be generalized to equations (e.g. algebraic e-quation, differential equation, difference equation and so on), if both sides of the equation are regarded as points in the Hilbert space. In this way, we can derive an algorithm to characterize the left-hand side by using some important features of the right-hand side. For example, for the difference equation $(u_{j+1} - u_j) = F(u_j)$, based on Definition 2.1, we will solve this equation by collecting and using the key features of $F(u_j)$ such as a soft point of order $(N+M)$ and setting them in the iteration. This means that $(u_{j+1} - u_j)$ satisfies the distribution coincident with some key features (information about generating function) of the soft point $F(u_j)$ for any j. We call this technique as 'the soft matching of the equation', which is denoted as $(u_{j+1} - u_j)@F(u_j)$.

We will give some simple cases about the soft point as follows. The mathematical point is in fact a soft point of order $(1+0)$, where its position is defined to be the geometric center $P_g = (N_j/N_0)$, $|j| \leq 1$. The physical point is in fact a soft point of order $(0+1)$, where its position is defined to be the mass barycenter $P_m = (M_j/M_0)$, $|j| \leq 1$.

To represent the mass-point located at the position P with the mass m, we would like to introduce the Grassmann coordinate $\binom{mP}{m}$ (see [11–13]). Then the superposition of mass-points is

$$\sum_j \binom{m_j P_j}{m_j} = \binom{\sum\limits_j m_j P_j}{\sum\limits_j m_j} = \binom{m P_m}{m},$$

where $m = \sum\limits_j m_j$ is the total amount of the masses and

$$P_m = \frac{\sum\limits_j m_j P_j}{\sum\limits_j m_j}$$

is exactly the position of the barycentre. The physical meaning of the superposition is that all the mass-points $\binom{m_j P_j}{m_j}$ accumulate on the mass barycentre P_m. Therefore, Definition 2.1 of the soft point is a generalization of Grassmann mass-point $\binom{mP_m}{m}$.

Now there are two concepts of position of a point: The geometric center P_g and the mass barycentre P_m. A serious problem is raised, which position should be taken? Nevertheless, these two positions are usually not the same.

If we have decided to use P_g or P_m, based on the second mechanical law, the motion equation should be $v'(t) = g$ for the free falling body. The solution can be written as $v(t) = v(0) + gt$ and $P(t) = P(0) + v(0)t + \frac{1}{2}gt^2$. Since the body is not in vacuum, Stokes takes the resistance into account which is proportional to the velocity of the falling body. Then an ODE can be gotten in the form of $v'(t) = \lambda(v_T - v(t))$. The velocity will tend to a constant v_T which is called the terminal velocity. Figure 2 shows the motion pattern of the body in the framework of soft point of order $(0+1)$ or $(1+0)$.

The trajectories of $P_g(t)$ and $P_m(t)$ are usually different, however, $v_g(t) \equiv v_m(t)$ and $P_g(t) - P_m(t) \equiv P_g(0) - P_m(0)$ always hold.

As just mentioned, mathematicians ignore the mass and take it to be uniformly distributed, while physicists focus on the mass barycenter and usually ignore the geometry. It would be better to combine two concepts of mass and geometry. From a mesoscopic point of view, it requires to consider at least two essential factors: The geometric center and the mass barycentre (P_g, P_m), which is involved in the concept of soft point of order $(1+1)$.

If the mass distribution of a point P is $m(x)$ on a geometric region Ω and $u(x)$ is the characteristic function $\chi_\Omega(x)$ of the region Ω, then the Grassmann coordinate of soft point of order $(1+1)$ could be represented as

$$\begin{pmatrix} M_1 \\ M_0 \end{pmatrix} = \begin{pmatrix} mP_m \\ m \end{pmatrix} = \begin{pmatrix} \int_\Omega xm(x)\mathrm{d}x \\ \int_\Omega m(x)\mathrm{d}x \end{pmatrix}, \quad \begin{pmatrix} N_1 \\ N_0 \end{pmatrix} = \begin{pmatrix} |\Omega|P_g \\ |\Omega| \end{pmatrix} = \begin{pmatrix} \int_\Omega xu(x)\mathrm{d}x \\ \int_\Omega u(x)\mathrm{d}x \end{pmatrix}.$$

An example of soft point of order $(1+1)$ is the egg model, whose geometry profile is difficult to be deformed, however, the mass barycenter $P_m = M_1/M_0$ satisfies the dynamical equation such as the Newton-Stokes equation. What we have observed is the position $P_g = N_1/N_0$. The relation of two concepts P_g and P_m of the point can be described as pulling a jelly with non-Newtonian characters: P_m will take the regression to P_g because of the material memory. Then an oscillation will happen. $|P_m - P_g| \leq 1$, if the radius r is normalized to be 1, since the egg yolk oscillates only in the egg.

