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Extremal Kähler Metrics of Toric Manifolds*

An-Min LI[1] Li SHENG[1]

Abstract This paper is a survey of some recent developments concerning extremal Kähler metrics on Toric Manifolds.

Keywords Extremal Kähler metric, Toric manifolds, $\widehat{K}$-stability, Uniform stability
2000 MR Subject Classification 53C21

1 Introduction

Canonical metrics on Kähler manifolds are a fundamental problem in complex geometry. Given a compact Kähler manifold (M, ω_0), a Kähler metric $\omega \in [\omega_0]$ is called extremal if it is a critical point of the functional

$$f(\omega) = \int_M [\mathcal{S}(\omega)]^2 \frac{\omega^n}{n!},$$

where $\mathcal{S}(\omega)$ is the scalar curvature. The gradient of the scalar curvature being a holomorphic vector field is the Euler-Lagrange equation for this variational problem. If $\mathcal{S}(\omega) \equiv$ constant, then ω is an extremal metric.

Extremal metrics were introduced by Calabi and have been studied extensively for the last 30 years. Tian provided an analytic stability condition that he proved to be equivalent to the existence of a Kähler-Einstein metric, and also defined the algebro-geometric notion of K-stability (see [24]). Donaldson extended Tian's definition of K-stability by giving an algebro-geometric definition of the Futaki invariant and conjectured that it is equivalent to the existence of a constant scalar curvature Kähler (cscK for short) metric (see [12]). The Yau-Tian-Donaldson conjecture states that a manifold M admits a cscK metric in the class $c_1(L)$ if and only if (M, L) is K-stable. This conjecture was generalized to extremal metrics.

In recent years, several researchers have made progress towards this conjecture, including Chen-Cheng, Darvas-Lu and Li (see [3, 8–9, 10–11, 18, 20]).

Donaldson [12] initiated a program to study extremal metrics on toric manifolds and formulated K-stability for polytopes. He conjectured that stability implies the existence of the cscK metric on toric manifolds, and solved the problem for cscK metrics on toric surfaces (see [14]). Chen, Li and Sheng solved the problem for extremal metrics for toric surfaces (see [7]) and generalized the result to homogeneous toric bundles (see [5]). Recently, Chen, Cheng and Li proved the existence of cscK metrics under uniform stability conditions for toric manifolds of any dimension. However, the conjecture is still open in general.

Manuscript received May 3, 2023. Revised July 9, 2023.
[1]Department of Mathematics, Sichuan University, Chengdu 610065, China.
E-mail: anminliscu@126.com lsheng@scu.edu.cn
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Examples from Apostolov, Calderbank, Gauduchon and Friedman [2] suggest that the condition of K-stability may need to be modified for polarized manifolds. Székelyhidi introduced $\widehat{K}$-stability and proved that (M, L) is $\widehat{K}$-stable when M admits a cscK metric in $c_1(L)$ and has a discrete automorphism group (see [23]). He formulated a variant of the Yau-Tian-Donaldson conjecture, which states that the manifold M admits an extremal metric in $c_1(L)$ if and only if (M, L) is $\widehat{K}$-stable.

Donaldson suggested in [15] that Szekelyhidi's definition (see [23]) gives the correct formulation of the YTD conjecture. Recently, Li–Lian–Sheng solved the Yau-Tian-Donaldson conjecture of the filtration version for toric manifolds (see [19]).

Theorem 1.1 *Let (M, ω) be an n-dimensional compact toric manifold, and Δ be its Delzant polytope. A is a smooth function on $\overline{\Delta}$. Then (Δ, A) is $\widehat{K}$-stable if and only if there exists a smooth $\mathbb{T}^n$-invariant metric $\mathfrak{g}$ on M such that the scalar curvature of $\mathfrak{g}$ is A.*

Let A be a constant or a linear function on Δ. As a consequence we have solved the Yau-Tian-Donaldson conjecture of the filtration version for n-dimensional toric manifolds (see also the feature article [16]).

2 Delzant Polytope and Toric Manifolds

2.1 Delzant polytope

A toric manifold is a symplectic manifold (M, ω) of dimension $2n$ that admits an effective n-torus $\mathbb{T}^n$-Hamiltonian action.

The torus action gives rise to a moment map $\tau : M \to \mathfrak{t}^*$, where $\mathfrak{t} \cong \mathbb{R}^n$ is the Lie algebra of $\mathbb{T}^n$ and $\mathfrak{t}^*$ is its dual. The image $\overline{\Delta} = \tau(M)$ is a convex polytope in $\mathfrak{t}^*$ known as the Delzant polytope of M.

