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Survey on Path-Dependent PDEs*

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Abstract In this paper, the authors provide a brief introduction of the path-dependent partial differential equations (PDEs for short) in the space of continuous paths, where the path derivatives are in the Dupire (rather than Fréchet) sense. They present the connections between Wiener expectation, backward stochastic differential equations (BSDEs for short) and path-dependent PDEs. They also consider the well-posedness of path-dependent PDEs, including classical solutions, Sobolev solutions and viscosity solutions.

Keywords Path-Dependent, Wiener expectation, BSDEs, Classical solution, Sobolev solution, Viscosity solution
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1 Introduction

Let us begin with the space of d-dimensional continuous paths $\Omega = C_0([0,T];\mathbb{R}^d)$ starting from origin, and equipped with the uniform norm

$$\|\omega\| := \max_{s\in[0,T]} |\omega_s|, \quad \omega \in \Omega,$$

where $T > 0$ is a fixed constant. Suppose that $\mathcal{B}(\Omega)$ is the Borel σ-algebra of Ω and $B_t(\omega) := \omega_t$ is the canonical process. In 1923, Wiener [41] introduced a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{B}(\Omega))$, under which the canonical process is a d-dimensional standard Brownian motion, i.e., B is incrementally stable and incrementally independent such that $\mathbb{E}[B_s B_t^\top] = \min(s,t) I_d$ for any $s,t \in [0,T]$. The probability measure $\mathbb{P}$ is called the Wiener measure, the space Ω is called the Wiener space and a function defined on Ω is called a Wiener functional. Afterwards, many interesting and important probability models has been realized as Wiener functionals on Wiener space, for example, diffusion processes.

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The present paper is a survey to the recent developments of the following type of path-dependent PDEs,

$$\begin{cases} D_t u(t,\omega) + \frac{1}{2}\operatorname{tr}(D_x^2 u(t,\omega)) = 0, \quad (t,\omega) \in [0,T) \times \Omega, \\ u(T,\omega) = \xi(\omega), \end{cases} \tag{1.1}$$

and its generalization to quasilinear cases, see (6.3)–(6.4). Here D_t, D_x and D_x^2 are not in the Fréchet sense but in the Dupire's horizontal/vertical sense (see Section 3 for details). Moreover, the solution $u = u(t,\omega)$ depends only on the past history path rather than the whole path at time t, i.e.,

$$u(t,\omega) = u(t,\omega_{\cdot\wedge t}) \text{ is non-anticipative.}$$

To our best knowledge, the (first-order) path-dependent PDE has been studied by Lukoyanov [21], which was called functional Hamilton-Jacobi equation in the paper. Independently, the term "path-dependent PDE" was proposed in Peng's talk in ICM2010 (see [28, p. 400]), which provides a one-to-one correspondence between path-dependent PDEs and BSDEs (or G-martingales). In fact, a Markovian BSDE can be seen as a semi-linear parabolic PDE via the nonlinear Feynman-Kac formula introduced by Peng [24] and Pardoux and Peng [23]. A main observation of [28] is that the solution to a general BSDE is in fact a non-anticipative process on Wiener space. Thus a very interesting and long-standing problem is to interpret a classical BSDE in the sense of Pardoux and Peng [22] as a path-dependent PDE.

The presentation of Dupire's path derivatives have promoted the development of path-dependent PDEs. Indeed, with the help of Dupire's derivatives and BSDEs theory, Peng and Wang [32] established the well-posedness of classical solutions to systems of semi-linear path-dependent PDEs provided that some regularity conditions are satisfied. A more general framework of semi-linear path-dependent parabolic integro-differential equations have been studied in [16, 40]. However, the path-dependent PDEs rarely have classical solutions due to the absence of any regularizing effect, which is different from the finite-dimensional parabolic case. Therefore, much research is devoted to various types of weaker notions of solutions of the path-dependent PDEs.

Based on a space of smooth, cylindrical and non-anticipative processes, Peng and Song [31] introduced $\mathbb{P}$-weighted Sobolev spaces and the corresponding Sobolev path derivatives. In this framework, each solution of the BSDE is identified with the Sobolev solution of the corresponding semi-linear path-dependent PDE. In particular, it removes the smooth assumptions on the terminal conditions compared with the classical solutions case.

On the other hand, the theory of viscosity solutions for Hamilton-Jacobi-Bellman equations introduced by Crandall and Lions [9] is a fundamentally important approach in the research of PDEs theory. However, the extension of the viscosity solution to the second order path-dependent PDEs is still to be further explored. In this case the space of Ω is infinite dimensional and lacks local compactness, which results in the main difficulty. Various types of notions of viscosity solutions to deal with the uniqueness and existence results have been introduced, see Section 7 for details.

The paper is organized as follows. In Section 2, we start with a revisit to Wiener expectation to illustrate the main idea of path-depend PDEs. Section 3 is devoted to the introduction of Dupire's derivatives. In Sections 4–5, we introduce classical solutions and Sobolev solutions of (systems of) path-dependent heat equations, respectively. In Section 6, we discuss semi-linear path-dependent PDEs via BSDEs in Wiener space and the development of the notion of fully nonlinear path-dependent PDEs and the related BSDEs. In Section 7, we give a brief introduction of the developments of viscosity solutions.

2 Revisit to Wiener Expectation

In this paper, we denote by $\langle\cdot,\cdot\rangle$ and $|\cdot|$ the scalar product and the associated norm of an Euclidian space, respectively. For a given set of parameters α, $C(\alpha)$ will denote a positive constant only depending on these parameters which may change from line to line.

In what follows, we provide an elementary way to introduce the Wiener expectation $\mathbb{E}[\cdot]$ (expectation associated to the Wiener measure) on the measurable space $(\Omega,\mathcal{B}(\Omega))$ from the view of path-dependent PDE. Many more powerful nonlinear expectations such as G-expectations may be established in this way (see [26–27, 29]).

Set $\Omega_t := \{\omega_{\cdot\wedge t} : \omega \in \Omega\}$ and denote by $\mathcal{B}(\Omega)$ (resp. $\mathcal{B}(\Omega_t)$) the Borel σ-algebra of Ω (resp. Ω_t) for each $t\in[0,T]$. Consider the following space of cylindrical functions on Wiener space:

$$L_{\mathrm{ip}}(\Omega_t) := \{\varphi(B_{t_1},\cdots,B_{t_k}) : k\in\mathbb{N},\ t_1<\cdots<t_k\in[0,t],\ \varphi\in C_{b.\mathrm{Lip}}(\mathbb{R}^{k\times d})\},$$

and $L_{\mathrm{ip}}(\Omega) := L_{\mathrm{ip}}(\Omega_T)$, where $C_{b.\mathrm{Lip}}(\mathbb{R}^{k\times d})$ denotes the space of all bounded and Lipschitz functions on $\mathbb{R}^{k\times d}$. Then we could define the Wiener expectation $\mathbb{E}[\xi]$ for each $\xi\in L_{\mathrm{ip}}(\Omega)$.

Indeed, for each $\xi\in L_{\mathrm{ip}}(\Omega)$ with the form of

$$\xi = \varphi(B_{t_1},B_{t_2},\cdots,B_{t_k}),\quad 0=t_0<t_1<\cdots<t_k=T,$$

and for each $t\in[t_{i-1},t_i]$, $i=1,\cdots,k$, we define the conditional expectation by

$$\mathbb{E}_t[\xi] := u_i(t,B_t;B_{t_1},\cdots,B_{t_{i-1}}). \tag{2.1}$$

Here, the function $u_i(t,x;x_1,\cdots,x_{i-1})$ with parameters $(x_1,\cdots,x_{i-1})\in\mathbb{R}^{(i-1)\times d}$ is the solution of the following heat equation:

$$\partial_t u_i(t,x;x_1,\cdots,x_{i-1}) + \frac{1}{2}\,\mathrm{tr}(\partial_x^2 u_i(t,x;x_1,\cdots,x_{i-1})) = 0,\quad (t,x)\in[t_{i-1},t_i)\times\mathbb{R}^d \tag{2.2}$$

with terminal conditions

$$u_i(t_i,x;x_1,\cdots,x_{i-1}) = u_{i+1}(t_i,x;x_1,\cdots,x_{i-1},x)\quad \text{for } i<k,$$

and $u_k(t_k, x; x_1, \cdots, x_{k-1}) = \varphi(x_1, \cdots, x_{k-1}, x)$. The Wiener expectation of ξ is defined by $\mathbb{E}[\xi] = \mathbb{E}_0[\xi]$. From the properties of PDE (2.2), we have the following result.

