Chin. Ann. Math. Ser. B
44(5), 2023, 753–764
DOI: 10.1007/s11401-023-0042-9

Chinese Annals of
Mathematics, Series B

On Geometric Realization of the General Manakov System*

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Abstract It is well-known that the general Manakov system is a 2-components nonlinear Schrödinger equation with 4 nonzero real parameters. The analytic property of the general Manakov system has been well-understood though it looks complicated. This paper devotes to exploring geometric properties of this system via the prescribed curvature representation in the category of Yang-Mills' theory. Three models of moving curves evolving in the symmetric Lie algebras $u(2,1) = \mathbf{k}_\alpha \oplus \mathbf{m}_\alpha$ ($\alpha = 1, 2$) and $u(3) = \mathbf{k}_3 \oplus \mathbf{m}_3$ are shown to be simultaneously the geometric realization of the general Manakov system. This reflects a new phenomenon in geometric realization of a partial differential equation/system.

Keywords Manakov system, Geometric realization, Prescribed curvature representation
2000 MR Subject Classification 53C30, 53E30, 35Q55, 37K25, 35Q60

1 Introduction

The general Manakov system reads (see [13])

$$\begin{cases} \mathrm{i}\varphi_{1t} + \varphi_{1xx} + (b_1|\varphi_1|^2 + b_2|\varphi_2|^2)\varphi_1 = 0, \\ \mathrm{i}\varphi_{2t} + \varphi_{2xx} + (c_1|\varphi_1|^2 + c_2|\varphi_2|^2)\varphi_2 = 0, \end{cases} \tag{1.1}$$

where $\varphi_1 = \varphi_1(x,t), \varphi_2 = \varphi_2(x,t)$ are two unknown complex functions, the subscript with respect to the variable t or x stands for the derivative indicated and b_1, b_2, c_1, c_2 are nonzero real parameters. The system (1.1) is also called a 2-components nonlinear Schrödinger equation in literature and has important applications in nonlinear optics, superfluid, plasma, Bose-Einstein condensed matter physics etc (refer to [1–4, 11, 15, 17, 21–22]). Although it involves 4 free real parameters and looks complicated, the analytic properties of the system (1.1) have been explored deeply and be summarized in [10]. For example, if φ is a solution to the nonlinear Schrödinger equation: $\mathrm{i}\varphi_t + \varphi_{xx} + a_2|\varphi|^2\varphi = 0$, where a_2 is a real parameter, then $(\varphi_1, \varphi_2) =$

Manuscript received April 14, 2023.
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*This work was supported by the National Natural Science Foundation of China (Nos. 12071080, 12141104), the Science Technology Project of Jiangxi Educational Committee (No. GJJ2201202) and the Natural Science Foundation of Jiangxi Province (Nos. 20212BAB211005, 20232BAB201006).

$(\varphi, \sigma\varphi)$ with σ being a complex number are solutions to the system (1.1), where $a_2 = c_1 + c_2|\sigma|^2, b_1 = c_1 + (c_2 - b_2)|\sigma|^2$. The system (1.1) has explicit t- and x-dependent "travelling" solutions: $\varphi_1 = A_0 e^{i[A_1(t-t_0)+A_2(x-x_0)+\phi_1]}, \varphi_2 = B_0 e^{i[B_1(t-t_0)+B_2(x-x_0)+\phi_2]}$, where $A_1 = -A_2^2 + A_0^2 b_1 + B_0^2 b_2, B_1 = -B_2^2 + A_0^2 c_1 + B_0^2 c_2,\ A_0, B_0, A_2, B_2, t_0, x_0, \phi_1$ and ϕ_2 are arbitrarily real parameters. There are tanh-sech dark-bright soliton solutions and algebraic geometry solutions expressed in Weierstrass elliptic functions to the system (1.1) (refer to [10, Chapter 7] for details). However, contrast to its analytic properties, the geometric aspect of the system (1.1) has not been well investigated. One notes that, with parameters being suitably chosen, the system (1.1) contains three integrable equations:

$$\begin{cases} i\varphi_{1t} + \varphi_{1xx} + 2(|\varphi_1|^2 + |\varphi_2|^2)\varphi_1 = 0, \\ i\varphi_{2t} + \varphi_{2xx} + 2(|\varphi_1|^2 + |\varphi_2|^2)\varphi_2 = 0, \end{cases}$$
$$\begin{cases} i\varphi_{1t} + \varphi_{1xx} - 2(|\varphi_1|^2 + |\varphi_2|^2)\varphi_1 = 0, \\ i\varphi_{2t} + \varphi_{2xx} - 2(|\varphi_1|^2 + |\varphi_2|^2)\varphi_2 = 0 \end{cases}$$
$$\begin{cases} i\varphi_{1t} + \varphi_{1xx} \pm 2(|\varphi_1|^2 - |\varphi_2|^2)\varphi_1 = 0, \\ i\varphi_{2t} + \varphi_{2xx} \pm 2(|\varphi_1|^2 - |\varphi_2|^2)\varphi_2 = 0, \end{cases}$$

in which the two systems in the third one are actually equivalent to each other by $\varphi_1 \to \varphi_2$ and $\varphi_2 \to \varphi_1$. By using the geometric concept of Schrödinger flows, the authors in [8] proved that the three integrable systems are respectively gauge equivalent to the equation $\widetilde{\gamma}_t = -[\widetilde{\gamma}, \widetilde{\gamma}_{xx}]$ of Schrödinger flows from $\mathbb{R}^1$ to the projective spaces $U(3)/U(2) \times U(1)$, $U(2,1)/U(2) \times U(1)$ and $U(2,1)/U(1,1) \times U(1)$. We would point out that $U(2,1)/U(1,1) \times U(1)$ is a pseudo-Kähler manifold. This gives a unified geometric interpretations for these three integrable systems. The result for $U(3)/U(2) \times U(1)$ was obtained by Terng and Uhlenback in [20] as a special case.

