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# Pseudo-Effective Vector Bundles with Vanishing First Chern Class on Astheno-Kähler Manifolds*

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**Abstract** Let $E$ be a holomophic vector bundle over a compact Astheno-Kähler manifold $(M,\omega)$. The authors would prove that $E$ is a numerically flat vector bundle if $E$ is pseudo-effective and the first Chern class $c_1^{BC}(E)$ is zero.

**Keywords** Pseudo-effective, Astheno-Kähler, Numerically flatness
**2000 MR Subject Classification** 32L05, 32Q99

## 1 Introduction

In [8], Demailly, Peternell and Schneider introduced the conception of numerically flat vector bundles. Let $E$ be a vector bundle over a compact Kähler manifold $(M,\omega)$. Demailly, Peternell and Schneider [8, Theorem 1.8] proved that $E$ is numerically flat if and only if there exists a filtration

$$0 = E_0 \subseteq \cdots \subseteq E_s = E$$

by subbundles whose quotients are Hermitian flat. Meanwile, they raised an interesting question whether the above characterization holds in non-Kähler case and pointed out the difficulty is to show the second Chern number of a numerically flat vector bundle is zero. Recently, Li, Nie and the second author (see [14, Theorem 1.4]) proved that the conjecture of Demailly, Peternell and Schneider holds on Astheno-Kähler manifolds and they established some other equivalent descriptions about numerically flatness.

In [4], Campana, Cao and Matsumura showed that a pseudo-effective vector bundle over a projective manifold with vanishing first Chern class is numerically flat (see also [12, Theorem 3.4]). This is a key lemma in the classification theory of compact Kähler manifolds with nef anticanonical line bundle and projective manifolds with pseudo-effecitve tangent bundle. Recently, Wu [15] generalized this theorem to compact Kähler manifolds. In this paper, we would generalized this theorem to Astheno-Kähler manifolds.

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**Theorem 1.1** *Let $E$ be a holomophic vector bundle over a compact Astheno-Kähler manifold $(M, \omega)$. If $E$ is pseudo-effective and the first Chern class $c_1^{BC}(E)$ is zero. Then $E$ is a numerically flat vector bundle.*

## 2 Preliminary

In this section, firstly, we recall some definitions about positivity of vector bundles.

**Definition 2.1** (see [6, 8]) *Let $(M, \omega)$ be a compact Hermitian manifold and $L$ be a line bundle over $M$. $L$ is called numerically effective (short for nef) if for every $\varepsilon > 0$, there exists a smooth hermitian metric $h_\varepsilon$ on $L$ such that the curvature satisfies $\sqrt{-1}\Theta(L, h_\varepsilon) \geq -\varepsilon\omega$. A singular Hermitian metric on a line bundle $L$ is a hermitian metric $h$ which is given in any trivialization by a weight function $\mathrm{e}^{-\varphi}$ such that $\varphi$ is locally integrable. $L$ is called pseudo-effective if there exists a singular metric $h$ on $L$ such that the curvature $\sqrt{-1}\Theta(L, h)$ is a closed positive $(1,1)$-current.*

**Definition 2.2** (see [3, 8]) *We say that a holomorphic vector bundle $E$ is nef over $M$ if $\mathcal{O}_E(1)$ is nef over $\mathbb{P}(E)$. Furthermore, we say that $E$ is numerically flat if both $E$ and the dual bundle $E^*$ is nef. $E$ is called pseudo-effective when $\mathcal{O}_E(1)$ is a pseudo-effective line bundle and additionally requires that the image of the non-nef locus of $\mathcal{O}_E(1)$ is properly contained in $M$.*

In [8], Demailly, Peternell and Schneider study the fundamental properties about nef vector bundles in detail and give the structure theorem of compact Kähler manifolds with nef tangent bundle. In [15], Wu gives the following equivalent definition of pseudo-effective vector bundles.