Lemma 2.1 *The mappings* $\mathbb{E}_t[\cdot] : L_{\text{ip}}(\Omega) \to L_{\text{ip}}(\Omega_t)$ *satisfy the following*: *For each* $\xi, \eta \in L_{\text{ip}}(\Omega)$,

(i) *monotonicity*: $\mathbb{E}_t[\xi] \geq \widehat{\mathbb{E}}_t[\eta]$ *if* $\xi \geq \eta$;

(ii) *constant preserving*: $\mathbb{E}_t[\xi] = \xi$ *for* $\xi \in L_{\text{ip}}(\Omega_t)$;

(iii) *linear additivity*: $\mathbb{E}_t[\xi + \eta] = \mathbb{E}_t[\xi] + \mathbb{E}_t[\eta]$;

(iv) *consistency*: $\mathbb{E}_s[\mathbb{E}_t[\xi]] = \widehat{\mathbb{E}}_s[\xi]$ *for* $s \leq t$.

Finally, for each given $p \geq 1$, we denote by $L^p(\Omega_t)$ (resp., $L^p(\Omega)$) the completion of $L_{\text{ip}}(\Omega_t)$ (resp., $L_{\text{ip}}(\Omega)$) under the norm

$$\|\xi\|_{L^p} := (\mathbb{E}[|\xi|^p])^{\frac{1}{p}}.$$

Then the canonical process $B_t = (B_t^i)_{i=1}^d$ is a d-dimensional Brownian motion on the Wiener space $(\Omega, L^1(\Omega), \mathbb{E}[\cdot])$. Moreover, $\mathbb{E}_t[\cdot]$ can be continuously extended to the mapping from $L^1(\Omega)$ to $L^1(\Omega_t)$. For the sake of convenience, denote by $L^p(\Omega_t; \mathbb{R}^d)$ (resp., $L^p(\Omega; \mathbb{R}^d)$) the $\mathbb{R}^d$-valued random vector such that each component belongs to $L^p(\Omega_t)$ (resp., $L^p(\Omega)$).

In the above construction of Wiener expectation, (2.2) can be viewed as a discrete type of path-dependent PDE. Indeed, for each $t \in [t_{i-1}, t_i)$, $i = 1, \cdots, k$, we denote

$$u(t, \omega) := u_i(t, \omega_t; \omega_{t_1}, \cdots, \omega_{t_{i-1}})$$

and set

$$\begin{aligned}
D_t u(t, \omega) &:= \partial_{t+} u_i(t, x; x_1, \cdots, x_{i-1})|_{x=\omega_t, x_1=\omega_{t_1}, \cdots, x_{i-1}=\omega_{t_{i-1}}}, \\
D_x u(t, \omega) &:= \partial_x u_i(t, x; x_1, \cdots, x_{i-1})|_{x=\omega_t, x_1=\omega_{t_1}, \cdots, x_{i-1}=\omega_{t_{i-1}}}, \\
D_x^2 u(t, \omega) &:= \partial_x^2 u_i(t, x; x_1, \cdots, x_{i-1})|_{x=\omega_t, x_1=\omega_{t_1}, \cdots, x_{i-1}=\omega_{t_{i-1}}}.
\end{aligned}$$

Then, for each $\xi \in L_{\text{ip}}(\Omega)$, it follows from (2.1) and (2.2) that

$$u(t, \omega) = \mathbb{E}_t[\xi]$$

is a classical solution to the path-dependent heat equation (1.1). It is clear that $u(t, \omega) = u(t, \omega_{\cdot \wedge t})$ since $\mathbb{E}_t[\xi]$ is progressively measurable (with respect to the filtration $\mathcal{B}(\Omega_t)$).

Remark 2.1 Suppose $\xi = \varphi(B_T)$, then it is easy to check that

$$u(t, \omega) = \mathbb{E}_t[\varphi(B_T)] = v(t, \omega_t),$$

in which v is the solution to the following heat equation

$$\begin{cases} \partial_t u(t, x) + \dfrac{1}{2} \operatorname{tr}(\partial_x^2 u(t, x)) = 0, \quad (t, x) \in [0, T) \times \mathbb{R}^d, \\ u(T, x) = \varphi(x). \end{cases} \tag{2.3}$$

In this case the path-dependent PDE (1.1) reduces to

$$\begin{cases}\partial_t v(t,\omega_t)+\dfrac{1}{2}\operatorname{tr}(\partial_x^2 v(t,\omega_t))=0, & (t,\omega_t)\in[0,T)\times\mathbb{R}^d,\\ v(T,\omega)=\varphi(\omega_T).\end{cases}$$

A natural question is how to characterize the path-dependent heat equation (1.1) for a general Wiener functional $\xi(\omega)$. For instance, how to define the corresponding time and space derivatives $D_t u(t,\omega)$ and $D_x u(t,\omega)$ for a progressively measurable function $u(t,\omega)$.

3 Dupire's Derivatives

In the seminal paper [10], Dupire introduced the path derivatives of progressively measurable functions. An important advantage of Dupire's derivative is that it emphasizes simply the perturbation of the point ω_t instead of the whole path ω at time t. As a trade-off, one needs to define the derivatives on a larger space of RCLL (right continuous with left limit) paths, instead of on Wiener space Ω.

Let $\widetilde{\Omega}$ be the space of all $\mathbb{R}^d$-valued RCLL functions $\widetilde{\omega}$ on $[0,T]$. As mentioned above, we are interested in progressively measurable functions. Thus, we introduce the following distance on $[0,T]\times\widetilde{\Omega}$. For each $(t^i,\widetilde{\omega}^i)\in[0,T]\times\widetilde{\Omega}$, $i=1,2$, set

$$\|\widetilde{\omega}^i\|_t:=\sup_{s\in[0,t]}|\widetilde{\omega}^i_s|,\quad d_\infty((t^1,\widetilde{\omega}^1),(t^2,\widetilde{\omega}^2)):=|t^1-t^2|^{\frac{1}{2}}+\sup_{s\in[0,T]}|\widetilde{\omega}^1_{s\wedge t^1}-\widetilde{\omega}^2_{s\wedge t^2}|.$$

Now we shall introduce the Dupire's path derivatives. Consider a progressively measurable function $\widetilde{u}$ on $[0,T]\times\widetilde{\Omega}$.

Definition 3.1 *The function $\widetilde{u}$ is said to be continuous at $(t,\widetilde{\omega})\in[0,T]\times\widetilde{\Omega}$, if for any $\varepsilon>0$ there exists $\delta>0$ such that for each $(t',\widetilde{\omega}')\in[0,T]\times\widetilde{\Omega}$ with $d_\infty((t,\widetilde{\omega}),(t',\widetilde{\omega}'))<\delta$, we have $|\widetilde{u}(t,\widetilde{\omega})-\widetilde{u}(t',\widetilde{\omega}')|<\varepsilon$. $\widetilde{u}$ is said to be in $C([0,T]\times\widetilde{\Omega})$ if it is continuous at each $(t,\widetilde{\omega})\in[0,T]\times\widetilde{\Omega}$.*

Definition 3.2 *The function $\widetilde{u}$ is said to be vertically differentiable at $(t,\widetilde{\omega})\in[0,T]\times\widetilde{\Omega}$, if there exists a vector $p\in\mathbb{R}^d$, such that*

$$\widetilde{u}(t,\widetilde{\omega}+h\mathbf{1}_{[t,T]})=\widetilde{u}(t,\widetilde{\omega})+\langle p,h\rangle+o(|h|)\quad \text{as } h\to0,\ h\in\mathbb{R}^d. \tag{3.1}$$

Set $D_x\widetilde{u}(t,\widetilde{\omega}):=p$. $\widetilde{u}$ is said to be vertically differentiable if $D_x\widetilde{u}(t,\widetilde{\omega})$ exists for each $(t,\widetilde{\omega})\in[0,T]\times\widetilde{\Omega}$. We can similarly define $D_x^2\widetilde{u}(t,\widetilde{\omega})$.

Note that the vertical derivatives involve the RCLL function $\mathbf{1}_{[t,T]}$ and then the space $\widetilde{\Omega}$ is necessary in this framework.