Definition 2.1 *A convex polytope Δ in $\mathfrak{t}^*$ is a Delzant polytope if*:

(I) *There are n edges meeting at each vertex*;

(II) *the edges meeting at the vertex p are rational, i.e., each edge is of the form $p + tv_i$, $0 \le t \le \infty$, where $v_i \in \mathbb{Z}^n$*;

(III) *the $v_1, \cdots, v_n$ in item* (II) *can be chosen to be a basis of $\mathbb{Z}^n$.*

Example 2.1 The following is some examples of Delzant polotopes and a non-Delzant polotope.

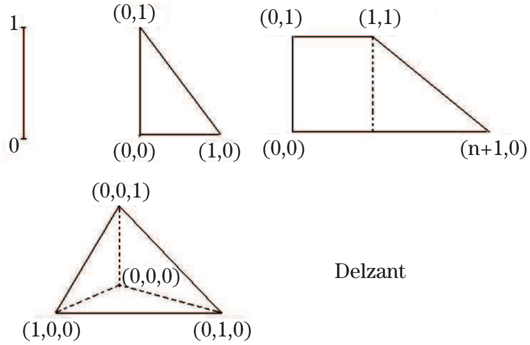

Figure 1 Delzant polotope.

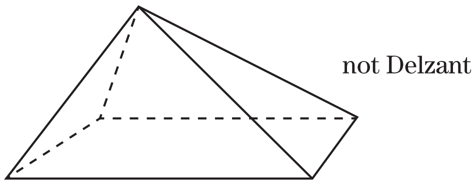

Figure 2 Non-Delzant polotope.

Delzant's theorem is a fundamental result in symplectic geometry that relates toric manifolds to Delzant polytopes. It provides a powerful tool for studying toric manifolds and their geometry, and the connection has important applications in various areas of mathematics and physics.

Delzant's theorem establishes a correspondence between toric manifolds and Delzant polytopes. Given a Delzant polytope Δ, Delzant associates a closed connected symplectic manifold (M_Δ, ω) of dimension $2n$ together with a Hamiltonian $\mathbb{T}^n$-action $\tau : T^n \to \mathrm{Diff}(M_\Delta, \omega)$ whose moment map $\mu_\tau : M_\Delta \to \mathfrak{t}^*$ satisfies $\mu_\tau(M_\Delta) = \Delta$.

The Delzant theorem states that toric manifolds are classified by Delzant polytopes, which means that there is a one-to-one correspondence between toric manifolds and Delzant polytopes given by the map

$$\{\text{Toric manifolds}\} \leftrightarrow \{\text{Delzant polytopes}\}$$
$$\{M^{2n}, \omega, T^n, \tau\} \mapsto \tau(M).$$

Example 2.2 The following is some examples of toric manifolds and Delzant polytope.

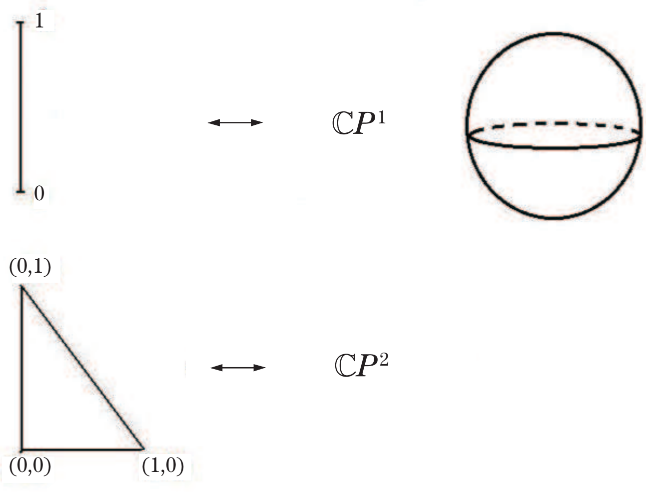

Figure 3 $\mathbb{C}P^1$ and $\mathbb{C}P^2$.