**Proposition 2.1** (see [15]) *Let $(M, \omega)$ be a compact Hermitian manifold and $E$ be a holomorphic vector bundle over $X$. Then $E$ is pseudo-effective if and only if for every $\varepsilon > 0$ there exists a singular metric $h_\varepsilon$ with analytic singularities on $\mathcal{O}_E(1)$, the curvature current $i\Theta(\mathcal{O}_E(1), h_\varepsilon) \geq -\varepsilon\pi^*\omega$, and the projection $\pi(Sing(h_\varepsilon))$ of the singular set of $h_\varepsilon$ is not equal to $X$.*

A plurisubharmonic function $u$ is said to have analytic singularities if $u$ can be written locally as

$$u = \frac{\alpha}{2}\log(|f_1|^2 + \cdots + |f_1|^N) + v,$$

where $v$ is a smooth function, $f_i$ are holomorphic functions and $\alpha$ is a positive constant. A singular hermitian metric $h$ on a line bundle has analytic singularities if $\varphi$ has analytic singularities where $\mathrm{e}^{-\varphi}$ is the local weight function for $h$.

The Bott-Chern cohomology and the Aeppli cohomology provide important invariants for the study of the geometry of compact (especially, non-Kähler) complex manifolds. These cohomology groups have been introduced by Bott and Chern in [2] and Aeppli in [1].

**Definition 2.3** *The Bott-Chern cohomology of a complex manifold $M$ is the bi-graded al-*

*gebra*

$$H^{\bullet,\bullet}_{BC}(M) = \frac{\ker \partial \cap \ker \overline{\partial}}{\operatorname{im} \partial\overline{\partial}}. \tag{2.1}$$

*The Aeppli cohomology of a complex manifold $M$ is the bi-graded $H^{\bullet,\bullet}_{A}(M)$-module*

$$H^{\bullet,\bullet}_{A}(M) = \frac{\ker \partial\overline{\partial}}{\operatorname{im} \partial + \operatorname{im} \overline{\partial}}. \tag{2.2}$$

**Definition 2.4** *Let $\omega$ be a Hermitian metric on a compact complex manifold.*

• *$\omega$ is said to be Gauduchon if $\partial\overline{\partial}\omega^{n-1} = 0$. In this case, we can define the first Chern number of a vector bundle $E$ as $c_1^{BC}(E) \cdot [\omega^{n-1}]$.*

• *$\omega$ is said to be Astheno-Kähler if $\partial\overline{\partial}\omega^{n-2} = 0$. In this case, we can define the second Chern number of a vector bundle $E$ as $c_2^{BC}(E) \cdot [\omega^{n-2}]$.*

Gauduchon [10] proved that given any Hermitian form $\omega$ there exists a conformal factor $\mathrm{e}^{\Phi}$ such that the new form $\mathrm{e}^{\Phi}\omega$ is Gauduchon metric. Astheno-Kähler metric was introduced by Jost and Yau [13] in their study of Hermitian harmonic maps from Hermitian manifolds to general Riemmanian manifolds.

Now, we wish to introduce the pushforward formula of Segre forms which was proved by Guler [11] for projective manifolds and by Diverio [9] for general compact complex manifolds. Let $E$ be a rank $r$ holomorphic vector bundle on a complex manifold $X$ and

$$c_{\bullet}(E) = 1 + c_1(E) + \cdots + c_r(E) \in H^{\bullet}(X, \mathbb{Z})$$

be the total Chern class of $E$. The inverse of $c_{\bullet}(E)$ is defined by the total Segre class

$$s_{\bullet}(E) = 1 + s_1(E) + \cdots + s_r(E) \in H^{\bullet}(X, \mathbb{Z}).$$

Given a Hermitian metric $H$ on $E$, then these Segre forms $s_k(E, H)$ can be defined by the following relation:

$$s_k(E, H) + c_1(E, H)s_{k-1}(E, H) + \cdots + c_k(E, H) = 0, \quad 0 \leq k \leq \min(r, n).$$

For example,

$$s_1(E, H) = -c_1(E, H)$$

and

$$s_2(E, H) = c_1(E, H)^2 - c_2(E, H).$$

Let $\pi : \mathbb{P}(E) \to M$ be the projectivized bundle of hyperplanes of $E$, and $\mathcal{O}_E(1)$ be the associated canonical line bundle. Denote $h$ the induced metric on $\mathcal{O}_E(1)$ and $\alpha = \frac{\sqrt{-1}}{2\pi}\Theta(\mathcal{O}_E(1), h)$. We have the following formula of Segre forms which is proved by Diverio.