Definition 3.3 *The function $\widetilde{u}$ is said to be horizontally differentiable at $(t,\widetilde{\omega})\in[0,T)\times\widetilde{\Omega}$, if there exists a constant $a\in\mathbb{R}$, such that*

$$\widetilde{u}(t+h,\widetilde{\omega}_{\cdot\wedge t})=\widetilde{u}(t,\widetilde{\omega})+ah+o(h)\quad \text{as } h\downarrow0,\ h\in\mathbb{R}^+. \tag{3.2}$$

We denote $D_t\widetilde{u}(t,\widetilde{\omega}) := a$. $\widetilde{u}$ *is said to be horizontally differentiable if* $D_t\widetilde{u}(t,\widetilde{\omega})$ *exists for each* $(t,\widetilde{\omega}) \in [0,T) \times \widetilde{\Omega}$.

It is clear that Dupire's derivatives just involve the perturbation of a single point and they also satisfy the classic properties: Linearity, product and chain rule.

Example 3.1 Suppose $\widetilde{u}(t,\widetilde{\omega}) = f(t,\widetilde{\omega}_t)$ for some function $f \in C^{1,1}([0,T]\times\mathbb{R}^d)$, then

$$D_t\widetilde{u}(t,\widetilde{\omega}) = \partial_t f(t,\widetilde{\omega}_t), \quad D_x\widetilde{u}(t,\widetilde{\omega}) = \partial_x f(t,\widetilde{\omega}_t),$$

which are the classic derivatives.

Example 3.2 Suppose $\widetilde{u}(t,\widetilde{\omega}) = \int_0^t \varphi(\widetilde{\omega}_s)\mathrm{d}s$ for some function $\varphi \in C^1(\mathbb{R}^d)$, then

$$D_t\widetilde{u}(t,\widetilde{\omega}) = \varphi(\widetilde{\omega}_t), \quad D_x\widetilde{u}(t,\widetilde{\omega}) = 0.$$

On the other hand, the Fréchet's derivative $D_\omega\widetilde{u}(t,\widetilde{\omega})$ (with respect to ω) is given by

$$\widetilde{\omega}' \to D_\omega\widetilde{u}(t,\widetilde{\omega})\widetilde{\omega}' := \int_0^t \partial_x\varphi(\widetilde{\omega}_s)\widetilde{\omega}'_s\mathrm{d}s.$$

It is obvious that

$$D_x\widetilde{u}(t,\widetilde{\omega}) = D_\omega\widetilde{u}(t,\widetilde{\omega})\mathbf{1}_{[t,T]} = 0.$$

The following collections shall be used frequently in this paper.

Definition 3.4 *Let* $\widetilde{u}$ *be progressively measurable function on* $[0,T]\times\widetilde{\Omega}$.

(i) $\widetilde{u}$ *is said to be in* $C_{b.\mathrm{Lip}}([0,T]\times\widetilde{\Omega}) \subset C([0,T]\times\widetilde{\Omega})$ *if* $\widetilde{u}$ *is bounded and satisfies*

$$|\widetilde{u}(t,\widetilde{\omega}) - \widetilde{u}(t',\widetilde{\omega}')| \le C(\widetilde{u})d_\infty((t,\widetilde{\omega}),(t',\widetilde{\omega}')), \quad (t,\widetilde{\omega}),(t',\widetilde{\omega}') \in [0,T]\times\widetilde{\Omega}.$$

(ii) $\widetilde{u}$ *is said to be in* $C_{l.\mathrm{Lip}}([0,T]\times\widetilde{\Omega}) \subset C([0,T]\times\widetilde{\Omega})$ *if*

$$|\widetilde{u}(t,\widetilde{\omega}) - \widetilde{u}(t',\widetilde{\omega}')| \le C(\widetilde{u})(1 + (\|\widetilde{\omega}\|_t)^{C(\widetilde{u})} + (\|\widetilde{\omega}'\|_{t'})^{C(\widetilde{u})})d_\infty((t,\widetilde{\omega}),(t',\widetilde{\omega}'))$$

for any $(t,\widetilde{\omega}),(t',\widetilde{\omega}') \in [0,T]\times\widetilde{\Omega}$.

(iii) $\widetilde{u}$ *is said to be in* $C^{1,2}_{b.\mathrm{Lip}}([0,T]\times\widetilde{\Omega}) \subset C([0,T]\times\widetilde{\Omega})$ *if* $\widetilde{u}, D_t\widetilde{u}, D_x\widetilde{u}, D_x^2\widetilde{u}$ *are in* $C_{b.\mathrm{Lip}}([0,T]\times\widetilde{\Omega})$. *Similarly, we can define* $C^{0,1}_{b.\mathrm{Lip}}([0,T]\times\widetilde{\Omega})$, $C^{0,2}_{b.\mathrm{Lip}}([0,T]\times\widetilde{\Omega})$ *and* $C^{1,2}_{l.\mathrm{Lip}}([0,T]\times\widetilde{\Omega})$.

Definition 3.5 *Given two progressively measurable functions* $\widetilde{u} : [0,T]\times\widetilde{\Omega} \to \mathbb{R}$ *and* $u : [0,T]\times\Omega \to \mathbb{R}$. *Then*

(i) u *is said to be consistent with* $\widetilde{u}$ *on* $[0,T]\times\Omega$ *if*

$$u(t,\omega) = \widetilde{u}(t,\omega), \quad (t,\omega) \in [0,T]\times\Omega. \tag{3.3}$$

(ii) u *is said to be in* $C^{1,2}_{b.\mathrm{Lip}}([0,T]\times\Omega)$ *if there exists* $\widetilde{u} \in C^{1,2}_{b.\mathrm{Lip}}([0,T]\times\widetilde{\Omega})$ *such that* (3.3) *holds and we denote*

$$D_t u(t,\omega) := D_t\widetilde{u}(t,\omega), \quad D_x u(t,\omega) := D_x\widetilde{u}(t,\omega), \quad D_x^2 u(t,\omega) := D_x^2\widetilde{u}(t,\omega). \tag{3.4}$$

Similarly, we can define $C^{1,2}_{l.\mathrm{Lip}}([0,T]\times\Omega)$.

Remark 3.1 By Zhang [43, Lemma 9.4.2], the path derivatives in (3.4) is independent of the choice of $\widetilde{u}$, i.e., if $\widetilde{u}'$ is another smooth function consistent with u, then

$$D_t\widetilde{u}' = D_t\widetilde{u}, \quad D_x\widetilde{u}' = D_x\widetilde{u}, \quad D_x^2\widetilde{u}' = D_x^2\widetilde{u} \quad \text{on } \Omega.$$

With the help of the above path derivatives, Dupire [10] introduced the the following functional Itô formula. Then it was generalized by Cont and Fournié [2–4] to a more general formulation (see also [17]).

Theorem 3.1 *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ be a probability space. If X is a continuous semimartingale and u is in $C^{1,2}_{l.\mathrm{Lip}}([0,T]\times\Omega)$, then for each $t \in [0,T]$,*

$$u(t,X) - u(0,0) = \int_0^t D_s u(s,X)\,\mathrm{d}s + \int_0^t D_x u(s,X)\mathrm{d}X_s + \frac{1}{2}\int_0^t \mathrm{tr}[D_x^2 u(s,X)d\langle X\rangle_s] \quad a.s.$$

In particular, if X is a Brownian motion, then each $t \in [0,T]$,

$$u(t,B) - u(0,0) = \int_0^t D_s u(s,B)\,\mathrm{d}s + \int_0^t D_x u(s,B)\mathrm{d}B_s + \frac{1}{2}\int_0^t \mathrm{tr}[D_x^2 u(s,B)]\mathrm{d}s \quad a.s.$$

Finally, we give the non-Markovian Feynman-Kac formula based on functional Itô's formula.

Lemma 3.1 *Consider the progressively measurable function $u(t,\omega) = \mathbb{E}_t[\xi]$ for a Wiener functional ξ. Suppose $u(t,\omega) \in C^{1,2}_{l.\mathrm{Lip}}([0,T]\times\Omega)$. Then $u(t,\omega)$ is the unique $C^{1,2}_{l.\mathrm{Lip}}([0,T]\times\Omega)$-solution of path-dependent PDE* (1.1).

Proof Applying functional Itô's formula (Theorem 3.1) yields that for each $t \in [0,T]$,

$$u(t,\omega) - u(0,0) = \int_0^t D_s u(s,\omega)\,\mathrm{d}s + \int_0^t D_x u(s,\omega)\mathrm{d}B_s + \frac{1}{2}\int_0^t \mathrm{tr}[D_x^2 u(s,\omega)]\mathrm{d}s,$$

which together with the fact that $u(t,\omega)$ is a martingale indicates that

$$D_s u(s,\omega) + \frac{1}{2}\,\mathrm{tr}[D_x^2 u(s,\omega)] = 0.$$

Note that $u(T,\omega) = \xi(\omega)$. Thus u is a $C^{1,2}_{l.\mathrm{Lip}}([0,T]\times\Omega)$-solution of path-dependent PDE (1.1).