It is not a surprise that the system (1.1) becomes non-integrable for general real parameters b_1, b_2, c_1 and c_2. However, one may still ask that, as mentioned in [8], does there exist an analogous geometric interpretation for the general Manakov system (1.1)? This leads us to fall back on the geometric realization of a (given) PDE introduced by Langer and Perline in [12]. It is shown in [12] that the moving equation: $\gamma_t = [\gamma_x, \gamma_{xx}]$ in the symmetric Lie algebra $u(n)$ $(n \geq 2)$ is a geometric realization of the matrix nonlinear Schrödinger equation, while, the moving equation: $\gamma_t = \gamma_{xxx} + \frac{3}{2}[\gamma_{xx}, [\gamma_x, \gamma_{xx}]]$ in the symmetric Lie algebra $sl(2n, \mathbb{R})$ $(n \geq 1)$ is a geometric realization of the matrix-KdV equation (see [5]). The result for the KdV equation is referred to [6]. According to this terminology, [8] shows actually that the (three) models: $\gamma_t = -[\gamma_x, \gamma_{xx}]$ of moving curves in $u(3)$ and $u(2,1)$ (which are respectively equivalent to the equations of Schrödinger flows to $U(3)/U(2) \times U(1)$, $U(2,1)/U(2) \times U(1)$ or $U(2,1)/U(1,1) \times U(1)$ by taking $\gamma = \int^x \widetilde{\gamma}$) are respectively the geometric realizations of the three integrable Manakov systems indicated above. Here two direct sum decompositions of the Lie algebra $u(2,1)$ (see below) are used.

The aim of this paper is to give a geometric realization of the general Manakov system (1.1) in the symmetric Lie algebras $u(3) = \mathbf{k}_3 \oplus \mathbf{m}_3$ and $u(2,1) = \mathbf{k}_\alpha \oplus \mathbf{m}_\alpha$ ($\alpha = 1, 2$) that preserves the integrability of (1.1) in three cases: 1) $b_1 = b_2 = c_1 = c_2 = 2$; 2) $b_1 = b_2 = c_1 = c_2 = -2$ and 3) $b_1 = c_1 = 2$, $b_2 = c_2 = -2$. The geometric interpretations of the three integrable Manakov systems by Schrödinger flows mentioned above will play an important role in this process. The main result is Theorem 3.1 below which indicates an interesting new phenomenon, says that more Lie algebras have to be used simultaneously in geometric realization of a partial differential equation/system.

The paper is organized as follows. Section 2 gives a brief preliminary discussion about the general Manakov system (1.1) for different situations of the 4 real parameters. In Section 3, we determine models of moving curves in three symmetric Lie algebras $u(2,1) = \mathbf{k}_\alpha \oplus \mathbf{m}_\alpha$ ($\alpha = 1, 2$) and $u(3) = \mathbf{k}_3 \oplus \mathbf{m}_3$ that is a geometric realization of the general Manakov system (1.1), according to the signs of its parameters. Some remarks are given.

2 Preliminaries

In order to characterize geometrically the general Manakov system (1.1), let's do some preliminary works. First of all, let $U(2,1)$ be the pseudo-unitary Lie group consisting of linear transformations on $\mathbb{C}^3$ that preserve the pseudo-metric $\mathrm{d}s^2 = -|\mathrm{d}z_1|^2 + |\mathrm{d}z_2|^2 + |\mathrm{d}z_3|^2$ or $\mathrm{d}s^2 = |\mathrm{d}z_1|^2 + |\mathrm{d}z_2|^2 - |\mathrm{d}z_3|^2$ invariant and $u(2,1)$ be the corresponding Lie algebra. Obviously, the above two Lie groups are actually isomorphism. Therefore, in the case of no confusion, the same notation $U(2,1)$ or $u(2,1)$ is used in the paper. As usual, $U(3)$ denotes the unitary Lie group of degree 3 and $u(3)$ its Lie algebra. We now set

$$\sigma_3 = \frac{\mathrm{i}}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{2.1}$$

One sees that the Lie algebra $u(2,1)$ has two direct sum decompositions as follows. One is that $u(2,1) = \mathbf{k_1} \oplus \mathbf{m_1}$ with

$$\mathbf{k_1} = \mathrm{Kernel}(ad_{\sigma_3}) = \left\{ \begin{pmatrix} \mathrm{i}a & 0 \\ 0 & B \end{pmatrix} \in u(2,1) \Big|\ a \in \mathbb{R},\ B \in u(2) \right\} \cong u(2) \times u(1)$$

and

$$\mathbf{m_1} = \left\{ \begin{pmatrix} 0 & \overline{\varphi}_1 & \overline{\varphi}_2 \\ \varphi_1 & 0 & 0 \\ \varphi_2 & 0 & 0 \end{pmatrix} \in u(2,1) \Big|\ \varphi_1, \varphi_2 \in \mathbb{C} \right\}.$$

Another is that $u(2,1) = \mathbf{k_2} \oplus \mathbf{m_2}$ with

$$\mathbf{k_2} = \mathrm{Kernel}(ad_{\sigma_3}) = \left\{ \begin{pmatrix} \mathrm{i}a & 0 \\ 0 & B \end{pmatrix} \in u(2,1) \Big|\ a \in \mathbb{R},\ B \in u(1,1) \right\} \cong u(1,1) \times u(1)$$

and

$$\mathbf{m_2} = \left\{ \begin{pmatrix} 0 & \overline{\varphi}_1 & \overline{\varphi}_2 \\ -\varphi_1 & 0 & 0 \\ \varphi_2 & 0 & 0 \end{pmatrix} \in u(2,1) \Big|\ \varphi_1, \varphi_2 \in \mathbb{C} \right\}.$$