**Lemma 2.1** (see [9, 11]) *For each $0 \leq k \leq n$, we have the equality*

$$\pi_*(\alpha^{r-1+k}) = s_k(E, H).$$

Recently, Li, Nie and the second author established some equivalent descriptions about numerically flat vector bundles on Astheno-Kähler manifolds.

**Theorem 2.1** (see [14, Theorem 1.4]) *Let $(M,\omega)$ be an $n$-dimensional compact Astheno-Kähler manifold, $\widetilde{\omega}$ be a Gauduchon metric conformal to $\omega$, and $E$ be a holomorphic vector bundle over $M$. Then the following statements on $E$ are equivalent*:

- *$E$ is numerically flat.*
- *$E$ is $\widetilde{\omega}$-semistable and $c_1^{BC}(E)\cdot[\widetilde{\omega}^{n-1}]=\mathrm{ch}_2^{BC}(E)\cdot[\omega^{n-2}]=0$.*
- *$E$ is approximate Hermitian flat.*
- *There exists a filtration*

$$0=E_0\subseteq\cdots\subseteq E_s=E$$

*by subbundles whose quotients are Hermitian flat.*

## 3 Proof of Theorem 1.1

In this section, we would prove Theorem 3.1 which generalizes [5, Theorem 1]. Theorem 1.1 is a corollary of the following theorem.

**Theorem 3.1** *Let $E$ be a pseudo-effective vector bundle over a compact Astheno-Kähler manifold $(M,\omega)$. Let $\widetilde{\omega}$ be a Gauduchon metric conformal to $\omega$. If $c_1^{BC}(E)\cdot[\widetilde{\omega}^{n-1}]=0$. Then $E$ is a numerically flat vector bundle.*

**Proof** By Theorem 2.1, we just need to prove that $E$ is $\widetilde{\omega}$-semistable and $\mathrm{ch}_2^{BC}(E)\cdot[\omega^{n-2}]=0$. The proof of semistable is the same as the proof of [5, Theorem 1]. The key point is the vanishing of the second Chern number. Since $E$ is pseudo-effective, $\det(E)$ is a pseudo-effective line bundle (see [15, Corollary 1]). We know that $c_1^{BC}(E)$ is zero since $\det(E)$ is pseudo-effective and the first Chern number vanishes. By the Bogomolov inequality (see [14, Proposition 2.6]), we obtain

$$c_2^{BC}(E)\cdot[\omega^{n-2}]\geq\frac{r-1}{2r}c_1^{BC}(E)^2\cdot[\omega^{n-2}]=0. \tag{3.1}$$

On the other hand, we have

$$s_2^{BC}(E)\cdot[\omega^{n-2}]\geq 0, \tag{3.2}$$

which will be proved in Proposition 3.1. Combining the two Chern number inequalities, we conclude that $\mathrm{ch}_2^{BC}(E)\cdot[\omega^{n-2}]=\frac{1}{2}(c_1^{BC}(E)^2-2c_2^{BC}(E))\cdot[\omega^{n-2}]=0$. This completes the proof.