Suppose $u' \in C^{1,2}_{l.\mathrm{Lip}}([0,T]\times\Omega)$ is also a solution of path-dependent PDE (1.1). It follows from Theorem 3.1 that u' is a martingale,

$$\mathrm{d}u'(t,\omega) = D_x u'(t,\omega)\mathrm{d}B_t,$$

i.e.,

$$u'(t,\omega) = \mathbb{E}_t[\xi] = u(t,\omega),$$

which completes the proof.

Remark 3.2 Note that $u(t,\omega) = \mathbb{E}_t[\xi] = \mathbb{E}[\xi(B^{t,\omega})]$ for each $(t,\omega) \in [0,T]\times\Omega$, in which

$$B_s^{t,\omega} := \omega_s \mathbf{1}_{[0,t]} + (B_s - B_t + \omega_t)\mathbf{1}_{(t,T]}, \quad s \in [0,T].$$

Then we could prove that $u(t,\omega)$ is in $C^{1,2}_{l.\mathrm{Lip}}([0,T]\times\Omega;\mathbb{R}^d)$ under some appropriate assumptions on the terminal condition ξ, which will be stated in Section 4.

4 Classical Solutions of Path-Dependent Heat Equations

In this section, we will prove the path-dependent PDE (1.1) admits a classical $C^{1,2}$-solution when the terminal condition is smooth enough.

Remark 4.1 In order to simplify the presentation, in this section we consider only $C^{1,2}_{b.\mathrm{Lip}}$-solutions. By slightly more involved estimates, we can extend our results to the case of $C^{1,2}_{l.\mathrm{Lip}}$-solutions (see [32]).

The following directional derivatives will be used frequently in the subsequent discussions.

Definition 4.1 *Given an $\mathbb{R}$-valued function $\widetilde{\xi}$ defined on $\widetilde{\Omega}$.*

(i) *$\widetilde{\xi}$ is said to be in $\mathbb{C}^2(\widetilde{\Omega})$, if for each $\widetilde{\omega} \in \widetilde{\Omega}$ and $t \in [0,T]$, there exists a vector $p \in \mathbb{R}^d$ and a symmetric matrix $A \in \mathbb{R}^{d\times d}$ such that,*

$$\widetilde{\xi}(\widetilde{\omega} + h\mathbf{1}_{[t,T]}) = \widetilde{\xi}(\widetilde{\omega}) + \langle p, h\rangle + \frac{1}{2}\langle Ah, h\rangle + o(|h|^2) \quad \text{as } h \to 0,\ h \in \mathbb{R}^d.$$

We denote $\mathbb{D}\widetilde{\xi}(t,\widetilde{\omega}) := p$ and $\mathbb{D}^2\widetilde{\xi}(t,\widetilde{\omega}) := A$. Similarly, we can define $\mathbb{C}^2_b(\widetilde{\Omega})$.

(ii) *$\widetilde{\xi}$ is said to be in $\mathbb{C}^2_{b.\mathrm{Lip}}(\widetilde{\Omega}) \subset \mathbb{C}^2_b(\widetilde{\Omega})$ if there exist some constants $C > 0$ depending only on $\widetilde{\xi}$ such that for each $\widetilde{\omega}, \widetilde{\omega}' \in \widetilde{\Omega}, t, t' \in [0,T]$,*

$$\begin{aligned}&|\widetilde{\xi}(\widetilde{\omega}) - \widetilde{\xi}(\widetilde{\omega}')| + |\mathbb{D}\widetilde{\xi}(t,\widetilde{\omega}) - \mathbb{D}\widetilde{\xi}(s,\widetilde{\omega}')| + |\mathbb{D}^2\widetilde{\xi}(t,\widetilde{\omega}) - \mathbb{D}^2\widetilde{\xi}(s,\widetilde{\omega}')| \\ &\le C(|t-t'| + \|\widetilde{\omega} - \widetilde{\omega}'\|).\end{aligned}$$

Similarly, we can define $\mathbb{C}_{b.\mathrm{Lip}}(\widetilde{\Omega})$ and $\mathbb{C}^1_{b.\mathrm{Lip}}(\widetilde{\Omega})$.

Lemma 4.1 *Suppose that $\widetilde{\xi} \in \mathbb{C}^2_{b.\mathrm{Lip}}(\widetilde{\Omega})$. Then the function*

$$\widetilde{u}(t,\widetilde{\omega}) := \mathbb{E}[\widetilde{\xi}(B^{t,\widetilde{\omega}})] = \mathbb{E}[\widetilde{\xi}(\widetilde{\omega}_s\mathbf{1}_{[0,t]} + (B_s - B_t + \widetilde{\omega}_t)\mathbf{1}_{(t,T]})]$$

is in $C^{0,2}_{b.\mathrm{Lip}}([0,T]\times\widetilde{\Omega})$.

Proof The proof will be divided into the following two steps.

(1) $\widetilde{u} \in C_{b,\mathrm{Lip}}([0,T]\times\widetilde{\Omega})$. For any $\widetilde{\omega}, \widetilde{\omega}' \in \widetilde{\Omega}, t, t' \in [0,T]$,

$$\begin{aligned}&|\widetilde{\xi}(\widetilde{\omega}_s\mathbf{1}_{[0,t]} + (B_s - B_t + \widetilde{\omega}_t)\mathbf{1}_{(t,T]}) - \widetilde{\xi}(\widetilde{\omega}_s\mathbf{1}_{[0,t']} + (B_s - B_t' + \widetilde{\omega}_{t'})\mathbf{1}_{(t',T]})| \\ &\le C(\widetilde{\xi})\Big(\|\widetilde{\omega}_{\cdot\wedge t} - \widetilde{\omega}'_{\cdot\wedge t'}\| + \sup_{t\wedge t'\le s\le t\vee t'}|B_s - B_{t\wedge t'}|\Big).\end{aligned}$$

It follows that

$$\begin{aligned}|\widetilde{u}(t,\widetilde{\omega}) - \widetilde{u}(t',\widetilde{\omega}')| &\le C(\widetilde{\xi})\Big(\|\widetilde{\omega}_{\cdot\wedge t} - \widetilde{\omega}'_{\cdot\wedge t'}\| + \mathbb{E}\Big[\sup_{t\wedge t'\le s\le t\vee t'}|B_s - B_{t\wedge t'}|\Big]\Big) \\ &= C(\widetilde{\xi})d_\infty((t,\widetilde{\omega}),(t',\widetilde{\omega}')),\end{aligned}$$

which is the desired result. In particular, $\widetilde{u}$ is progressively measurable.

(2) $D_x\widetilde{u}, D_x^2\widetilde{u} \in C_{b,\mathrm{Lip}}([0,T]\times\widetilde{\Omega})$. For each $h\in\mathbb{R}^d$, by Taylor's expansion we have

$$\begin{aligned}&\widetilde{\xi}(\widetilde{\omega}_s\mathbf{1}_{[0,t]}+(B_s-B_t+h+\widetilde{\omega}_t)\mathbf{1}_{(t,T]})\\=\,&\widetilde{\xi}(\widetilde{\omega}_s\mathbf{1}_{[0,t]}+(B_s-B_t+\widetilde{\omega}_t)\mathbf{1}_{(t,T]})\\&+h\int_0^1\mathbb{D}\widetilde{\xi}(t,\widetilde{\omega}_s\mathbf{1}_{[0,t]}+(B_s-B_t+\theta h+\widetilde{\omega}_t)\mathbf{1}_{(t,T]})\mathrm{d}\theta,\end{aligned}$$

which together with dominated convergence theorem indicates that

$$D_x\widetilde{u}(t,\widetilde{\omega})=\lim_{h\to0}\frac{\widetilde{u}(t,\widetilde{\omega}+h\mathbf{1}_{[t,T]})-\widetilde{u}(t,\widetilde{\omega})}{h}=\mathbb{E}[\mathbb{D}\widetilde{\xi}(t,B^{t,\widetilde{\omega}})].$$