The difference $P_m - P_g$ can be regarded as the deformation of the mass, which will cause a reacting force and can be written in the following equation about the internal force f_i:

$$\frac{f_i}{m} = P_g'' - P_m'' = -\omega^2(P_g - P_m). \tag{2.2}$$

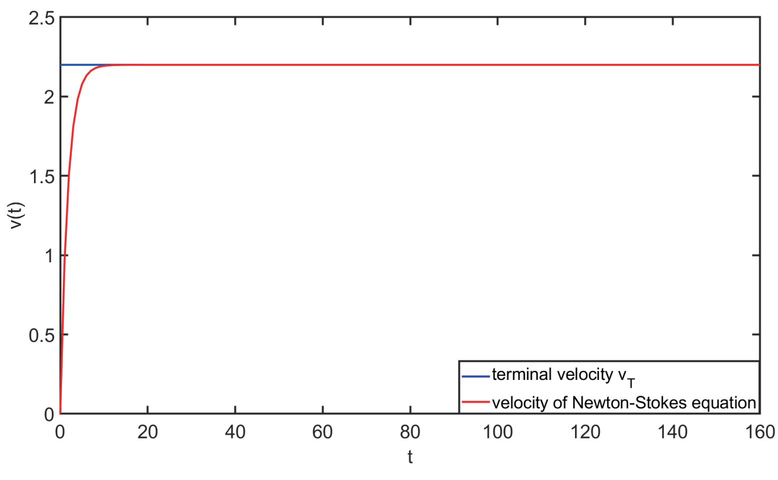


Figure 2 The Newton-Stokes trajectory of the velocity.

Then its solutions will be $v_g(t) = v_m(t) + A\omega\sin\omega t + B\omega\cos\omega t$ and $P_g(t) = P_m(t) - A\cos\omega t + B\sin\omega t$ $(A^2 + B^2 \leq 1)$, where $A = P_m(0) - P_g(0)$ and $B\omega = P_g'(0) - P_m'(0)$ depend on the initial relative position and velocity, respectively. From another side, $P_m(t)$ satisfies the Newton-Stokes equation with respect to the external force f_e:

$$\frac{f_e}{m} = v_m'(t) = \lambda(v_T - v_m(t)), \tag{2.3}$$

where f_e is composed of the gravity and the resistance. By solving the coupled equations (2.2)–(2.3), we obtain that $v_g(t) = v_T - (v_T - v_0)\mathrm{e}^{-\lambda t} + A\omega\sin\omega t + B\omega\cos\omega t$. Figure 3 shows the pattern of $v_g(t)$ with the green curve by the egg model and the pattern of $v_m(t)$ with the red curve. $v_m(t)$ is just the solution of Newton-Stokes equation which is shown in Figure 2. That means, $v_g(t)$ is a more mesoscopic description compared with $v_m(t)$.

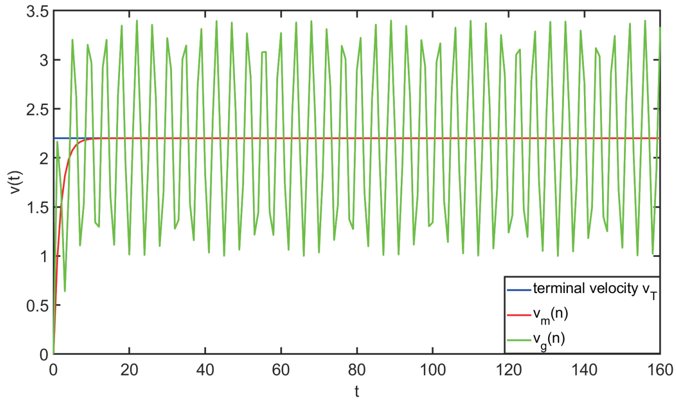


Figure 3 A special example of egg model.