By the definition of pseudo-effectivity (Proposition 2.1), for every $\varepsilon>0$, there exists a singular metric $h_\varepsilon$ with analytic singularity on $\mathcal{O}_{\mathbb{P}(E)}(1)$, such that the curvature current

$$\frac{\mathrm{i}}{2\pi}\Theta(\mathcal{O}_{\mathbb{P}(E)}(1),h_\varepsilon)\geq-\varepsilon\pi^*(\mathrm{e}^{(n-1)\phi}\omega)$$

and $\pi(\mathrm{Sing}(h_\varepsilon))$ is not equal to $M$. By a classical result of Gauduchon [10], we can find a unique smooth function $\phi$ such that $\max_M \phi = 0$ and $\widetilde{\omega} = \mathrm{e}^{\phi}\omega$ is Gauduchon. Choose a smooth Hermitian metric $H$ on $E$, denoted by $h$ the induced metric on $\mathcal{O}_{\mathbb{P}(E)}(1)$, let $\alpha = c_1(\mathcal{O}_{\mathbb{P}(E)}(1), h)$ and consider the singular metric $h_\varepsilon = h\mathrm{e}^{-\varphi_\varepsilon}$, where $\varphi_\varepsilon$ is a function in $\mathbb{P}(E)$. Then we have

$$\alpha + \mathrm{i}\partial\overline{\partial}\varphi_\varepsilon + \varepsilon\pi^*(\mathrm{e}^{(n-1)\phi}\omega) \geq 0.$$

When the metric on $\mathcal{O}_{\mathbb{P}(E)}(1)$ is singular, we could not just follow Lemma 2.1, we define new currents $S_{\delta,\varepsilon}$ with bounded potential functions:

$$S_{\delta,\varepsilon} = \alpha + \varepsilon\pi^*(\mathrm{e}^{(n-1)\phi}\omega) + \mathrm{i}\partial\overline{\partial}\log(\mathrm{e}^{\varphi_\varepsilon} + \delta),$$

where $\delta$ is a positive constant. The $S_{\delta,\varepsilon}$ have a positive lower bound that does not depend on $\delta$, we have

$$S_{\delta,\varepsilon} + K\pi^*(\mathrm{e}^{(n-1)\phi}\omega) \geq 0.$$

By the definition of Monge-Ampère operators (see [7]), we know that

$$(S_{\delta,\varepsilon} + K\pi^*(\mathrm{e}^{(n-1)\phi}\omega))^{r+1} \geq 0.$$

By the positivity of $(S_{\delta,\varepsilon} + K\pi^*(\mathrm{e}^{(n-1)\phi}\omega))^{r+1}$ and some careful calculation, we have the following proposition.

**Proposition 3.1** *Let $E$ be a pseudo-effective vector bundle over a compact Astheno-Kähler manifold $(M,\omega)$. Let $\widetilde{\omega}$ be a Gauduchon metric conformal to $\omega$. If $c_1^{BC}(E)\cdot[\widetilde{\omega}^{n-1}] = 0$. Then $s_2^{BC}(E)\cdot[\omega^{n-2}] \geq 0$.*

**Proof**

$$\begin{aligned}
0 &\leq \int_{\mathbb{P}(E)} \pi^*\eta_\varepsilon(S_{\delta,\varepsilon} + K\pi^*(\mathrm{e}^{(n-1)\phi}\omega))^{r+1} \wedge \pi^*\omega^{n-2} \\
&= \int_{\mathbb{P}(E)} \pi^*\eta_\varepsilon S_{\delta,\varepsilon}^{r+1} \wedge \pi^*\omega^{n-2} + (r+1)K\int_{\mathbb{P}(E)} \pi^*\eta_\varepsilon S_{\delta,\varepsilon}^{r} \wedge \pi^*(\mathrm{e}^{\phi}\omega)^{n-1} \\
&\quad + \frac{r(r+1)K^2}{2}\int_M \eta_\varepsilon \mathrm{e}^{2(n-1)\phi}\omega^n.
\end{aligned} \tag{3.3}$$

For each $\varepsilon > 0$, we can choose a smooth function $0 \leq \eta_\varepsilon \leq 1$, which is equal to 1 in a domain of $\pi(Z_\varepsilon)$ ($Z_\varepsilon$ are singularities of $\varphi_\varepsilon$) such that

$$\frac{r(r+1)K^2}{2}\int_M \eta_\varepsilon \mathrm{e}^{2(n-1)\phi}\omega^n < \varepsilon. \tag{3.4}$$