Then for any $\widetilde{\omega},\widetilde{\omega}'\in\widetilde{\Omega}, t,t'\in[0,T]$, we have

$$\begin{aligned}&|\mathbb{D}\widetilde{\xi}(t,\widetilde{\omega}_s\mathbf{1}_{[0,t]}+(B_s-B_t+\widetilde{\omega}_t)\mathbf{1}_{(t,T]})-\mathbb{D}\widetilde{\xi}(t',\widetilde{\omega}_s\mathbf{1}_{[0,t']}+(B_s-B_t'+\widetilde{\omega}_{t'})\mathbf{1}_{(t',T]})|\\\le\,&C(\widetilde{\xi})\Big(|t-t'|+\|\widetilde{\omega}_{\cdot\wedge t}-\widetilde{\omega}'_{\cdot\wedge t'}\|+\sup_{t\wedge t'\le s\le t\vee t'}|B_s-B_{t\wedge t'}|\Big),\end{aligned}$$

which indicates that

$$\begin{aligned}&|D_x\widetilde{u}(t,\widetilde{\omega})-D_x\widetilde{u}(t',\widetilde{\omega}')|\\\le\,&C(\widetilde{\xi})(|t-t'|+d_\infty((t,\widetilde{\omega}),(t',\widetilde{\omega}')))\\\le\,&C(\widetilde{\xi},T)d_\infty((t,\widetilde{\omega}),(t',\widetilde{\omega}')).\end{aligned}$$

By a similar analysis, we could deduce that

$$D_x^2\widetilde{u}(t,\widetilde{\omega})=\mathbb{E}[\mathbb{D}^2\widetilde{\xi}(t,B^{t,\widetilde{\omega}})]$$

and $D_x^2\widetilde{u}\in C_{b,\mathrm{Lip}}([0,T]\times\widetilde{\Omega})$. The proof is complete.

Compared with the case of vertical derivatives, it is difficult to prove u is horizontally differentiable due to the path-dependency. In order to prove the existence of $D_t\widetilde{u}$, we need the following results.

First, for a fixed constant $t\in[0,T)$ we introduce an approximation procedure: For any $\widetilde{\omega}\in\widetilde{\Omega}$, we set

$$\widetilde{\omega}_s^{(n)}=\widetilde{\omega}_s\mathbf{1}_{[0,t)}(s)+\sum_{i=1}^n\widetilde{\omega}_{t_i^n}\mathbf{1}_{[t_{i-1}^n,t_i^n)}(s)+\widetilde{\omega}_T\mathbf{1}_{\{T\}}(s)$$

with $t_i^n=t+\frac{i}{n}(T-t)$, $i=0,\cdots,n$. Then set

$$\widetilde{\xi}^{(n)}(\widetilde{\omega}):=\widetilde{\xi}(\widetilde{\omega}^{(n)})$$

and

$$\widetilde{u}^{(n)}(r,\widetilde{\omega}):=\mathbb{E}[\widetilde{\xi}^{(n)}(B^{r,\widetilde{\omega}})],\quad(r,\widetilde{\omega})\in[t,T]\times\widetilde{\Omega}.$$

Lemma 4.2 *Suppose that $\widetilde{\xi} \in \mathbb{C}^2_{b.\mathrm{Lip}}(\widetilde{\Omega})$. Then $D_x\widetilde{u}^{(n)}(r,\widetilde{\omega})$ and $D_x^2\widetilde{u}^{(n)}(r,\widetilde{\omega})$ exist for any $(r,\widetilde{\omega}) \in [t,T] \times \widetilde{\Omega}$. Moreover*

$$\begin{aligned}&|\widetilde{u}^{(n)}(r,\widetilde{\omega}) - \widetilde{u}(r,\widetilde{\omega})| + |D_x\widetilde{u}^{(n)}(r,\widetilde{\omega}) - D_x\widetilde{u}(r,\widetilde{\omega})| + |D_x^2\widetilde{u}^{(n)}(r,\widetilde{\omega}) - D_x^2\widetilde{u}(r,\widetilde{\omega})|\\ &\le C(\widetilde{\xi},T)(n^{-\frac{1}{4}} + \|\widetilde{\omega}^{(n)}_{\cdot\wedge r} - \widetilde{\omega}_{\cdot\wedge r}\|).\end{aligned}$$

Proof The proof is similar as in Lemma 4.1 (see also [32, Lemma 3.13]) and we omit it.

Lemma 4.3 *Suppose that $\widetilde{\xi} \in \mathbb{C}^2_{b.\mathrm{Lip}}(\widetilde{\Omega})$. Then $\widetilde{u}^{(n)} \in C^{1,2}_{b.\mathrm{Lip}}([0,T] \times \widetilde{\Omega})$. Moreover*

$$\begin{cases} D_t u^{(n)}(r,\widetilde{\omega}) + \dfrac{1}{2}\operatorname{tr}(D_x^2 u^{(n)}(r,\widetilde{\omega})) = 0, \quad (r,\widetilde{\omega}) \in [t,T) \times \widetilde{\Omega},\\ u^{(n)}(T,\widetilde{\omega}) = \widetilde{\xi}^{(n)}(\widetilde{\omega}). \end{cases} \tag{4.1}$$

Proof Note that $\widetilde{\xi}^{(n)}(\widetilde{\omega})$ depends only on $\widetilde{\omega}_{\cdot\wedge t}, \widetilde{\omega}_{t_1^n}, \widetilde{\omega}_{t_2^n}, \cdots, \widetilde{\omega}_T$, which can be seen as a cylindrical function on $[t,T]$. By a similar analysis as in Section 2, (4.1) is a discrete type of path-dependent PDE on $[t,T]$ and then $\widetilde{u}^{(n)}(r,\widetilde{\omega})$ is the unique $C^{1,2}_{b.\mathrm{Lip}}([0,T] \times \widetilde{\Omega})$-solution.

Now we are in a position to prove that path-dependent PDE (1.1) has a $C^{1,2}_{b.\mathrm{Lip}}([0,T] \times \Omega)$-solution.

Theorem 4.1 *Denote $u(t,\omega) = \widetilde{u}(t,\omega)$ for each $(t,\omega) \in [0,T] \times \Omega$. Then u is the unique $C^{1,2}_{b.\mathrm{Lip}}([0,T] \times \Omega)$-solution of the path-dependent PDE* (1.1).

Proof For each $\delta > 0$ satisfying $t + \delta \le T$ and for any $\omega \in \Omega$, according to the independent increment of Brownian motion, we get

$$\begin{aligned} u(t,\omega) &= \mathbb{E}[\mathbb{E}_{t+\delta}[\widetilde{\xi}(\omega_s \mathbf{1}_{[0,t]} + (B_s - B_t + \omega_t)\mathbf{1}_{(t,T]})]]\\ &= \mathbb{E}[\mathbb{E}[\widetilde{\xi}(\omega'_s \mathbf{1}_{[0,t+\delta]} + (B_s - B_{t+\delta} + \omega'_{t+\delta})\mathbf{1}_{(t+\delta,T]})]_{\omega' = B^{t,\omega}_{\cdot\wedge(t+\delta)}}]\\ &= \mathbb{E}[u(t+\delta, B^{t,\omega}_{\cdot\wedge(t+\delta)})]. \end{aligned}$$

It follows that

$$u(t+\delta, \omega_{\cdot\wedge t}) - u(t,\omega) = u(t+\delta, \omega_{\cdot\wedge t}) - \mathbb{E}[u(t+\delta, B^{t,\omega}_{\cdot\wedge(t+\delta)})].$$

With the help of Lemma 4.2, we obtain

$$|\widetilde{u}^{(n)}(t+\delta, \omega_{\cdot\wedge t}) - u(t+\delta, \omega_{\cdot\wedge t})| \le C(\widetilde{\xi},T)n^{-\frac{1}{4}}$$

and

$$\begin{aligned}&\mathbb{E}[|\widetilde{u}^{(n)}(t+\delta, B^{t,\omega}_{\cdot\wedge(t+\delta)}) - u(t+\delta, B^{t,\omega}_{\cdot\wedge(t+\delta)})|]\\ &\le C(\widetilde{\xi},T)\Big(n^{-\frac{1}{4}} + \mathbb{E}\Big[\sup_{s\in[t,T)}\Big|B_s - \sum_{i=1}^n B_{t_k^n}\mathbf{1}_{[t_{k-1}^n, t_k^n)}(s)\Big|\Big]\Big)\\ &\le C(\widetilde{\xi},T)n^{-\frac{1}{4}}.\end{aligned}$$

Thus, we conclude that

$$u(t+\delta,\omega_{\cdot\wedge t})-u(t,\omega)=\lim_{n\to\infty}\mathbb{E}[\widetilde{u}^{(n)}(t+\delta,\omega_{\cdot\wedge t})-\widetilde{u}^{(n)}(t+\delta,B^{t,\omega}_{\cdot\wedge(t+\delta)})].$$