Of course, the Lie algebra $u(3)$ has the direct sum decomposition: $u(3) = \mathbf{k_3} \oplus \mathbf{m_3}$ with

$$\mathbf{k_3} = \mathrm{Kernel}(ad_{\sigma_3}) = \left\{ \begin{pmatrix} \mathrm{i}a & 0 \\ 0 & B \end{pmatrix} \in u(3) \middle| \ a \in \mathbb{R},\ B \in u(2) \right\} \cong u(2) \times u(1)$$

and

$$\mathbf{m_3} = \left\{ \begin{pmatrix} 0 & \overline{\varphi}_1 & \overline{\varphi}_2 \\ -\varphi_1 & 0 & 0 \\ -\varphi_2 & 0 & 0 \end{pmatrix} \in u(3) \middle| \ \varphi_1, \varphi_2 \in \mathbb{C} \right\}.$$

All the above decompositions satisfy the symmetric conditions:

$$[\mathbf{k}_\alpha, \mathbf{k}_\alpha] \subset \mathbf{k}_\alpha, \quad [\mathbf{k}_\alpha, \mathbf{m}_\alpha] \subset \mathbf{m}_\alpha, \quad [\mathbf{m}_\alpha, \mathbf{m}_\alpha] \subset \mathbf{k}_\alpha,$$

where $\alpha \in \{1, 2, 3\}$.

Let G be one of the Lie groups $U(2,1)$ and $U(3)$ and g its Lie algebra. We set the following orbit space

$$M = \{E^{-1}\sigma_3 E \in g \mid \forall E \in G\}, \tag{2.2}$$

and define the action of G on M by

$$\Phi : G \times M \to M, \quad (X, \gamma) \mapsto \Phi(X, \gamma) = X \circ \gamma = X\gamma X^{-1}, \quad X \in G,\ \gamma \in M.$$

Obviously, we have

$$\begin{aligned} &I_{3\times 3} \ \circ \ \gamma = \gamma, \quad \forall \gamma \in M, \quad (I_{3\times 3} \text{ is the 3-unit matrix, i.e., unit element in } G), \\ &(XY) \circ \gamma = (XY)\gamma(XY)^{-1} = X \circ (Y \circ \gamma), \quad \forall X, Y \in G \end{aligned}$$

and the action is transitive, i.e., $\forall \gamma_1 = E_1^{-1}\sigma_3 E_1$, $\gamma_2 = E_2^{-1}\sigma_3 E_2 \in M$, there exists $X = E_2^{-1}E_1 \in G$ such that $X \circ \gamma_1 = \gamma_2$. Furthermore, according to the direct sum decompositions mentioned above, when $G = U(2,1)$ and its Lie algebra $g = u(2,1) = \mathbf{k_1} \oplus \mathbf{m_1}$, one obtains that the isotropic subgroup at $\sigma_3 \in M$ is

$$G_{\sigma_3} = \{X \in U(2,1) \mid X \circ \sigma_3 = \sigma_3\} = \{X \in U(2,1) \mid X\sigma_3 = \sigma_3 X\} = U(2) \times U(1).$$

Hence the orbit space M in this case is symmetric space $M = U(2,1)/U(2) \times U(1)$. Similarly, in the case of $g = u(2,1) = \mathbf{k_2} \oplus \mathbf{m_2}$, the orbit space is $M = U(2,1)/U(1,1) \times U(1)$ which is a pseudo-Kähler manifold mentioned previously. And in the case of $g = u(3) = \mathbf{k_3} \oplus \mathbf{m_3}$, the orbit space is $M = U(3)/U(2) \times U(1)$. We point out that the three symmetric spaces obtained above are actually the projective spaces, but we need not to use this character in this paper.

For any $U \in \mathbf{m}_\alpha$, one sees that $\mathbf{m}_\alpha = \mathbf{m}_{\alpha\mathbf{1}} \oplus \mathbf{m}_{\alpha\mathbf{2}}$ with

$$\mathbf{m}_{\alpha\mathbf{1}} = \left\{ \begin{pmatrix} 0 & \overline{\varphi} & 0 \\ \varepsilon_{\alpha 1}\varphi & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ \middle| \ \varphi \in \mathbb{C} \right\},$$

$$\mathbf{m}_{\alpha\mathbf{2}} = \left\{ \begin{pmatrix} 0 & 0 & \overline{\varphi} \\ 0 & 0 & 0 \\ \varepsilon_{\alpha 2}\varphi & 0 & 0 \end{pmatrix} \Big| \varphi \in \mathbb{C} \right\},$$

where $\varepsilon_{11} = \varepsilon_{12} = 1$; $\varepsilon_{21} = -1, \varepsilon_{22} = 1; \varepsilon_{31} = \varepsilon_{32} = -1$. For the action $\Phi(X, U) = X \circ U$, where $X \in G$ and $U \in \mathbf{m}_\alpha$, we need the following definition.

Definition 2.1 *For $X \in G, U \in \mathbf{m}_\alpha$, we define*

$$\mathrm{Pr}_j(X \circ U) = X \circ U^{(\mathbf{m}_{\alpha\mathbf{j}})}, \tag{2.3}$$

where $U^{(\mathbf{m}_{\alpha\mathbf{j}})} = U|_{\mathbf{m}_{\alpha\mathbf{j}}}$, $j = 1, 2$, and $\mathrm{Pr}_j(X \circ U)$ is called the projection of $X \circ U$ on $\mathbf{m}_{\alpha\mathbf{j}}$.