Since the support of $1 - \pi^*\eta_\varepsilon$ belongs to $\mathbb{P}(E)\setminus Z_\varepsilon$, by the continuity of Monge-Ampère operators (see [7, Corollary 3.6]) along the bounded decreasing sequences, we can choose $\delta > 0$ small enough such that

$$\int_{\mathbb{P}(E)} (1 - \pi^*\eta_\varepsilon) S_{\delta,\varepsilon}^{r+1} \wedge \pi^*\omega^{n-2} > -\frac{\varepsilon}{3} \tag{3.5}$$

and

$$\int_{\mathbb{P}(E)} (1-\pi^*\eta_\varepsilon) S_{\delta,\varepsilon}^r \wedge \pi^*\widetilde{\omega}^{n-1} > -\frac{\varepsilon}{3K(r+1)}. \tag{3.6}$$

The reader can refer to [5, Page 529] for details.

$$\begin{aligned}
\int_{\mathbb{P}(E)} S_{\delta,\varepsilon}^r \wedge \pi^*(\mathrm{e}^{\phi}\omega)^{n-1} &= \int_{\mathbb{P}(E)} S_{\delta,\varepsilon}^r \wedge \pi^*(\widetilde{\omega})^{n-1} \\
&= \int_{\mathbb{P}(E)} (\alpha + \sqrt{-1}\partial\overline{\partial}\log(\mathrm{e}^{\varphi_\varepsilon}+\delta) + \varepsilon\pi^*(\mathrm{e}^{(n-1)\phi}\omega))^r \wedge \pi^*\widetilde{\omega}^{n-1} \\
&= \int_{\mathbb{P}(E)} (\alpha + \sqrt{-1}\partial\overline{\partial}\log(\mathrm{e}^{\varphi_\varepsilon}+\delta))^r \wedge \pi^*\widetilde{\omega}^{n-1} \\
&\quad + r\varepsilon \int_{\mathbb{P}(E)} (\alpha + \sqrt{-1}\partial\overline{\partial}\log(\mathrm{e}^{\varphi_\varepsilon}+\delta))^{r-1} \wedge \pi^*(\mathrm{e}^{\frac{2(n-1)}{n}\phi}\omega)^n \\
&= \int_{\mathbb{P}(E)} \alpha^r \wedge \pi^*\widetilde{\omega}^{n-1} + r\varepsilon \int_{\mathbb{P}(E)} \alpha^{r-1} \wedge \pi^*(\mathrm{e}^{\frac{2(n-1)}{n}\phi}\omega)^n \\
&= \int_M s_1(E,H) \wedge \widetilde{\omega}^{n-1} + r\varepsilon \int_M \mathrm{e}^{2(n-1)\phi}\omega^n \\
&= r\varepsilon \int_M \mathrm{e}^{2(n-1)\phi}\omega^n.
\end{aligned} \tag{3.7}$$

The fourth equality comes from $\partial\overline{\partial}\pi^*(\mathrm{e}^{\frac{2(n-1)}{n}\phi}\omega)^n = \pi^*\partial\overline{\partial}(\mathrm{e}^{\frac{2(n-1)}{n}\phi}\omega)^n = 0$, $\partial\overline{\partial}\widetilde{\omega}^{n-1} = 0$ and Stoke's theorem. The fifth equality comes from Theorem 2.1, in the last equality, we use the condition: $s_1^{BC}(E)\cdot[\widetilde{\omega}^{n-1}] = -c_1^{BC}(E)\cdot[\widetilde{\omega}^{n-1}] = 0$. Combining with (3.3)–(3.7), we get

$$0 \le \varepsilon + r(r+1)K\varepsilon \int_M \mathrm{e}^{2(n-1)\phi}\omega^n + \frac{2\varepsilon}{3} + \int_{\mathbb{P}(E)} S_{\delta,\varepsilon}^{r+1} \wedge \pi^*\omega^{n-2}. \tag{3.8}$$