In view of Lemma 4.3 and Theorem 3.1, we have

$$\begin{aligned}
&\widetilde{u}^{(n)}(t+\delta,\omega_{\cdot\wedge t})-\widetilde{u}^{(n)}(t+\delta,B^{t,\omega}_{\cdot\wedge(t+\delta)})\\
=&\,\widetilde{u}^{(n)}(t+\delta,\omega_{\cdot\wedge t})-\widetilde{u}^{(n)}(t,\omega)+\widetilde{u}^{(n)}(t,\omega)-\widetilde{u}^{(n)}(t+\delta,B^{t,\omega}_{\cdot\wedge(t+\delta)})\\
=&\int_t^{t+\delta}D_s\widetilde{u}^{(n)}(s,\omega_{\cdot\wedge t})\mathrm{d}s-\int_t^{t+\delta}D_s\widetilde{u}^{(n)}(s,B^{t,\omega}_{\cdot\wedge s})\mathrm{d}s\\
&-\int_t^{t+\delta}D_x\widetilde{u}^{(n)}(s,B^{t,\omega}_{\cdot\wedge s})\mathrm{d}B_s-\frac{1}{2}\int_t^{t+\delta}\mathrm{tr}(D_x^2\widetilde{u}^{(n)}(s,B^{t,\omega}_{\cdot\wedge s}))\mathrm{d}s. \qquad (4.2)
\end{aligned}$$

Applying Lemmas 4.2–4.3 again, we get

$$\begin{aligned}
&\mathbb{E}[|D_x\widetilde{u}^{(n)}(s,B^{t,\omega}_{\cdot\wedge s})-D_xu(s,B^{t,\omega}_{\cdot\wedge s})|+|D_x^2\widetilde{u}^{(n)}(s,B^{t,\omega}_{\cdot\wedge s})-D_x^2u(s,B^{t,\omega}_{\cdot\wedge s})|]\\
\le&\, C(\widetilde{\xi},T)n^{-\frac{1}{4}},\\
&\mathbb{E}[|D_s\widetilde{u}^{(n)}(s,\omega_{\cdot\wedge t})-D_s\widetilde{u}^{(n)}(s,B^{t,\omega}_{\cdot\wedge s})|]\\
\le&\, C(\widetilde{\xi},T)\mathbb{E}\Big[\sup_{s\in[t,t+\delta]}|B_s-B_t|\Big]\\
\le&\, C(\widetilde{\xi},T)\delta^{\frac{1}{2}},
\end{aligned}$$

which together with (4.2) implies that

$$u(t+\delta,\omega_{\cdot\wedge t})-u(t,\omega)=-\frac{1}{2}\mathbb{E}\Big[\int_t^{t+\delta}\mathrm{tr}(D_x^2u(s,B^{t,\omega}_{\cdot\wedge s}))\mathrm{d}s\Big]+o(\delta).$$

Using dominated convergence theorem, we have

$$\lim_{\delta\downarrow 0}\frac{u(t+\delta,\omega_{\cdot\wedge t})-u(t,\omega)}{\delta}=-\frac{1}{2}\,\mathrm{tr}(D_x^2u(t,\omega)),$$

which implies that $D_tu\in C_{b,\mathrm{Lip}}([0,T]\times\Omega)$. Therefore, $u\in C^{1,2}_{b.\mathrm{Lip}}([0,T]\times\Omega)$ satisfies (1.1), which ends the proof.

Remark 4.2 Using the above argument, we can also deal with the classical solution of a system of path dependent PDE (1.1), namely, $u(t,\omega)$ can be $\mathbb{R}^n$-valued.

Example 4.1 Given a function $\xi:\Omega\to\mathbb{R}$ by

$$\xi(\omega)=\int_0^T\varphi(\omega_s)\,\mathrm{d}s$$

for some real valued function $\varphi\in C^2_{b.\mathrm{Lip}}(\mathbb{R}^d)$. Then, we set

$$\widetilde{\xi}(\widetilde{\omega})=\int_0^T\varphi(\widetilde{\omega}_s)\,\mathrm{d}s,\quad \widetilde{\omega}\in\widetilde{\Omega}.$$

It is easy to check $\widetilde{\xi} \in \mathbb{C}^2_{b.\mathrm{Lip}}(\widetilde{\Omega})$ with

$$\mathbb{D}\widetilde{\xi}(t,\widetilde{\omega}) = \int_t^T \partial_x\varphi(\omega_s)\,\mathrm{d}s \quad \text{and} \quad \mathbb{D}^2\widetilde{\xi}(t,\widetilde{\omega}) = \int_t^T \partial_x^2\varphi(\omega_s)\,\mathrm{d}s.$$

In this case,

$$u(t,\omega) = \mathbb{E}\Big[\int_0^T \varphi(B_s^{t,\omega})\,\mathrm{d}s\Big] = \int_0^t \varphi(\omega_s)\,\mathrm{d}s + \int_t^T \mathbb{E}[\varphi(\omega_t + B_s - B_t)]\,\mathrm{d}s.$$

On the other hand, by the classic Feynman-Kac formula, we deduce that

$$u^s(t,x) = \mathbb{E}[\varphi(x + B_s - B_t)], \quad (t,x) \in [0,s] \times \mathbb{R}^d$$

is the solution of the following parabolic PDE:

$$\begin{cases} \partial_t u^s(t,x) + \dfrac{1}{2}\,\mathrm{tr}[\partial_x^2 u^s(t,x)] = 0, \quad (t,x) \in [0,s) \times \mathbb{R}^d, \\ u^s(s,x) = \varphi(x). \end{cases}$$

It follows that

$$u(t,\omega) = \int_0^t \varphi(\omega_s)\,\mathrm{d}s + \int_t^T u^s(t,\omega_t)\,\mathrm{d}s.$$

According to the definitions of Dupire's path derivatives, we obtain

$$\begin{aligned} D_t u(t,\omega) &= \int_t^T \partial_t u^s(t,\omega(t))\,\mathrm{d}s, \quad D_x u(t,\omega) = \int_t^T \partial_x u^s(t,\omega(t))\,\mathrm{d}s, \\ D_x^2 u(t,\omega) &= \int_t^T \partial_x^2 u^s(t,\omega(t))\,\mathrm{d}s. \end{aligned}$$

Consequently,

$$D_t u(t,\omega) + \frac{1}{2}\,\mathrm{tr}(D_x^2 u(t,\omega)) = 0,$$

which is the path-dependent PDE (1.1).

Remark 4.3 Compared with the finite-dimensional parabolic case, it is difficult to prove the path regularities of solutions of the path-dependent PDE (1.1) for general terminal conditions due to the absence of any regularizing effect. Thus, various types of weaker notions of solutions were introduced to deal with the path-dependent PDE (1.1).

5 Sobolev Solutions for Path-Dependent Heat Equations

Based on Dupire's path-derivative, Peng and Song [31] introduced a new type of Sobolev path-functions and the corresponding path derivatives. One advantage of this framework is that it removes the smooth assumptions on the terminal conditions.

We first give some notations. Let $(\mathcal{F}_t)_{0\le t\le T}$ be the natural filtration generated by the Brownian motion B augmented with the family $\mathscr{N}^{\mathbb{P}}$ of $\mathbb{P}$-null sets of $\mathcal{F}_T$. Then, for each $p \ge 1$, we consider the following collections:

• $M^p(0,T)$ is the collection of $\mathbb{R}^d$-valued $\mathcal{F}$-progressively measurable processes $(z_t)_{0\le t\le T}$ satisfying

$$\|z\|_{M^p} = \mathbb{E}\Big[\int_0^T |z_t|^p \mathrm{d}t\Big]^{\frac{1}{p}} < \infty;$$

• $H^p(0,T)$ is the collection of $\mathbb{R}^d$-valued $\mathcal{F}$-progressively measurable processes $(z_t)_{0\le t\le T}$ satisfying

$$\|z\|_{H^p} = \mathbb{E}\Big[\Big(\int_0^T |z_t|^2 \mathrm{d}t\Big)^{\frac{p}{2}}\Big]^{\frac{1}{p}} < \infty;$$

• $S^p(0,T)$ is the collection of real-valued $\mathcal{F}$-adapted continuous processes $(y_t)_{0\le t\le T}$ satisfying

$$\|y\|_{S^p} = \mathbb{E}\Big[\sup_{t\in[0,T]} |y_t|^p\Big]^{\frac{1}{p}} < \infty.$$

• Denote by $M^p(0,T;\mathbb{R}^d)$ the $\mathbb{R}^d$-valued process such that each component belongs to $M^p(0,T)$. Similarly, we can define $H^p(0,T;\mathbb{R}^d)$ and $S^p_G(0,T;\mathbb{R}^d)$.