Obviously, $X \circ U = \mathrm{Pr}_1(X \circ U) + \mathrm{Pr}_2(X \circ U)$ and, meanwhile, the projections $\mathrm{Pr}_j(X \circ U)$ $(j = 1, 2)$ depend only on $X \circ U$.

Next, let us return to the general Manakov system (1.1). Since b_1 and c_2 are nonzero, it is easy to verify that, up to a suitable re-scaling: $\varphi_1 \to a\varphi_1$ and $\varphi_2 \to b\varphi_2$ for some real numbers a, b, the system (1.1) becomes

$$\begin{cases} \mathrm{i}\varphi_{1t} + \varphi_{1xx} + (2\varepsilon_{b_1}|\varphi_1|^2 + \widetilde{b}_2|\varphi_2|^2)\varphi_1 = 0, \\ \mathrm{i}\varphi_{2t} + \varphi_{2xx} + (\widetilde{c}_1|\varphi_1|^2 + 2\varepsilon_{c_2}|\varphi_2|^2)\varphi_2 = 0, \end{cases} \tag{2.4}$$

where $\widetilde{b}_2$ and $\widetilde{c}_1$ are the corresponding numbers after the re-scaling which will still be respectively written as b_2 and c_1 in the sequel, $\varepsilon_{b_1} = 1$ when $b_1 > 0$ and $\varepsilon_{b_1} = -1$ when $b_1 < 0$, and so is ε_{c_2}. According to the signs of ε_{b_1} and ε_{c_2}, one sees that system (2.4), and hence system (1.1), is divided into

- 1. $U(2,1)/U(2) \times U(1)$-type (when $\varepsilon_{b_1} = \varepsilon_{c_2} = -1$),

$$\begin{cases} \mathrm{i}\varphi_{1t} + \varphi_{1xx} - 2(|\varphi_1|^2 + |\varphi_2|^2)\varphi_1 = -(b_2 + 2)|\varphi_2|^2\varphi_1, \\ \mathrm{i}\varphi_{2t} + \varphi_{2xx} - 2(|\varphi_1|^2 + |\varphi_2|^2)\varphi_2 = -(c_1 + 2)|\varphi_1|^2\varphi_2; \end{cases} \tag{2.5}$$

- 2. $U(2,1)/U(1,1) \times U(1)$-type (when $\varepsilon_{b_1} = 1$ and $\varepsilon_{c_2} = -1$),

$$\begin{cases} \mathrm{i}\varphi_{1t} + \varphi_{1xx} + 2(|\varphi_1|^2 - |\varphi_2|^2)\varphi_1 = -(2 + b_2)|\varphi_2|^2\varphi_1, \\ \mathrm{i}\varphi_{2t} + \varphi_{2xx} + 2(|\varphi_1|^2 - |\varphi_2|^2)\varphi_2 = (2 - c_1)|\varphi_1|^2\varphi_2; \end{cases} \tag{2.6}$$

- 3. $U(2,1)/U(1,1) \times U(1)$-type (when $\varepsilon_{b_1} = -1$ and $\varepsilon_{c_2} = 1$),

$$\begin{cases} \mathrm{i}\varphi_{1t} + \varphi_{1xx} - 2(|\varphi_1|^2 - |\varphi_2|^2)\varphi_1 = (2 - b_2)|\varphi_2|^2\varphi_1, \\ \mathrm{i}\varphi_{2t} + \varphi_{2xx} - 2(|\varphi_1|^2 - |\varphi_2|^2)\varphi_2 = -(2 + c_1)|\varphi_1|^2\varphi_2; \end{cases} \tag{2.7}$$

- 4. $U(3)/U(2) \times U(1)$-type (when $\varepsilon_{b_1} = \varepsilon_{c_2} = 1$),

$$\begin{cases} \mathrm{i}\varphi_{1t} + \varphi_{1xx} + 2(|\varphi_1|^2 + |\varphi_2|^2)\varphi_1 = (2 - b_2)|\varphi_2|^2\varphi_1, \\ \mathrm{i}\varphi_{2t} + \varphi_{2xx} + 2(|\varphi_1|^2 + |\varphi_2|^2)\varphi_2 = (2 - c_1)|\varphi_1|^2\varphi_2, \end{cases} \tag{2.8}$$

in which, (2.6) and (2.7) are actually equivalent to each other by $(\varphi_1, \varphi_2) \to (\varphi_2, \varphi_1)$. Thus the general Manakov system (1.1) locates in one of the three types systems (2.5), (2.6) and

(2.8). One notes that system (2.5) is integrable when $b_2 = c_1 = -2$, so are system (2.6) when $b_2 = -2$, $c_1 = 2$ and system (2.8) when $b_2 = c_1 = 2$. This shows, as mentioned in Introduction, the importance of the three integrable Manakov systems in the geometric study of the general Manakov system (1.1).

3 Geometric Realizations of the Manakov Systems

In this section, we come to find models of moving curves in $u(2,1) = \mathbf{k}_\alpha \oplus \mathbf{m}_\alpha$ $(\alpha = 1, 2)$ and $u(3) = \mathbf{k_3} \oplus \mathbf{m_3}$ that are equivalent to systems (2.5)–(2.6) and (2.8), respectively. From which we then obtain a geometric realization of the general Manakov system (1.1).

In order to do this, we have to apply the idea of PDEs with prescribed representation introduced in [9] in the category of Yang-Mills' theory (see [7] also). This is somewhat a power tool in transforming a non-integrable PDE into its equivalent form. Now we first come to treat the $U(2,1)/U(1,1)\times U(1)$-type system (2.6), as the manifold $U(2,1)/U(1,1)\times U(1)$ is a pseudo-Kähler manifold.