Now we would calculate the second Segre number $s_2^{BC}(E)\cdot[\omega^{n-2}]$. Since $\partial\overline{\partial}\omega^{n-2} = 0$, we have

$$s_2^{BC}(E)\cdot[\omega^{n-2}] = \int_M s_2(E,H) \wedge \omega^{n-2}, \tag{3.9}$$

where $H$ is a arbitrary Hermitian metric on $E$ and the second Segre number does not depend on the Hermitian metric $H$.

$$\begin{aligned}
\int_M s_2(E,H) \wedge \omega^{n-2} &= \int_{\mathbb{P}(E)} \alpha^{r+1} \wedge \pi^*\omega^{n-2} \\
&= \int_{\mathbb{P}(E)} (\alpha + \mathrm{i}\partial\overline{\partial}\log(\mathrm{e}^{\varphi_\varepsilon}+\delta))^{r+1} \wedge \pi^*\omega^{n-2} \\
&= \int_{\mathbb{P}(E)} (S_{\delta,\varepsilon} - \varepsilon\pi^*(\mathrm{e}^{(n-1)\phi}\omega))^{r+1} \wedge \pi^*\omega^{n-2} \\
&= \int_{\mathbb{P}(E)} S_{\delta,\varepsilon}^{r+1} \wedge \pi^*\omega^{n-2} - (r+1)\varepsilon \int_{\mathbb{P}(E)} S_{\delta,\varepsilon}^r \wedge \pi^*(\mathrm{e}^{\phi}\omega)^{n-1} \\
&\quad + \frac{r(r+1)}{2}\varepsilon^2 \int_M S_{\delta,\varepsilon}^{r-1} \wedge \pi^*(\mathrm{e}^{2(n-1)\phi}\omega^n).
\end{aligned} \tag{3.10}$$

It is easy to see that

$$\int_{\mathbb{P}(E)} S_{\delta,\varepsilon}^{r-1} \wedge \pi^*(\mathrm{e}^{2(n-1)\phi}\omega^n) = \int_{\mathbb{P}(E)} \alpha^{r-1} \wedge \pi^*(\mathrm{e}^{2(n-1)\phi}\omega^n) = \int_M \mathrm{e}^{2(n-1)\phi}\omega^n. \tag{3.11}$$

So the integration above is bounded. Combining with (3.10)–(3.11) and (3.7), we obtain

$$\int_M s_2(E,H) \wedge \omega^{n-2} = \int_{\mathbb{P}(E)} S_{\delta,\varepsilon}^{r+1} \wedge \pi^*\omega^{n-2} - r(r+1)\varepsilon^2 \int_M \mathrm{e}^{2(n-1)\phi}\omega^n + \frac{r(r+1)}{2}\varepsilon^2 \int_M \mathrm{e}^{2(n-1)\phi}\omega^n. \tag{3.12}$$

Conmbining with (3.12) and (3.8)–(3.9), let $\varepsilon \to 0$, we have

$$s_2^{BC}(E) \cdot [\omega^{n-2}] \geq 0. \tag{3.13}$$

**Remark 3.1** By [14, Theorem 1.4], there exists a filtration

$$0 = E_0 \subseteq \cdots \subseteq E_s = E$$

by subbundles whose quotients are Hermitian flat. Hence, all Chern classes $c_k(E)$ and Segre classes $s_k(E)$ vanish. Unlike the Gauduchon metrics the Astheno-Kähler metrics impose some constraints on the underlying manifold. It is still unknown that is a numerically flat vector bundle equivalent to the existence of a filtration

$$0 = E_0 \subseteq \cdots \subseteq E_s = E$$

by subbundles whose quotients are Hermitian flat in non-Astheno-Kähler complex manifold.

## Declarations

**Conflicts of interest** The authors declare no conflicts of interest.

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