Another example of soft point of order $(1 + 1)$ is the tennis model, whose geometry profile will be deformed to spheroid, while the mass keeps uniformly distributed $m(x) = u(x)$. This will be discussed more in details in the next section.

3 Tennis Model–A Further Discussion

The equilibrium state is an important concept in the study of mechanical systems, as it represents a relative stable situation of a moving body. The dynamical equilibrium state is the common state in many fields, such as chemistry, physics, economics and physiology. Chaos phenomenon is ubiquitous in dynamical equilibrium. The ball falling in the non-Newtonian fluid in equilibrium state shows chaotic behavior without exception, that is, the motion of the ball behaves randomly but satisfies some physical laws. An example of chaos is the logistic equation $x_{t+1} = kx_t(1 - x_t)$.

During the falling, the motion patten of the ball in equilibrium state demonstrates the amplitude modulated oscillations, sometimes with sudden accelerations (e.g. Figure 1). These sudden accelerations indicate sudden decreases of the resistance. The research in [7] has proposed that the non-transient oscillation is caused by the flow-induced structure (FIS for short) formed in the area around the ball, and the sudden acceleration of the ball is due to the rupture of the FIS. Similar oscillation phenomenon (see [14]) has occurred with a bubble rising in the fluid, that is, its velocity also oscillates non-transiently. This oscillation phenomenon may be due to the deformation of the bubble to form the cusps as it rises through the fluid (see [15]). At the moment the cusp forms, the bubble suddenly "jumps" to release the cusp, which then creates a strong negative wake behind the bubble.

Now we use the data sampled from the experiment in [7] as the entry point to study the ball falling in non-Newtonian fluid. For these data, we take $t > t_{\text{equilibrium}}$ as the equilibrium state of the motion and the sudden accelerations occur in the time interval $[t_{\text{sudden}-}, t_{\text{sudden}+}]$. The velocity of the sudden acceleration in $[t_{\text{sudden}-}, t_{\text{sudden}+}]$ is distributed in $[v_{\min}, v_{\max}]$, otherwise, is distributed in the interval of $[v_{\min}, v_{\text{mid}}]$, and $v_{\min} < v_{\text{mid}} < v_{\max}$. People often use statistics such as the mean value, the median value, the variance and so on to describe the statistical characters of physical properties or to find new physical relations and equations.

The purpose of this section is to establish a mathematical model based on the Newton-Stokes equation, discuss the physical relation of the ball falling in a non-Newtonian fluid in details, and construct proper iterative algorithm to simulate the mechanical behavior of the velocity. We hope that the simulated velocity trajectory can exhibit chaotic phenomena including continuous random oscillations and sudden accelerations, and reproduce some of the statistical characters of the experimental data.

The most researchers considered FIS model (e.g. [7]), that is, the non-uniformity of the fluid in the area around the ball. From another perspective, we take the fluid as a reference object and relatively regard the variation of the fluid around the ball as the deformation of the ball itself. That is, we take the falling ball as a tennis of soft point of order $(1+1)$ mesoscopically.

The Tennis Model: The geometric profile will be deformed easily when a tennis with radius r after a huge hit, then it will change into a spheroid with equatorial radius $r_{\text{equat}}(t)$ and polar radius $r_{\text{polar}}(t)$. However, the mass will always keep uniformly distributed in the geometry of the spheroid. Therefore, $P_g = P_m$ and $v_g = v_m$, which is a special case of soft point of order $(1+1)$. The conservation of the volume of the spheroid keeps

$$r_{\text{equat}}^2(t) r_{\text{polar}}(t) \equiv r^3.$$

Now we use the tennis model to analyse the falling ball. The deformation of the ball to the spheroid is caused by the difference of the gravity and the resistance. An oscillation of geometry will happen, that the spheroid would always intent to regress to the original ball due to the material memory.