Proposition 3.1 *For the Lie group $U(2,1)$ corresponding to the Lie algebra $u(2,1) = \mathbf{k_2} \oplus \mathbf{m_2}$, the following equation of a map $\gamma : \mathbb{R}^1 \times \mathbb{R}^1 \to U(2,1)/U(1,1)\times U(1)$:*

$$\gamma_t = -[\gamma, \gamma_{xx}] + \Big[\gamma, \int_0^x \widehat{\gamma}_{\widetilde{x}} \mathrm{d}\widetilde{x}\Big], \tag{3.1}$$

is gauge equivalent to the $U(2,1)/U(1,1)\times U(1)$-type system (2.6), where γ is presented by $\gamma = \gamma(x,t) = E^{-1}\sigma_3 E$ with $E = E(x,t) \in U(2,1)$ and $E_x = UE$, $U \in \mathbf{m_2}$, and $\widehat{\gamma}_x = -\frac{2+b_2}{2}\mathrm{tr}[\mathrm{Pr}_2(\gamma_x)]^2\mathrm{Pr}_1(\gamma_x) - \frac{2-c_1}{2}\mathrm{tr}[\mathrm{Pr}_1(\gamma_x)]^2\mathrm{Pr}_2(\gamma_x)$ depending only on γ_x and hence on γ.

Proof First of all, we show that the $U(2,1)/U(1,1)\times U(1)$-type system (2.6) can be gauge transformed to (3.1). In order to do this, we come to construct a connection 1-form $\widetilde{A}$ and a curvature 2-form $\widetilde{K}$ on the trivial bundle $\mathbb{R}^2 \times U(2,1)$ such that the system (2.6) is of the prescribed curvature representation

$$F_{\widetilde{A}} = \mathrm{d}\widetilde{A} + \widetilde{A} \wedge \widetilde{A} = \widetilde{K}. \tag{3.2}$$

In fact, by noting that $u(2,1) = \mathbf{k_2} \oplus \mathbf{m_2}$, for a given solution (φ_1, φ_2) to the system (2.6) we define

$$\widetilde{A} = (\lambda\sigma_3 - U)\mathrm{d}x + (-\lambda^2\sigma_3 + \lambda U - V)\mathrm{d}t, \tag{3.3}$$

where

$$U = \begin{pmatrix} 0 & \overline{\varphi}_1 & \overline{\varphi}_2 \\ -\varphi_1 & 0 & 0 \\ \varphi_2 & 0 & 0 \end{pmatrix} \in \mathbf{m_2}, \quad V = -2\sigma_3(U_x - U^2).$$

It is a direct calculation that

$$F_{\widetilde{A}} = \mathrm{d}\widetilde{A} + \widetilde{A} \wedge \widetilde{A} = (U_t - V_x + [U, V])\mathrm{d}x \wedge \mathrm{d}t. \tag{3.4}$$

Thus, we choose a 2-form given by

$$\widetilde{K} = \widetilde{R}\mathrm{d}x \wedge \mathrm{d}t, \tag{3.5}$$

in which

$$\widetilde{R} = \begin{pmatrix} 0 & \overline{R}_1 & \overline{R}_2 \\ -R_1 & 0 & 0 \\ R_2 & 0 & 0 \end{pmatrix}$$

with $R_1 = \mathrm{i}(2 + b_2)|\varphi_2|^2\varphi_1, R_2 = -\mathrm{i}(2 - c_1)|\varphi_1|^2\varphi_2$. One verifies easily that system (2.6) possesses the prescribed curvature representation (3.2) with $\widetilde{A}$ and $\widetilde{K}$ being given by (3.3) and (3.5), respectively.

For the above solution $(\varphi_1(x,t), \varphi_2(x,t))$ to system (2.6), we may take an $E(x,t) \in U(2,1)$ by solving the differential equation: $E_x(x,t) = U(x,t)E(x,t)$, where $U = U(x,t)$ is the one given in (3.3) and the dependance of E with respect to t will be determined later. Now we make gauge transformation for the connection $\widetilde{A}$ given in (3.3) by

$$\widetilde{A} \longmapsto A = E^{-1}\mathrm{d}E + E^{-1}\widetilde{A}E. \tag{3.6}$$

It is well-known from the theory of Yang-Mills that under the gauge transformation (3.6), the curvature F_A of A satisfies

$$F_A = E^{-1}F_{\widetilde{A}}E = E^{-1}\widetilde{K}E. \tag{3.7}$$

By a direct calculation, we obtain from (3.6) that

$$\begin{aligned} A &= E^{-1}\mathrm{d}E + E^{-1}\widetilde{A}E \\ &= \lambda\gamma\mathrm{d}x + (E^{-1}E_t - \lambda^2\gamma - \lambda[\gamma, \gamma_x] - E^{-1}VE)\mathrm{d}t, \end{aligned} \tag{3.8}$$

where $\gamma = E^{-1}\sigma_3 E$ and $E^{-1}E_t$ independent of λ is to be determined later. Furthermore, by using (3.8) and a direct computation shows that

$$\begin{aligned} F_A &= \mathrm{d}A + A \wedge A \\ &= \{(E^{-1}E_t - \lambda^2\gamma - \lambda[\gamma, \gamma_x] - E^{-1}VE)_x - \lambda\gamma_t \\ &\quad + [\lambda\gamma, E^{-1}E_t - \lambda^2\gamma - \lambda[\gamma, \gamma_x] - E^{-1}VE]\}\mathrm{d}x \wedge \mathrm{d}t \\ &= \{\lambda(-2(\gamma\gamma_x)_x - \gamma_t + [\gamma, E^{-1}E_t] - [\gamma, E^{-1}VE]) \\ &\quad + (E^{-1}E_t)_x - (E^{-1}VE)_x\}\mathrm{d}x \wedge \mathrm{d}t. \end{aligned} \tag{3.9}$$