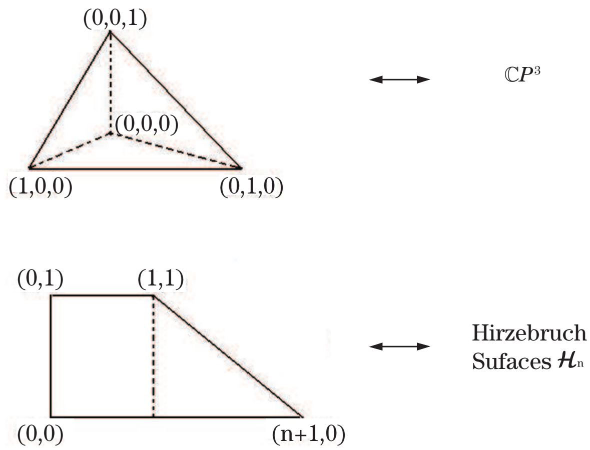

Figure 4 $\mathbb{C}P^3$ and Hirzebruch surfaces.

2.2 Abreu's equation

There are two natural types of local coordinates on a toric manifold: Complex log affine coordinates $z \in \mathbb{C}^n$ and symplectic (Darboux) coordinates $\xi \in \Delta \subset \mathbb{R}^n$. The open dense subset M° of M is defined as the set of points where the $\mathbb{T}^n$-action is free. In complex coordinates, M° is given by

$$M^\circ = \mathbb{R}^n \times i\mathbb{T}^n = x + iy : x \in \mathbb{R}^n, y \in \mathbb{R}^n/\mathbb{Z}^n,$$

where the $\mathbb{T}^n$-action is given by $t \cdot (x + iy) = x + i(y + t)$ and the complex structure J is multiplication by i. The Kähler form is given by a potential $f \in C^\infty(M^\circ)$, which depends only on the x coordinates: $f = f(x) \in C^\infty(\mathbb{R}^n)$. Since f is a smooth strictly convex function on $\mathfrak{t}$, its gradient defines a normal map ∇^f from $\mathfrak{t}$ to $\mathfrak{t}^*$, given by

$$\xi = \nabla^f(x) = \Big(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}\Big).$$

Let $u = L(f)$ be the Legendre transformation of f.

In symplectic coordinates, M° can be described as

$$M^\circ = \{(\xi, y) : \xi \in \Delta \subset \mathbb{R}^n, y \in \mathbb{R}^n/\mathbb{Z}^n\},$$

where the $\mathbb{T}^n$-action is given by $t \cdot (\xi, y) = (\xi, y + t)$. The moment map, restricted to Δ, is given by

$$\tau_{\mathfrak{f}} : \Delta \to \mathfrak{t} \xrightarrow{\nabla^f} \Delta,$$

$$(z_1, \cdots, z_n) \mapsto (x_1, \cdots, x_n) \mapsto (\xi_1, \cdots, \xi_n).$$

Let M be a compact complex manifold equipped with a torus action by $\mathbb{T}^n$. Let $C^\infty_{\mathbb{T}^n}(M)$ denote the set of smooth $\mathbb{T}^n$-invariant functions on M, and let ω_g be a Kähler form on M.

Define

$$\mathcal{C}^\infty(M,\omega_g)=\Big\{\phi\in C^\infty_{\mathbb{T}^n}(M)\,\Big|\,\omega_g+\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\phi>0\Big\}.$$

Given a polytope Δ in $\mathbb{R}^n$ with facets $\ell_1,\cdots,\ell_d$, let ν_i be the inward pointing normal vector to the facet ℓ_i, and let $l_i(\xi)=\sum\limits_j \xi_j\nu_i^j-\lambda_i$ be the equation for the facet ℓ_i. Then

$$\Delta=\{\xi\mid l_i(\xi)>0,0\le i\le d-1\}.$$

Let $v(\xi)=\sum\limits_i l_i(\xi)\log l_i(\xi)$ be the Guillemin symplectic potential function. Define

$$\mathcal{C}^\infty(M,\omega_g)=\Big\{\phi\in C^\infty_{\mathbb{T}^n}(M)\,\Big|\,\omega_g+\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\phi>0\Big\}.$$

We say a convex function u satisfies "Guillemin's boundary condition" if $u\in\mathcal{C}^\infty(\Delta,v)$.

In terms of coordinates ξ and Legendre transform function u of f, the scalar curvature can be written as

$$R(u)=-\Sigma U^{ij}w_{ij},$$

where (U^{ij}) is the cofactor matrix of the Hessian matrix (u_{ij}), and $w=(\det(u_{ij}))^{-1}$.