For a given $p>1$, the classical Sobolev space $W^{1,p}(\mathbb{R}^d)$ is the completion of $C_0^\infty(\mathbb{R}^d)$ under the norm $\|u\|_{L^p(\mathbb{R}^d)} + \|\partial_x u\|_{L^p(\mathbb{R}^d)}$, where $C_0^\infty(\mathbb{R}^d)$ is the space of all infinitely differentiable real functions u with compact supports. From the view of this point, we introduce the counterpart in the space of continuous paths.

Definition 5.1 *A progressively measurable function u is said to be in $\mathcal{C}^\infty(0,T)$ if there exists a time partition $(t_i)_{i=0}^n$ with $0=t_0<t_1<\cdots<t_n=T$, such that for each $k=0,1,\cdots,n-1$ and $t\in[t_k,t_{k+1})$,*

$$u(t,\omega) = u_k(t,\omega_t;\omega_{t_1},\cdots,\omega_{t_k}).$$

Here for each k, the function $u_k:[t_k,t_{k+1}]\times\mathbb{R}^{d\times(k+1)}\to\mathbb{R}$ is a C^∞-function with

$$u_k(t_k,x;x_1,\cdots,x_{k-1},x) = u_k(t_k,x;x_1,\cdots,x_{k-1}),$$

all of whose derivatives have at most polynomial growth.

For any $u\in\mathcal{C}^\infty(0,T)$, by Itô's formula, we have

$$u(t,\omega) = u(0,\omega) + \int_0^t \mathcal{A}u(s,\omega)\mathrm{d}s + \int_0^t D_x u(s,\omega)\mathrm{d}B_s, \quad t\in[0,T],$$

where $\mathcal{A}$ is the following heat differential operator:

$$\mathcal{A}(s,\omega) := D_t u(s,\omega) + \frac{1}{2}\operatorname{tr}(D_x^2 u(s,\omega)). \tag{5.1}$$

Then we define the following norm:

$$\|u\|_{W_{\mathcal{A}}^{\frac{1}{2},1;p}} := \{\|u\|_{S^p}^p + \|\mathcal{A}u\|_{M^p}^p + \|D_x u\|_{H^p}^p\}^{\frac{1}{p}}.$$

Denote by $W_{\mathcal{A}}^{\frac{1}{2},1;p}(0,T)$ the completion of $\mathcal{C}^{\infty}(0,T)$ with respect to the norm $\|\cdot\|_{W_{\mathcal{A}}^{\frac{1}{2},1;p}}$. It is easy to check that $W_{\mathcal{A}}^{\frac{1}{2},1;p}(0,T)$ is a subspace of $S^p(0,T)$. Moreover, the differential operators D_x and $\mathcal{A}$ can be continuously extended to this space:

$$D_x : W_{\mathcal{A}}^{\frac{1}{2},1;p}(0,T) \to H^p(0,T), \quad \mathcal{A} : W_{\mathcal{A}}^{\frac{1}{2},1;p}(0,T) \to M^p(0,T).$$

Then $W_{\mathcal{A}}^{\frac{1}{2},1;p}(0,T)$ is said to be a $\mathbb{P}$-weighted Sobolev space. We have the following result.

Theorem 5.1 (see [31, Theorem 2.9]) *For each given $u \in S^p(0,T)$, the following two conditions are equivalent*:

(i) $u \in W_{\mathcal{A}}^{\frac{1}{2},1;p}(0,T)$;

(ii) *there exists $(u_0, \eta, v) \in \mathbb{R} \times M^p(0,T) \times H^p(0,T)$ such that*

$$u(t,\omega) = u_0 + \int_0^t \eta_s \mathrm{d}s + \int_0^t v_s \mathrm{d}B_s. \tag{5.2}$$

Moreover, we have

$$\mathcal{A}u(t,\omega) = \eta(t,\omega), \quad D_x u(t,\omega) = v(t,\omega).$$

From the above theorem, the Itô's process u of the form (5.2) gives us a generalized path-dependent Itô's formula:

$$u(t,\omega) = u_0 + \int_0^t \mathcal{A}u(s,\omega)\mathrm{d}s + \int^t D_x u(t,\omega)\mathrm{d}B_s.$$

Based on the above discussions, in the $\mathbb{P}$-Sobolev space $W_{\mathcal{A}}^{\frac{1}{2},1;p}(0,T)$ the path-dependent counterpart of (1.1) is formulated as:

$$\begin{cases} \mathcal{A}u(t,\omega) = 0, & (t,\omega) \in [0,T) \times \Omega, \\ u(T,\omega) = \xi(\omega). \end{cases} \tag{5.3}$$

Now we are in a position to show the well-posedness of the path-dependent PDE (5.3).

Theorem 5.2 *Assume $\xi \in L^p(\Omega)$ for some $p > 1$. Then $u(t,\omega) := \mathbb{E}_t[\xi]$ is the unique $W_{\mathcal{A}}^{\frac{1}{2},1;p}(0,T)$-solution to the path-dependent heat equation* (5.3).

Proof It follows from martingale representation theorem that there is a process $Z \in H^p(0,T)$ such that

$$u(t,\omega) = \mathbb{E}[\xi] + \int_0^t Z_s \mathrm{d}B_s, \quad \forall t \in [0,T].$$

Applying Theorem 5.1, we have $u \in W_{\mathcal{A}}^{\frac{1}{2},1;p}(0,T)$ and

$$\mathcal{A}u(t,\omega) = 0, \quad D_x u(t,\omega) = Z(t,\omega).$$

On the other hand, assume u' is also a $W_{\mathcal{A}}^{\frac{1}{2},1;p}(0,T)$-solution to the path-dependent PDE (5.3). It follows that

$$u'(t,\omega) = u'_0 + \int_0^t D_x u'(s,\omega)\mathrm{d}B_s,$$

which implies that $u'(t,\omega)$ is a martingale, i.e., $u'(t,\omega) = \mathbb{E}_t[\xi](\omega)$. The proof is complete.

Remark 5.1 Remark that in the $\mathbb{P}$-Sobolev space we do not need to define the path derivatives on a larger space of right continuous paths with left limit. Moreover, the path-dependent PDE (5.3) can be also a system of PDEs.

6 Nonlinear Path-Dependent PDEs

In this section, we will generalize the path-dependent heat equation (1.1) to the nonlinear linear case through BSDEs theory (see Peng [28]).

Consider the following backward stochastic differential equation (BSDE for short):

$$Y_t = \xi + \int_t^T g(s,\omega,Y_s,Z_s)\,\mathrm{d}s - \int_t^T Z_s\,\mathrm{d}B_s, \quad t\in[0,T], \tag{6.1}$$

where $\xi \in L^p(\Omega;\mathbb{R}^n)$ and the driver $g:\Omega\times[0,T]\times\mathbb{R}^n\times\mathbb{R}^{n\times d}\to\mathbb{R}^n$ is an $\mathcal{F}$-progressively measurable process satisfying the following conditions:

(i) For each $(y,z)\in\mathbb{R}\times\mathbb{R}^d$, $g(\cdot,y,z)\in H^p(0,T;\mathbb{R}^n)$;

(ii) there exists some positive constant L such that for any $(y,z),(y',z')\in\mathbb{R}^n\times\mathbb{R}^{n\times d}$,

$$|g(t,y,z)-g(t,y',z')|\le L(|y-y'|+|z-z'|).$$

Theorem 6.1 (see [22]) *Suppose that conditions* (i) *and* (ii) *are fulfilled. Then, the BSDE* (6.1) *admits a unique solution* $(Y,Z)\in\mathcal{S}^p(0,T;\mathbb{R}^n)\times\mathcal{H}^p(0,T;\mathbb{R}^{n\times d})$.