Recalling the Newton-Stokes equation, the force acting on the ball will be macroscopically written into two terms. One is caused by the gravity which relates to the volume V of the ball, while the other is caused by the resistance which relates to the radius r of the ball (see [1]). More in details,

$$mv'(t) = (\rho_b - \rho_f)gV - 6\pi\mu r v(t). \tag{3.1}$$

The deformation will lead to the oscillation of the equatorial radius $r_{\text{equat}}(t)$ and the polar radius $r_{\text{polar}}(t)$. The oscillation of the velocity will respond to the oscillation of the geometric shape of the ball. Because the volume V of the ball is conserved during the deformation, (3.1) can be reformed as

$$mv'(t) = (\rho_b - \rho_f)gV - 6\pi\mu r_{\text{equat}}(t) v(t). \tag{3.2}$$

Denoting the ratio

$$a(t) = \frac{r_{\text{equat}}(t)}{r},$$

(3.2) can be simplified to

$$v'(t) = \lambda(v_T - a(t)v(t)), \tag{3.3}$$

where λ and v_T are the same as in (1.2). The terminal velocity v_T is a macroscopic description of the ball in the equilibrium state. Based on Ergodic theory, v_T could be taken as the mean value (e.g. geometric mean, arithmetic mean, harmonic mean and so on). Pay attention to the balance of the number of the data, we can also choose the median value as v_T.

The ratio $a(t)$ describes the deformation of the ball and is called the deformation factor. From macroscopic point of view, Newton-Stokes equation (3.1) ignores the deformation of the ball and always takes $a(t) \equiv 1$.

If we know $a(t)$, then coupled with (3.3), we can simulate the falling motion by an implicit form of discrete iterative algorithm

$$v_n - v_{n-1} = \lambda(v_T - a_n v_n) \tag{3.4}$$

and an explicit form of iterative algorithm

$$v_n = \frac{v_{n-1} + \lambda v_T}{1 + \lambda a_n}. \tag{3.5}$$

The next step at n of velocity will be a convex combination of v_{n-1} and the terminal velocity v_T with the weight of $1 : \lambda$ when $a_n \equiv 1$ in Newton-Stokes equation, now it will be collaborated with a deformation factor of a_n.

In the following, based on the idea of data-driven, we construct the model of $a(t)$ by the dual concept of $v(t)$ with respect to the Newton-Stokes equation. More in details, two methods are proposed, one is the soft matching model, while the other is by using the deterministic frequency with the amplitude modulation.

Dual Relation Now we will discuss the relation of the velocity and the deformation factor of the falling ball. Denoting

$$\widehat{v}(t) = \frac{v(t)}{v_T},$$

Newton-Stokes equation (3.3) can be reformed as

$$\widehat{v}'(t) = \lambda(1 - a(t)\widehat{v}(t)). \tag{3.6}$$

When the ball enters the equilibrium state, that is, the resultant force acting on the ball is equal to zero and the acceleration is zero macroscopically. This derives a macroscopic dual relation of

$$a(t)\widehat{v}(t) = 1. \tag{3.7}$$

Taking the logarithm on both sides of (3.7) gives the following equation

$$\ln a(t) + \ln \widehat{v}(t) = 0. \tag{3.8}$$

It can be concluded that under the macroscopic framework of Newton-Stokes equation, when the ball enters the equilibrium state, $\ln a(t)$ and $\ln \widehat{v}(t)$ will keep a mirrored relation.

Definition 3.1 (Mirrored random variables) *Two random variables ξ and η are called mirrored with each other, if their density functions $f(t)$ and $g(t)$ satisfy*

$$f(t) = g(-t).$$

Remark 3.1 If the density function $f(t)$ of a random variable ξ is an even function, then ξ is called self-mirrored. For example, the uniform distribution $\frac{1}{2}\chi_{[-1,1]}(t)$ on $[-1,1]$ and the Gaussian distribution $\frac{1}{\sqrt{2\pi}\sigma}\mathrm{e}^{-\frac{t^2}{2\sigma^2}}$ are self-mirrored.

Corollary 3.1 *If ξ and η are mirrored random variables, then their distribution functions $F(x)$ and $G(x)$ satisfy*

$$F(x) + G(-x) = 1. \tag{3.9}$$

Proof

$$\begin{aligned} F(x) &= \int_{-\infty}^{x} f(t)\mathrm{d}t = \int_{-x}^{+\infty} f(-t)\mathrm{d}t = \int_{-x}^{+\infty} g(t)\mathrm{d}t \\ &= 1 - \int_{-\infty}^{-x} g(t)\mathrm{d}t = 1 - G(-x). \end{aligned} \tag{3.10}$$

Remark 3.2 If ξ and η are mirrored random variables, then $E(\xi+\eta) = 0$, in other words, $\xi+\eta$ is a soft point of zero macroscopically or ξ and η are a pair of symmetric points with respect to zero mesoscopically.