It is well known that ω_f gives an extremal metric if and only if $R(u)$ is a linear function of Δ. Let A be a smooth function on $\overline{\Delta}$. The Abreu equation in [1] is given by

$$-\Sigma U^{ij}w_{ij}=A.$$

2.3 Extremal metric on toric manifolds

Guillemin constructed a natural Kähler form ω_g on a toric symplectic manifold M and denoted its class by $[\omega_g]$. This metric is known as the Guillemin metric. Let $v=L(g)$, where g is the potential function of the Guillemin metric.

The following theorem is due to Guillemin.

Theorem 2.1 (see Guillemin [17]) *Suppose that Δ is defined by linear inequalities $\langle\xi,v_i\rangle-\lambda_i>0$, where v_i is the inward pointing normal vector to the facet F_i of Δ, and $\langle\xi,v_i\rangle-\lambda_i=0$ defines the facet. Write $l_i(\xi)=\langle\xi,v_i\rangle-\lambda_i$. Then*

$$v(\xi)=\sum_i l_i\log l_i.$$

The problem of prescribing scalar curvature for the Guillemin metric reduces to finding a smooth convex solution u in Δ for the 4-th order PDE

$$-\Sigma U^{ij}w_{ij}=A \tag{2.1}$$

subject to the boundary condition

$$u-\sum_i l_i\log l_i\in C^\infty(\overline{\Delta})$$

(see [13, 22]).

3 K-stability and Existence of Extremal Metrics

3.1 K-stability and existence of extremal metrics

For any smooth function A on $\overline{\Delta}$, Donaldson defines a functional on $\mathcal{C}^\infty(\Delta)$ as follows:

$$\mathcal{F}_A(u) = -\int_\Delta \log\det(u_{ij})\mathrm{d}\mu + \mathcal{L}_A(u),$$

where $\mathcal{L}_A$ is the linear functional

$$\mathcal{L}_A(u) = \int_{\partial\Delta} u\mathrm{d}\sigma - \int_\Delta Au\mathrm{d}\mu.$$

When A is constant, the functional $\mathcal{F}_A$ is called the Mabuchi functional, and $\mathcal{L}_A$ is the Futaki invariants.

We introduce several classes of functions.

Denote $\mathcal{P}$ as the set of rational piecewise linear convex functions on $\overline{\Delta}$.
Set

$$\begin{aligned}
\mathcal{C} &= \{u \in C(\overline{\Delta}) :\ u \text{ is convex on } \overline{\Delta} \text{ and smooth in } \Delta\},\\
\mathbf{S} &= \left\{u \in C(\overline{\Delta}) :\ u \text{ is convex on } \overline{\Delta} \text{ and } u - v \text{ is smooth on } \overline{\Delta}\right\}.
\end{aligned}$$

For a fixed point $p_o \in \Delta$, we consider

$$\mathcal{P}_{p_o} = \{u \in \mathcal{P} : u \geq u(p_o) = 0\}, \tag{3.1}$$
$$\mathcal{C}_{p_o} = \{u \in \mathcal{C} : u \geq u(p_o) = 0\}, \tag{3.2}$$
$$\mathbf{S}_{p_o} = \{u \in \mathbf{S} : u \geq u(p_o) = 0\}. \tag{3.3}$$

We say functions in $\mathcal{P}_{p_o}$, $\mathcal{C}_{p_o}$ and $\mathbf{S}_{p_o}$ are normalized at p_o. Let

$$\mathcal{C}_* = \left\{ \begin{array}{l} u \mid \text{there exists a constant } C > 0 \text{ and a sequence of } u_k \text{ in } \mathcal{P}_{p_o} \text{ such} \\ \text{that } \int_{\partial\Delta} u_k\mathrm{d}\sigma < C \text{ and } u_k \text{ locally uniformly converges to } u \text{ in } \Delta \end{array} \right\}.$$

Let $P > 0$ be a constant, we define

$$\mathcal{C}_*^P = \left\{u \in \mathcal{C}_* \middle| \int_{\partial\Delta} u\mathrm{d}\sigma \leq P\right\}.$$

Definition 3.1 *Let $A \in C^\infty(\overline{\Delta})$ be a smooth function on $\overline{\Delta}$. (Δ, A) is called K-stable if $\mathcal{L}_A(u) \geq 0$ for all rational piecewise-linear convex functions u and*

$$\mathcal{L}_A(u) = 0$$

if and only if u is a linear function.

Donaldson also introduced a stronger version of stability, which we call uniform stability. We fix a point $p \in \Delta$ and say that u is normalized at p if $u \geq u(p) = 0$.