Substituting (3.9) into (3.7) and identifying the coefficients of λ^0 in both sides of (3.7), we obtain that

$$E^{-1}E_t = E^{-1}VE + \int_0^x E^{-1}\widetilde{R}E\mathrm{d}\widetilde{x} = E^{-1}VE + \int_0^x \widehat{\gamma}_{\widetilde{x}}\mathrm{d}\widetilde{x}, \tag{3.10}$$

where

$$\widehat{\gamma}_x = -\frac{2+b_2}{2}\mathrm{tr}[\mathrm{Pr}_2(\gamma_x)]^2\mathrm{Pr}_1(\gamma_x) - \frac{2-c_1}{2}\mathrm{tr}[\mathrm{Pr}_1(\gamma_x)]^2\mathrm{Pr}_2(\gamma_x)$$

depending only on γ_x. Here we have used the following facts: $\gamma_x = E^{-1}[\sigma_3, U]E$ and

$$\mathrm{Pr}_1(\gamma_x) = E^{-1}\begin{pmatrix} 0 & \mathrm{i}\overline{\varphi}_1 & 0 \\ \mathrm{i}\varphi_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}E, \quad \mathrm{Pr}_2(\gamma_x) = E^{-1}\begin{pmatrix} 0 & 0 & \mathrm{i}\overline{\varphi}_2 \\ 0 & 0 & 0 \\ -\mathrm{i}\varphi_2 & 0 & 0 \end{pmatrix}E$$

in obtaining the second equality in (3.10). Meanwhile, the coefficient of λ^1 in the left-hand-side of (3.7) is zero implies that

$$-\gamma_t - 2(\gamma\gamma_x)_x + [\gamma, E^{-1}E_t] - [\gamma, E^{-1}VE] = 0. \tag{3.11}$$

Substituting (3.10) into (3.11) and by noting $\gamma_x\gamma_x = -\frac{1}{2}(\gamma_{xx}\gamma + \gamma\gamma_{xx})$, we have

$$\gamma_t = -2(\gamma\gamma_x)_x + \Big[\gamma, \int_0^x \widehat{\gamma}_{\widetilde{x}}\mathrm{d}\widetilde{x}\Big] = -[\gamma, \gamma_{xx}] + \Big[\gamma, \int_0^x \widehat{\gamma}_{\widetilde{x}}\mathrm{d}\widetilde{x}\Big],$$

which is exactly (3.1). This proves that system (2.6) is gauge transformed to (3.1). In addition, (3.1) also possesses the prescribed curvature representation:

$$F_A = \mathrm{d}A + A\wedge A = \Big\{\lambda\Big(-\gamma_t - [\gamma, \gamma_{xx}] + \Big[\gamma, \int_0^x \widehat{\gamma}_{\widetilde{x}}\mathrm{d}\widetilde{x}\Big]\Big) + \widehat{\gamma}_x\Big\}\mathrm{d}x\wedge\mathrm{d}t = K,$$

where the connection A and the curvature K are respectively given by

$$A = \lambda\gamma\mathrm{d}x + \Big(-\lambda^2\gamma - \lambda[\gamma, \gamma_x] + \int_0^x \widehat{\gamma}_{\widetilde{x}}\mathrm{d}\widetilde{x}\Big)\mathrm{d}t, \tag{3.12}$$

$$K = \widehat{\gamma}_x\mathrm{d}x\wedge\mathrm{d}t. \tag{3.13}$$

Next, conversely, we show that (3.1) can also be gauge transformed to system (2.6). For a given solution $\gamma = E^{-1}\sigma_3E$ to (3.1), where $E = E(x,t) \in U(2,1)$ which has the Lie algebra $u(2,1) = \mathbf{k_2}\oplus\mathbf{m_2}$. Without loss of generality, E is assumed to satisfy $E_x = UE$ for some

$$U = \begin{pmatrix} 0 & \overline{\varphi}_1 & \overline{\varphi}_2 \\ -\varphi_1 & 0 & 0 \\ \varphi_2 & 0 & 0 \end{pmatrix} \in \mathbf{m_2}.$$

In fact, if $E_x = PE$ holds for a general $P \in u(2,1)$ with $P = P^{(\mathbf{k_2})} + P^{(\mathbf{m_2})}$, one may obtain a $A \in K_2$ by solving the differential equation: $A_x = -AP^{(\mathbf{k_2})}$, where K_2 is the Lie subgroup of $U(2,1)$ corresponding to the Lie subalgebra $\mathbf{k_2}$. By taking the transform: $E \to \widetilde{E} = AE$, we have that

$$\widetilde{E}_x = A_xE + AE_x = AP^{(\mathbf{m_2})}A^{-1}\widetilde{E}$$

with $AP^{(\mathbf{m_2})}A^{-1} \in \mathbf{m_2}$. Hence, denoting $AP^{(\mathbf{m_2})}A^{-1}$ by $\widetilde{U}$ and by using the fact that $A \in K_2$ commute with σ_3, we arrive at

$$\gamma = E^{-1}\sigma_3 E = E^{-1}A^{-1}\sigma_3 AE = (AE)^{-1}\sigma_3(AE) = \widetilde{E}^{-1}\sigma_3\widetilde{E},$$

and $\widetilde{E}$ fulfills the required condition: $\widetilde{E}_x = \widetilde{U}\widetilde{E}$ with $\widetilde{U} \in \mathbf{m_2}$.