The BSDEs provide a probabilistic interpretation of a system of semi-linear PDEs. Indeed, when $\xi(\omega)=\varphi(\omega_T)$ and $g(t,\omega,y,z)=f(t,\omega_t,y,z)$ for some smooth functions ψ and f, it follows from Pardoux and Peng [23] that $(Y_t,Z_t)\equiv(u(t,B_t),\partial_x u(t,B_t))$, where u is the solution to the following semi-linear PDE:

$$\begin{cases}\partial_t u(t,x)+\dfrac{1}{2}\operatorname{tr}(\partial_x^2 u(t,x))+f(t,x,u,\partial_x u)=0, \quad (t,x)\in[0,T)\times\mathbb{R}^d,\\ u(T,x)=\varphi(x).\end{cases} \tag{6.2}$$

In particular, the authors introduced a probabilistic approach to prove the regularities of solutions to semi-linear PDEs in [23]. Thus a very natural problem is to interpret a general BSDE as the following semi-linear path-dependent PDE:

$$\begin{cases}D_t u(t,\omega)+\dfrac{1}{2}\operatorname{tr}(D_x^2 u(t,\omega))+g(t,\omega,u,D_x u)=0, \quad (t,\omega)\in[0,T)\times\Omega,\\ u(T,\omega)=\xi(\omega).\end{cases} \tag{6.3}$$

With the help of Dupire's derivatives, Peng and Wang [32] studied the well-posedness of classical solutions to the semi-linear path-dependent PDE (6.3). Under some regularity assumptions for the terminal condition and the driver, we first apply Kolmogorovs continuity criterion to obtain the differentiability of the solution to the BSDE (6.1) through the "frozen method", which is introduced by Peng [25–26] to construct nonlinear expectations. Next, using an approximation method based on discrete types of path-dependent PDEs as in Section 3, we obtain the path regularity of the solution Z. Finally, with the help of the aforementioned results, we prove that the BSDE (6.1) can be formulated as the solution of the semi-linear path-dependent PDE (6.3). Indeed, the path-dependent function $u(t,\omega) := Y_t(\omega)$ is the unique solution of the path-dependent PDE (6.3).

Theorem 6.2 (see [32]) *Under some regularity conditions on the terminal value ξ and the driver g, the path-dependent PDE* (6.3) *admits a unique solution $u \in C^{1,2}_{l.\mathrm{Lip}}([0,T]\times\Omega)$. Moreover,*

$$u(t,\omega) = Y_t(\omega), \quad D_x u(t,\omega) = Z_t(\omega), \quad (t,\omega)\in[0,T]\times\Omega,$$

where (Y,Z) is the solution to the BSDE (6.1).

The above theorem provides an alternative method for the study of the BSDEs theory. For instance, we could study more general semi-linear path-dependent PDE through forward and backward stochastic differential equations (FBSDEs for short) as in [23]. Indeed, Wang [40] obtained the uniqueness and existence of classical solutions to general semi-linear path-dependent parabolic integro-differential equations (see also [16]), which generalized the results of [23] to the path-dependent situation. However, we usually cannot obtain the classical solutions to the path-dependent PDE (6.3) when the coefficients are only Lipschitz functions.

Under more general conditions, Peng and Song [31] established the uniqueness and existence of Sobolev solutions to the following semi-linear path-dependent PDEs:

$$\begin{cases} \mathcal{A}u(t,\omega) + g(t,\omega,u,D_x u) = 0, \quad (t,\omega)\in[0,T)\times\Omega, \\ u(T,\omega) = \xi(\omega), \end{cases} \tag{6.4}$$

where $\mathcal{A}$ is the extended heat operator of $D_t u + \frac{1}{2}D^2_{xx}$.

Theorem 6.3 (see [31, Theorem 2.11]) *Assume that conditions* (i) *and* (ii) *are satisfied. Then the path-dependent PDE* (6.4) *admits a unique solution $u \in W^{\frac{1}{2},1;p}_{\mathcal{A}}(0,T)$. Moreover,*

$$u(t,\omega) = Y_t(\omega), \quad D_x u(t,\omega) = Z_t(\omega), \quad (t,\omega)\in[0,T]\times\Omega,$$

where (Y,Z) is the solution to the BSDE (6.1).

According to the above theorem, the BSDE (6.1) can be directly seen as a well-posed path-dependent PDE (6.4). Thus the results of existence, uniqueness, monotonicity and regularity of BSDEs, obtained in the past decades can be directly applied to obtain the corresponding properties in the PPDEs framework.

Remark 6.1 As far as we know, there is no such literature to study classical solutions to fully nonlinear path-dependent PDEs, which is much more difficult due to the complicated structure. On the other hand, in the framework of $\mathbb{P}$-Sobolev space, the time derivative $D_t u$ and the second order space derivative $D^2_x u$ are 'mixed' together to $\mathcal{A}u = D_t u + \frac{1}{2}\operatorname{tr}(D^2_x u)$. Thus only the derivatives $\mathcal{A}$ and $D_x u$ are well-defined, which cannot be applied to study a fully nonlinear PPDE.

In order to deal with fully nonlinear PPDEs, [31] and [37] introduce the G-expectation weighted Sobolev spaces, or "G-Sobolev spaces". In the second order G-Sobolev space $W_G^{1,2;p}(0, T)$, the derivatives $D_t u, D_x u$ and $D_x^2 u$ are all well defined separately due to the unique decomposition of G-Itô processes. Furthermore, the unique decomposition of generalized G-Itô processes makes it possible to well define the derivatives $\mathcal{A}_G u := D_t u + G(D_x^2 u)$ and $D_x u$ for u in the first order G-Sobolev space $W_G^{\frac{1}{2},1;p}(0,T)$ without requiring the existence of $D_t u$ and $D_x^2 u$. As an application, [37] proved the wellposedness in $W_G^{\frac{1}{2},1;p}(0,T)$ of the following fully nonlinear PPDE:

$$\mathcal{A}_G u + g(t, \omega, u, D_x u) = 0, \tag{6.5}$$

and established the one-to-one correspondence between fully nonlinear path-dependent PDE (6.5) and the BSDE driven by G-Brownian motion (see [15]).

7 Viscosity Solutions

The theory of viscosity solutions for Hamilton-Jacobi-Bellman equations was introduced by Crandall and Lions in [9]. Since then, important progresses have been made in the field of viscosity solutions theory, as it has rich connections with stochastic control, mathematical finance and so on (see [14, 42]). An important advantage of a viscosity solution is that we only need it to be a continuous function compared with classical solution (see [8]). Motivated by this, many experts have been devoted to the study of viscosity solutions of path-dependent PDEs.

In path-dependent case the main difficult comes from the fact that the space of Ω is infinite dimensional and thus lacks local compactness. Then the viscosity solutions theory introduced by Lions [18–20] cannot be applied to the path-dependent case, which is not a separable Hilbert space. Note that the viscosity solutions for first order path-dependent PDEs have been investigated in Lukoyanov [21] through adapting elegantly the compactness arguments. However, there is no an unified framework for viscosity solutions of second order path-dependent PDEs at the moment.

In [30], Peng firstly proposed a notion of viscosity solutions for path-dependent PDEs on the space of right continuous paths and established the comparison theorem using compactness argument. Then, Tang and Zhang [38] formulated a different notion of viscosity solutions for path-dependent PDEs on the space of RCLL paths. They proved that the value function of a non-Markovian stochastic optimal control problem is a viscosity solution of the related path-dependent Hamilton-Jacobi-Bellman equations. However, they did not consider the comparison theorem.

In [11], Ekren et al. firstly established the uniqueness and existence results of viscosity solutions for semi-linear path-dependent PDEs with the help of BSDEs theory. In this setting, they used the optimal stopping approach to define viscosity solutions in order to avoid the local compactness. Unlike the classical case, supersolutions and subsolutions are defined through a larger set of tangent test functions. Specifically, the tangency condition is not point-wise but in the sense of mean with respect to an appropriate class of probability measures. Afterwards, Ekren, Touzi and Zhang [12–13] extended the results of [11] to the fully nonlinear case by some more delicate and involved estimates. For more research on this topic, we refer to [33–36, 39] and the references therein.

On the other hand, Cosso and Russo [7] established the well-posedness for viscosity solutions to path-dependent heat equation (1.1) using classical Crandall-Lions notion. The arguments of [7] rely heavily on the so-called Borwein-Preiss smooth variant of Ekeland's variational principle, which exploits the completeness of the space instead of the missing local compactness. They constructed a gauge-type function and then obtained the comparison theorem of viscosity solutions to the path-dependent heat equations. We refer the reader to [5, 44] and the references therein for more research on this topic.

Finally, we would like to mention that Barrasso and Russo [1] proposed the notion of decoupled mild solutions to semilinear path-dependent equations, and Cosso and Russo [6] introduced the so-called strong-viscosity solution, which is quite similar to the notion of good solution for PDEs. There are still many questions and many difficulties to be solved in this field.

Declarations

Conflicts of interest Shige PENG is an editorial board member for Chinese Annals of Mathematics Series B and was not involved in the editorial review or the decision to publish this article. All authors declare that there are no conflicts of interest.

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