Definition 3.2 (Δ, A) *is called uniformly stable if there is a constant* $\lambda > 0$ *such that for any normalized convex function* $u \in \mathcal{C}^{\infty}(\Delta)$*, we have*

$$\mathcal{L}_A(u) \geq \lambda \int_{\partial\Delta} u.$$

Sometimes we say that Δ *is* (A, λ)*-stable.*

It is easy to show that uniform stability implies K-stability.

For any dimension, Chen, Li and Sheng have proved in [6] the following theorem.

Theorem 3.1 *If the Abreu equation* (2.1) *has a smooth solution in* $\mathbf{S}(\Delta, v)$*, then* (Δ, A) *is uniformly stable.*

3.2 Test configurations and K-stability

Consider a compact complex manifold X with an ample line bundle L.

Definition 3.3 *A test configuration for* (X, L) *of exponent* r *consists of*:

(1) *A scheme* χ *with a* $\mathbb{C}^*$*-action.*

(2) *A* $\mathbb{C}^*$*-equivariant line bundle* $\mathcal{L} \to \chi$.

(3) *A flat* $\mathbb{C}^*$*-equivariant map* $\pi : \chi \to \mathbb{C}$*, where* $\mathbb{C}^*$ *acts on* $\mathbb{C}$ *by multiplication in the standard way, such that any fibre* $\chi_t = \pi^{-1}(t)$ *for* $t \neq 0$ *is isomorphic to* X *and the pair* (X, L^r) *is isomorphic to* $(\chi_t, \mathcal{L}_{\chi_t})$.

The number r *is called the exponent of the test-configuration. The* $\mathbb{C}^*$*-action on* χ *induces an action on the central fibre*: *The scheme* $X_0 = \pi^{-1}(0)$.

A test configuration for (M, L) of exponent $r > 0$ can be constructed by embedding M into $\mathbb{C}P^{N_r}$ using a basis of sections of L^r and a $\mathbb{C}^*$-subgroup of $GL(N_r + 1, \mathbb{C})$.

For toric varieties, Donaldson showed that any rational piecewise linear convex function on the moment polytope gives rise to a test-configuration of the variety.

Definition 3.4 *The pair* (X, L) *is* K*-stable if for each test configuration for* (X, L)*, the Futaki invariant of the induced action on* $(\chi_0, \mathcal{L}_{X_0})$ *is* ≥ 0 *with equality if and only if the test configuration is trivial.*

3.3 Filtrations

Let (X, L) be a polarized manifold. We denote $R_k = H^0(X, L^k)$, and

$$R = \bigoplus_{k \geq 0} R_k = \bigoplus_{k \geq 0} H^0(X, L^k)$$

as the homogeneous coordinate ring of (X, L) (see [4, 21, 23]).

Definition 3.5 *A filtration of* R *is a chain of finite-dimensional subspaces*

$$C = F_0R \subset F_1R \subset F_2R \subset \cdots \subset R,$$

such that the following conditions hold:

(1) *The filtration is multiplicative, i.e.,*

$$(F_iR)(F_jR) \subset F_{i+j}R$$

for all $i, j \geq 0$;

(2) *The filtration is compatible with the grading* R_k *of* R, *i.e., if* $f \in F_i R$ *for some* $i \geq 0$, *then each homogeneous piece of* f *is also in* $F_i R$;

(3) $\bigcup_{i \geq 0} F_i R = R$.

Given a filtration χ of R, the Rees algebra of χ is defined as

$$\mathrm{Rees}(\chi) = \bigoplus_{i \geq 0} (F_i R) t^i \subset R[t].$$

The associated graded algebra of χ is defined as

$$gr(\chi) = \bigoplus_{i \geq 0} (F_i R)/(F_{i-1} R),$$

where $F_{-1} R = 0$.

The fiber of the Rees algebra of χ at non-zero t is isomorphic to R, while the fiber at $t = 0$ is isomorphic to $gr(\chi)$.

We call a filtration finitely generated if its Rees algebra is finitely generated. The main advantage of considering filtrations instead of test configurations is that filtrations are more general, as they are not all necessarily finitely generated.

3.4 Filtration for toric manifolds

For toric varieties, Donaldson showed that any rational piecewise linear convex function defined on the moment polytope corresponds to a test-configuration of the variety. Székelyhidi showed that any positive convex function on the polytope gives rise to a filtration of the homogeneous coordinate ring.