Since (3.1) possesses the prescribed curvature representation: $F_A = \mathrm{d}A + A \wedge A = K$, where A and K are given by (3.12) and (3.13) respectively. We make the following gauge transformation for the connection A via $G = E^{-1}$:

$$A \longmapsto \widetilde{A} = G^{-1}\mathrm{d}G + G^{-1}AG = -(\mathrm{d}E)E^{-1} + EAE^{-1}. \tag{3.14}$$

By a direct calculation displayed previously, we have from (3.14) that

$$\begin{aligned}
&\widetilde{A} = (\lambda\sigma_3 - U)\mathrm{d}x + (-\lambda^2\sigma_3 + \lambda U - V)\mathrm{d}t,\\
&F_{\widetilde{A}} = (U_t - V_x + [U, V])\mathrm{d}x \wedge \mathrm{d}t,
\end{aligned}$$

which are exactly the connection $\widetilde{A}$ and the curvature $F_{\widetilde{A}}$ given in (3.3) and (3.4), respectively. The details of calculation are omitted here. Hence the (φ_1, φ_2) obtained from $\gamma = E^{-1}\sigma_3 E$ is a solution to system (2.6). This proves that (3.1) can also be gauge transformed to system (2.6). The proof of Proposition 3.1 is completed.

In a similar way, we may also establish the following propositions.

Proposition 3.2 *For the Lie group $U(2,1)$ corresponding to the Lie algebra $u(2,1) = \mathbf{k_1} \oplus \mathbf{m_1}$, the following equation of a map $\gamma : \mathbb{R}^1 \times \mathbb{R}^1 \to U(2,1)/U(2) \times U(1)$:*

$$\gamma_t = -[\gamma, \gamma_{xx}] + \Big[\gamma, \int_0^x \widehat{\gamma}_{\widetilde{x}}\mathrm{d}\widetilde{x}\Big], \tag{3.15}$$

is gauge equivalent to the $U(2,1)/U(2) \times U(1)$-type system (2.5), where

$$\gamma = \gamma(x,t) = E^{-1}\sigma_3 E \in u(2,1)$$

with

$$E = E(x,t) \in U(2,1),$$

$$E_x = UE, \quad U = U(x,t) \in \mathbf{m_1},$$

and

$$\widehat{\gamma}_x = -\frac{b_2+2}{2}\mathrm{tr}[\mathrm{Pr}_2(\gamma_x)]^2\mathrm{Pr}_1(\gamma_x) - \frac{c_1+2}{2}\mathrm{tr}[\mathrm{Pr}_1(\gamma_x)]^2\mathrm{Pr}_2(\gamma_x)$$

depending only on γ_x and hence on γ.

Proposition 3.3 *For the Lie group $U(3)$, the following equation of a map $\gamma : \mathbb{R}^1 \times \mathbb{R}^1 \to U(3)/U(2) \times U(1)$, the model of moving curves in $u(3)$ which locates on $U(3)/U(2) \times U(1)$:*

$$\gamma_t = -[\gamma, \gamma_{xx}] + \Big[\gamma, \int_0^x \widehat{\gamma}_{\widetilde{x}} \mathrm{d}\widetilde{x}\Big], \tag{3.16}$$

is gauge equivalent to the $U(3)/U(2) \times U(1)$-type system (2.8)*, where*

$$\gamma = \gamma(x,t) = E^{-1}\sigma_3 E \in u(3)$$

with $E = E(x,t) \in U(3)$ and $E_x = UE$, $U = U(x,t) \in \mathbf{m_3}$, and

$$\widehat{\gamma}_x = -\frac{2-b_2}{2}\mathrm{tr}[\mathrm{Pr}_2(\gamma_x)]^2\mathrm{Pr}_1(\gamma_x) - \frac{2-c_1}{2}\mathrm{tr}[\mathrm{Pr}_1(\gamma_x)]^2\mathrm{Pr}_2(\gamma_x)$$

depending only on γ_x and hence on γ.

Remark 3.1 When the two parameters b_2 and c_1 are chosen such that the coefficients of two terms in $\widehat{\gamma}_x$ are zero and hence $\widehat{\gamma}_x \equiv 0$, (3.15), (3.1) and (3.16) return respectively to the equation of Schödinger flows from $\mathbb{R}^1$ to the manifold $U(2,1)/U(2) \times U(1)$ or $U(2,1)/U(1,1) \times U(1)$ or $U(3)/U(2) \times U(1)$, in other words, Proposition 3.1 returns to the result obtained in [8]. One knows that, when the target manifold is Kähler, Schrödinger flow is actually the Hamiltonian gradient flow of the energy functional. We believe that (3.15), (3.1) and (3.16) are the Hamiltonian gradient flows of some functionals of maps from $\mathbb{R}^1$ to $U(2,1)/U(2) \times U(1)$ or $U(2,1)/U(1,1) \times U(1)$ or $U(3)/U(2) \times U(1)$, respectively.

Now we come to determine models of moving curves in three symmetric Lie algebras $u(2,1) = \mathbf{k}_\alpha \oplus \mathbf{m}_\alpha$ ($\alpha = 1,2$) and $u(3) = \mathbf{k_3} \oplus \mathbf{m_3}$ that are the geometric realizations of the general Manakov system (1.1) according to the signs of the parameters. For the meaning of models of moving curves in Lie algebras, we refer to [16, 19].