Definition 3.2 (Duality of random variables) e^{ξ} *and* e^{η} *are called dual random variables with each other, if ξ and η possess a mirrored relation. At this time, $E\ln(\mathrm{e}^{\xi}\cdot\mathrm{e}^{\eta}) = 0$ from mesoscopic point of view.*

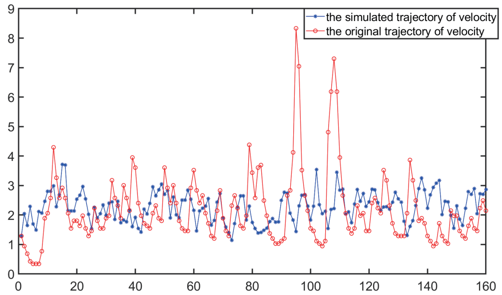
Soft matching model For the functions $v(t)$ and $a(t)$, mathematicians take them as the points in Hilbert space. After discretization, $(v_0,\cdots,v_n)$ and $(a_0,\cdots,a_n)$ are the points in R^{n+1}. Then recall the concept of soft point in Definitions 2.1 and 3.1–3.2, the problem to solve (3.5) will turn to be a soft matching problem (see Remark 2.1 in Section 2 of this paper). (3.7) and (3.8) will turn to be

$$a(t)@\frac{v_T}{v(t)} \tag{3.11}$$

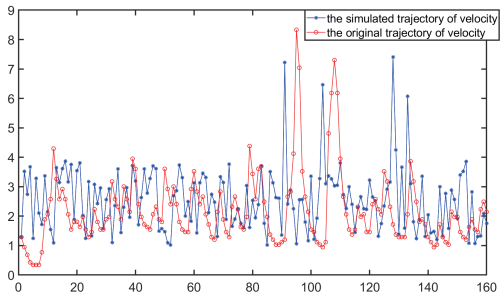
and

$$\ln a @ -\ln\widehat{v}. \tag{3.12}$$

Then from (3.11), we can construct a random number generator of a_n by using the empirical distribution of the velocity. Setting the generated a_n into the iterative algorithm (3.5), we get a simulated trajectory of the velocity of the falling ball. Figure 4(a) uses the empirical distribution of velocity and the terminal velocity is learned by LASSO. In order to show the sudden acceleration more clearly, a coefficient d with respect to v_T is designed and dv_T is used



(a) $v_T \approx 4$.



(b) $dv_T \approx 60, d = 15$.

Figure 4 Trajectory of v by generating a_n from empirical distribution of velocity.

instead of v_T in algorithm (3.5). We have chosen a lot of different d and found that when $d > 10$, the simulated trajectory is more consistent with the experimental data as shown in Figure 4(b).

Remark 3.3 The above example shows that if we choose a large value of d and use dv_T instead of v_T in generalized Newton-Stokes equation (3.3), then the dual relation between the deformation factor and the velocity should be

$$a(t)\frac{v(t)}{dv_T} = 1.$$

In this example, most of v_n are less than dv_T, that is,

$$\frac{v_n}{dv_T} < 1$$

and

$$a_n = \frac{dv_T}{v_n} > 1.$$

This means that the ball is mostly in a flattened state which is consistent with the reality. This phenomenon shows that the resistance is mostly less than the gravity during the falling. The resultant of gravity and resistance determines the falling of the ball, while the value of resistance determines the flattening of the ball. The phenomenon that taking a larger d will get a better trajectory also happens in the next model, and will be discussed in more details.

Amplitude-modulated model In logarithmic space, many physical processes will become more stationary. Meanwhile, in mathematics the dual problem can be easily solved by taking the logarithm and transforming it into a mirror problem. So we study the characteristic properties of $\ln v$ in logarithmic space. The trajectory of $w = \ln v$ which comes from the physical experiment in [7] is shown in Figure 5.