Suppose that $f : \overline{\Delta} \to \mathbb{R}$ is a positive convex function, where Δ is the moment polytope corresponding to the polarized toric variety (X, L). A basis of sections of $H^0(X, L^k)$ can be identified with the rational lattice points in $\Delta \bigcap \frac{1}{k}\mathbb{Z}^n$. If $\alpha \in \Delta \bigcap \frac{1}{k}\mathbb{Z}^n$, we write s_α for the corresponding section of L^k.

Now, on $R^k = H^0(X, L^k)$, define the filtration as follows:

$$F_i R^k = \mathrm{span}\{s_\alpha : k f(\alpha) \leq i\}.$$

The convexity of f ensures that the filtration of the graded ring of (X, L) defined in this way satisfies the multiplicative property. The other two conditions also follow easily.

Let $f_k : \Delta \to \mathbb{R}$ be the largest convex function that, on the points $\alpha \in \Delta \bigcap \frac{1}{k}\mathbb{Z}^n$, is defined by

$$f_k(\alpha) = \frac{1}{k} \lceil k f(\alpha) \rceil.$$

f_k is a rational piecewise-linear approximation to the function f.

Donaldson showed that the test-configuration corresponding to f_k has Futaki invariant

$$\mathcal{L}_a(f_k) = \int_{\partial \Delta} f_k \mathrm{d}\sigma - a \int_{\Delta} f_k \mathrm{d}\mu,$$

where $\mathrm{d}\sigma$ is a certain measure on the boundary and a is a normalizing constant. Since f_k is a decreasing sequence of functions converging to f pointwise, we have

$$\lim_{k\to\infty}\mathcal{L}_a(f_k)=\int_{\partial\Delta}f\mathrm{d}\sigma-a\int_{\Delta}f\mathrm{d}\mu.$$

Definition 3.6 *Let $A\in C^\infty(\overline{\Delta})$ be a smooth function on $\overline{\Delta}$. (Δ,A) is called $\widehat{K}$-stable if $\mathcal{L}_A(u)\geq 0$ for all convex functions $u:\overline{\Delta}\to\mathbb{R}$, and $\mathcal{L}_A(u)=0$ if and only if u is a linear function in Δ.*

Since test configurations and filtrations are algebraic objects, both K-stability and $\widehat{K}$-stability are algebraic conditions. It is evident that uniform stability implies $\widehat{K}$-stability. However, the converse is not straightforward.

4 Proof of Theorem 1.1

Let Δ be a n-dimensional convex polytope and A be a scalar function on $\overline{\Delta}$. We define $\mathcal{Q}$ to be the set of lower semi-continuous functions u in $\mathcal{C}_*^P$ such that

$$\int_{\partial\Delta}u\mathrm{d}\sigma=1,\quad u(o)=\inf_{\overline{\Delta}}u=0,$$

where o is the center of Δ. We have the following results (see [19]).

Theorem 4.1 *Suppose (Δ,A) is $\widehat{K}$-stable. If $u\in\mathcal{Q}$ and $\mathcal{L}_A(u)=0$, then $u\in L^\infty(\overline{\Delta})$.*

Theorem 4.2 *Let (M,ω) be an n-dimensional compact toric manifold with Delzant polytope Δ. Then (Δ,A) is $\widehat{K}$-stable if and only if there exists $\lambda>0$ such that (Δ,A) is uniformly K-stable.*

To prove Theorem 1.1, we use the continuity method. Let $I=[0,1]$ be the unit interval. At $t=0$, we start with the Guillemin metric, and let A_0 be its scalar curvature on Δ. Then Δ must be (A_0,λ_0)-stable for some constant $\lambda_0>0$.

We then define $A_t=tA+(1-t)A_0$ and $\lambda_t=t\lambda+(1-t)\lambda_0$. It is easy to verify that Δ is (A_t,λ_t)-stable for any $t\in[0,1]$. We define

$$\Lambda=\{t\mid\mathcal{S}(u)=A_t\text{ has a solution in }\mathbf{S}\},$$

and we need to show that Λ is both open and closed.

Openness of Λ is standard. To show that Λ is closed, we use Chen-Cheng's calculation to obtain the following theorem.

Theorem 4.3 *If Δ is (A,λ)-stable, then the entropy $\int_M F\mathrm{e}^F\omega_g^n$ is bounded, where $F=\log\frac{\omega_f^n}{\omega_g^n}$.*

Using the results of Chen-Cheng, we obtain $[0,1]\subset\Lambda$.

Declarations

Conflicts of interest The authors declare no conflicts of interest.

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