Theorem 3.1 *The following models of moving curves $\widetilde{\gamma}$ in the symmetric Lie algebras $\mathbf{g} = u(2,1)$ or $u(3)$ with the direct sum $\mathbf{g} = \mathbf{k}_\alpha \oplus \mathbf{m}_\alpha$, $\alpha = 1,2,3$,*

$$\widetilde{\gamma}_t = -[\widetilde{\gamma}_x, \widetilde{\gamma}_{xx}] + \Big[\widetilde{\gamma}, \int_0^x \widehat{\widetilde{\gamma}} \mathrm{d}\widetilde{x}\Big] - \int_0^x [\widetilde{\gamma}, \widehat{\widetilde{\gamma}}] \mathrm{d}\widetilde{x} \tag{3.17}$$

are respectively equivalent to (3.15), (3.1) *and* (3.16)*, where $\widetilde{\gamma} \in \mathbf{g}$ with $\widetilde{\gamma}_x \in \{E^{-1}\sigma_3 E \mid E \in \mathbf{G}, E_x = UE \ \&\ U \in \mathbf{m}_\alpha\}$, and $\widehat{\widetilde{\gamma}}$ is given by*

$$\widehat{\widetilde{\gamma}} = \begin{cases} -\dfrac{b_2+2}{2}\mathrm{tr}[\mathrm{Pr}_2(\widetilde{\gamma}_{xx})]^2\mathrm{Pr}_1(\widetilde{\gamma}_{xx}) - \dfrac{c_1+2}{2}\mathrm{tr}[\mathrm{Pr}_1(\widetilde{\gamma}_{xx})]^2\mathrm{Pr}_2(\widetilde{\gamma}_{xx}), & \text{when } \varepsilon_{b_1} = \varepsilon_{c_2} = -1, \\ -\dfrac{2+b_2}{2}\mathrm{tr}[\mathrm{Pr}_2(\widetilde{\gamma}_{xx})]^2\mathrm{Pr}_1(\widetilde{\gamma}_{xx}) - \dfrac{2-c_1}{2}\mathrm{tr}[\mathrm{Pr}_1(\widetilde{\gamma}_{xx})]^2\mathrm{Pr}_2(\widetilde{\gamma}_{xx}), & \text{when } \varepsilon_{b_1} = 1,\ \varepsilon_{c_2} = -1, \\ -\dfrac{2-b_2}{2}\mathrm{tr}[\mathrm{Pr}_2(\widetilde{\gamma}_{xx})]^2\mathrm{Pr}_1(\widetilde{\gamma}_{xx}) - \dfrac{2-c_1}{2}\mathrm{tr}[\mathrm{Pr}_1(\widetilde{\gamma}_{xx})]^2\mathrm{Pr}_2(\widetilde{\gamma}_{xx}), & \text{when } \varepsilon_{b_1} = \varepsilon_{c_2} = 1. \end{cases}$$

In other words, (3.17) *is a geometric realization of the general Manakov system* (1.1).

Proof From Propositions 3.1–3.3 and the correspondences of the signs of b_1, c_2 and the Lie algebras $\mathbf{g} = \mathbf{k}_\alpha \oplus \mathbf{m}_\alpha$ $(\alpha = 1, 2, 3)$, it is direct to verify, by taking the derivative with respect to x, that if $\widetilde{\gamma}$ solves (3.17), then $\gamma = \widetilde{\gamma}_x \in \{E^{-1}\sigma_3 E \mid E \in \mathbf{G}, E_x = UE \ \&\ U \in \mathbf{m}_\alpha\}$ satisfies respectively the systems (3.15), (3.1) and (3.16) according the signs of b_1, c_2. Conversely, if $\gamma(x,t) = E^{-1}(x,t)\sigma_3 E(x,t)$ satisfies respectively the systems (3.15), (3.1) and (3.16), then it is easy to see that $\widetilde{\gamma} = \int_0^x \gamma(s,t)\mathrm{d}s$ solves (3.17) and fulfills

$$(\widetilde{\gamma})_x = \gamma(x,t) \in \{E^{-1}\sigma_3 E \mid E \in \mathbf{G}, E_x = UE\&U \in \mathbf{m}_\alpha\}.$$

Theorem 3.1 also indicates that the geometric realization model (3.17) preserves the integrability of (1.1) in the cases that $b_1 = b_2 = c_1 = c_2 = -2$ or $b_1 = c_1 = 2$ & $b_2 = c_2 = -2$ or $b_1 = b_2 = c_1 = c_2 = 2$. Three different symmetric Lie algebras are used simultaneously to produce the model (3.17). This reflects a new phenomenon in geometric realization of a partial differential equation/system.

Finally, we would point out that for the general k-components nonlinear Schrödinger equation $(k \geq 3)$ (see [14, 18])

$$\begin{cases} \mathrm{i}\varphi_{1t} + \varphi_{1xx} + (a_{11}|\varphi_1|^2 + a_{12}|\varphi_2|^2 + \cdots + a_{1k}|\varphi_k|^2)\varphi_1 = 0, \\ \mathrm{i}\varphi_{2t} + \varphi_{2xx} + (a_{21}|\varphi_1|^2 + a_{22}|\varphi_2|^2 + \cdots + a_{2k}|\varphi_k|^2)\varphi_2 = 0, \\ \cdots \\ \mathrm{i}\varphi_{kt} + \varphi_{kxx} + (a_{k1}|\varphi_1|^2 + a_{k2}|\varphi_2|^2 + \cdots + a_{kk}|\varphi_k|^2)\varphi_k = 0, \end{cases}$$

where $a_{\alpha\beta}$ $(1 \leq \alpha, \beta \leq k)$ are nonzero real parameters, its geometric properties can also be similarly discussed and characterized.

Declarations

Conflicsts of interest The authors declare no conflicts of interest.

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