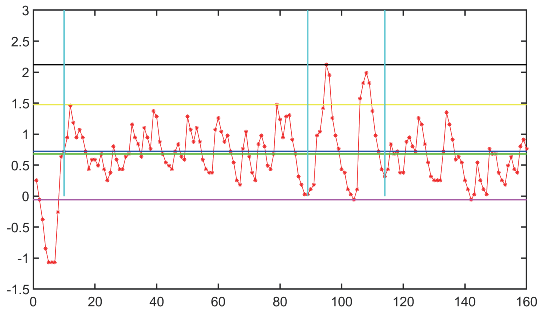


Figure 5 Trajectory of $w = \ln v$.

In Figure 5, the ball enters the equilibrium state when $n > t_{\text{equilibrium}}$. The five horizontal lines from bottom to top represent the lower bound $w_{\min}$, the expectation value w_a, the median value w_m, the middle bound w_{mid} and the upper bound $w_{\max}$ of the velocity in logarithmic space at the equilibrium state, respectively. It behaves as continuous random oscillators distributed in the interval $[w_{\min}, w_{\text{mid}}]$, with occasional sudden accelerations up to $w_{\max}$ when $n \in [t_{\text{sudden}-}, t_{\text{sudden}+}]$.

Since the behavior of $w(t) = \ln v(t)$ has mainly shown as a harmonic oscillation in the interval $[w_{\min}, w_{\text{mid}}]$, we first use the model

$$w_1^*(t) = C + A\sin(\omega_1 t) \tag{3.13}$$

to simulate it, where

$$C = \frac{(w_{\min} + w_{\text{mid}})}{2}$$

and

$$A = \frac{(w_{\text{mid}} - w_{\min})}{2}$$

are the median (mean) value and the amplitude of the oscillation, respectively. The frequency ω_1 can be estimated by many statistical methods, here we use ω_1 learned by LASSO (e.g. [9]). The model (3.13) maintains the lower bound $w_{\min}$, the middle bound w_{mid} and the median value of the data of $\ln v$. To improve the model to show the random behavior, a random factor ε_1 is added to serve as the amplitude modulation:

$$w_2^*(t) = C + A\varepsilon_1 \sin(\omega_1 t), \tag{3.14}$$

where the distribution of ε_1 is supported on the interval $[0, 1]$.

The location where the sudden acceleration occurs may be indeterministic. In our example, the sudden acceleration happens in $[t_{\text{sudden}-}, t_{\text{sudden}+}]$, and we can construct many different models to simulate this phenomenon. For example, we design the term

$$B\left(\frac{b - \sin(\omega_2 t)}{1 + b}\right)^{2k+1}$$

to cause an impulse to characterize the sudden acceleration. The values of frequency ω_2, the amplitude B and the parameters (b, k) characterize the position, the height and the width of the impulse, respectively. To demonstrate random phenomenon, we use the same method of the amplitude modulation, and the model

$$w_3^*(t) = C + A\varepsilon_1 \sin(\omega_1 t) + B\varepsilon_2 \Big(\frac{b - \sin(\omega_2 t)}{1 + b}\Big)^{2k+1} \tag{3.15}$$

is constructed to simulate the function $w(t)$, and $\mathrm{e}^{w_3^*(t)}$ to simulate the function $v(t)$ in the sense of the soft matching. The distribution of ε_2 is supported on the interval $[0, 1]$.

The sudden acceleration simulated by model (3.15) occurs periodically. However, to describe the sudden acceleration in more details is a more interesting and difficult problem. More precisely, why, when, where and how often the sudden acceleration will happen has attracted a lot of scientists. For example, if we take the sudden acceleration as a single extreme phenomenon, the Gaussian model with appropriate parameters can be used.

Based on the dual relation

$$a(t) \cdot \frac{\mathrm{e}^{w(t)}}{v_T} = 1, \tag{3.16}$$

we have

$$a(t) = v_T \mathrm{e}^{-w(t)}.$$

Recalling the concept of the soft matching in Remark 2.1, a generator of the deformation factor is constructed as:

$$a_n @ v_T \mathrm{e}^{-w(n)}. \tag{3.17}$$

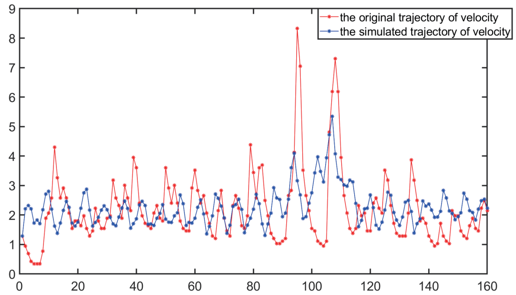
Then set a_n obtained by (3.17) into the iterative algorithm (3.5) to simulate the trajectory of the velocity of the falling ball. The result is shown in Figure 6(a) which exhibits the random oscillations, but lacks the sudden acceleration. We design a coefficient d with respect of v_T and the iterative algorithm (3.5) is suggested as

$$v_n = \frac{v_{n-1} + \lambda d v_T}{1 + \lambda a_n}. \tag{3.18}$$

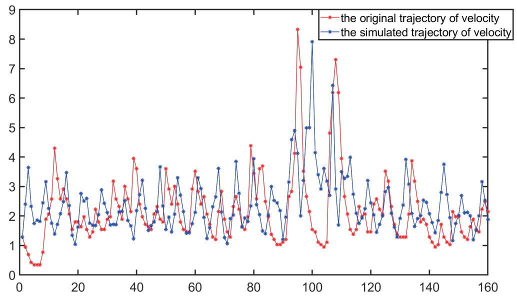
We have tried many different value of d and found that the simulated trajectory matches the data better when the larger d has been taken. The numerical experiments show satisfactory results when $d > 10$. The algorithm (3.18) with the new coefficient d enriches the representing ability of the algorithm (3.5), in particular, encouragingly consists of the sudden acceleration phenomenon. Figure 6(b) shows the result of $d = 15$.

We would like to point out that this phenomenon has already happened in the soft matching model of the paragraph above.

In fact, the total energy of the system consists of the energy of the inertial motion and oscillations, which will be larger than the energy of inertial motion described by the classical



(c) $v_T \approx 4$.



(d) $dv_T \approx 60, d = 15$.

Figure 6 The simulated trajectories of velocity.

Newton-Stokes equation. The reason for taking a larger d in (3.18) may be more in line with the law of conservation of energy. The value of d had better to be chosen larger than 10, which inspires us that the total energy $\frac{1}{2}md^2v_T^2$ of the system of generalized Newton-Stokes equation is larger than $\frac{1}{2}mv_T^2$ of the classical Newton-Stokes equation. This implies that the generalized Newton-Stokes equation brings back the factors which may not have been observed, detected and considered by classical Newton-Stokes equation. The violation of classical Newton-Stokes equation may be compared with the discovery of the dark matter. $d > 10$ means that the total energy should be 100 times greater than the energy of classical Newton-Stokes equation. According to the Einstein's mass-energy equivalence $E = mc^2$, this implies that the mass of dark matter accounts for at least 99% of the total mass of all matter in the universe.

4 Conclusion

From the mesoscopic point of view, the definition of soft point of order $(N+M)$ is introduced by considering the attributes of both geometric profile and mass distribution. Based on this concept, the soft matching technique is developed to simulate the chaotic behaviors of the equations. Especially, the tennis model of soft point of order $(1+1)$ is proposed and the generalized Newton-Stokes equation

$$v'(t) = \lambda(v_T - a(t)v(t))$$

with deformation factor $a(t)$ is derived. More in details, a dual concept of deformation factor $a(t)$ and velocity $v(t)$ with respect to the above equation is established. Two data-driven models of the deformation factor are provided, one is based on the concept of soft matching, while the other is by using amplitude modulation. Discrete iterative algorithm of coupled models of

$$a_n \cdot \frac{v_n}{v_T} = 1$$

and

$$v_n = \frac{v_{n-1} + \lambda v_T}{1 + \lambda a_n}$$

under the concept of soft point is developed to simulate the trajectory of the velocity. The algorithm is tested by using different values of coefficient d corresponding to the terminal velocity v_T. The results are more satisfactory when we use dv_T instead of v_T in the algorithm with a lager d. It may enable us to characterize the total energy of the dynamical system more accurately. Numerical examples show the behavior of chaos of the continual random oscillation and sudden acceleration satisfactorily. The concept of soft point and soft matching as well as the method developed in this paper are expected to be generalized into a wide range of applications, such as the study of trajectory of the high-speed aerocraft, the stock index and so on.

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Declarations

Conflicsts of interest The authors declare no conflicts of interest.

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