Chin. Ann. Math. Ser. B
44(6), 2023, 857–892
DOI: 10.1007/s11401-023-0049-2

Chinese Annals of Mathematics, Series B

Mean Curvature Flow of Arbitrary Codimension in Complex Projective Spaces*

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Abstract Recently, Pipoli and Sinestrari [Pipoli, G. and Sinestrari, C., Mean curvature flow of pinched submanifolds of $\mathbb{CP}^n$, *Comm. Anal. Geom.*, **25**, 2017, 799–846] initiated the study of convergence problem for the mean curvature flow of small codimension in the complex projective space $\mathbb{CP}^m$. The purpose of this paper is to develop the work due to Pipoli and Sinestrari, and verify a new convergence theorem for the mean curvature flow of arbitrary codimension in the complex projective space. Namely, the authors prove that if the initial submanifold in $\mathbb{CP}^m$ satisfies a suitable pinching condition, then the mean curvature flow converges to a round point in finite time, or converges to a totally geodesic submanifold as $t \to \infty$. Consequently, they obtain a differentiable sphere theorem for submanifolds in the complex projective space.

Keywords Mean curvature flow, Submanifolds of arbitrary codimension, Complex projective space, Convergence theorem, Differentiable sphere theorem
2000 MR Subject Classification 53C44, 53C40, 53C20, 58J35

1 Introduction

Let $F_0 : M^n \to N^{n+q}$ be an n-dimensional submanifold isometrically immersed in an $(n+q)$-dimensional Riemannian manifold N. The mean curvature flow with initial value F_0 is a smooth family of immersions $F : M \times [0, T) \to N$ satisfying

$$\begin{cases} \dfrac{\partial}{\partial t}F(x,t) = H(x,t), \\ F(\cdot,0) = F_0, \end{cases} \tag{1.1}$$

where $H(x,t)$ is the mean curvature vector of the submanifold $M_t = F_t(M)$, $F_t = F(\cdot, t)$.

In 1984, Huisken [13] first proved that uniformly convex hypersurfaces in Euclidean space will converge to a round point along the mean curvature flow. Later, Huisken [14–15] verified the beautiful convergence theorems for the mean curvature flows of convex hypersurfaces in certain Riemannian manifolds and pinched hypersurfaces in spheres.

In [10, 36], Gu and Xu proved a convergence theorem for the Ricci flow of submanifolds of arbitrary codimension in a Riemannian manifold. Consequently they obtained a differentiable

Manuscript received May 13, 2023. Revised August 17, 2023.
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*This work was supported by the National Natural Science Foundation of China (Nos. 12071424, 11531012, 12201087).

sphere theorem for submanifolds in the space form $\mathbb{F}^{n+q}(c)$ under the pinching condition $|h|^2 \leq 2c + \frac{|H|^2}{n-1}$. Notice that if $c = 0$, the pinching condition above is the best possible. Meanwhile, under the same pinching condition for the initial submanifold, Andrews-Baker-Liu-Xu-Ye-Zhao [1–2, 21] proved a convergence theorem for the mean curvature flow of submanifolds and a differentiable sphere theorem for submanifolds in the space form $\mathbb{F}^{n+q}(c)$. Afterwards, Liu, Xu and Zhao [22] obtained a convergence result for mean curvature flow of arbitrary codimension in certain Riemannian manifolds. Recently, inspired by the rigidity theory of submanifolds (see [29, 33–35, 39]), and by developing new techniques, Lei and Xu [16–18] verified an optimal convergence theorem for the mean curvature flow of submanifolds in hyperbolic spaces and a new convergence theorem for the mean curvature flow of submanifolds in spheres, which improve the convergence theorems due to Baker [2], Huisken [15] and Liu-Xu-Ye-Zhao [21]. For more results on rigidity, sphere and convergence theorems, we refer the readers to [3–5, 7–9, 11, 19–20, 23–25, 28, 30–32, 37–38, 40].

More recently, Pipoli and Sinestrari [26] first proved the following convergence theorem for mean curvature flow of small codimension in the complex projective space.

Theorem A *Let $F_0 : M^n \to \mathbb{CP}^{\frac{n+q}{2}}$ be a closed submanifold of dimension n and codimension q in the complex projective space with Fubini-Study metric. Suppose either $n \geq 5$ and $q = 1$, or $2 \leq q < \frac{n-3}{4}$. Let $F : M^n \times [0,T) \to \mathbb{CP}^{\frac{n+q}{2}}$ be the mean curvature flow with initial value F_0. If F_0 satisfies*

$$|h|^2 < \begin{cases} \dfrac{1}{n-1}|H|^2 + 2, & q = 1, \\ \dfrac{1}{n-1}|H|^2 + \dfrac{n-3-4q}{n}, & q \geq 2, \end{cases}$$

then F_t converges to a round point in finite time, or converges to a totally geodesic submanifold as $t \to \infty$. In particular, M is diffeomorphic to either $\mathbb{S}^n$ or $\mathbb{CP}^{\frac{n}{2}}$.

In this paper, we investigate the mean curvature flow of arbitrary codimensional submanifolds in the complex projective space, and prove the following theorems.

Theorem 1.1 *Let $F_0 : M^n \to \mathbb{CP}^{\frac{n+1}{2}}$ be an n-dimensional closed hypersurface in $\mathbb{CP}^{\frac{n+1}{2}}$ $(n \geq 3)$. Let $F : M^n \times [0,T) \to \mathbb{CP}^{\frac{n+1}{2}}$ be the mean curvature flow with initial value F_0. If F_0 satisfies*

$$|h|^2 < \varphi(|H|^2), \tag{1.2}$$

then F_t converges to a round point in finite time. Here $\varphi(|H|^2)$ is given by

$$\varphi(|H|^2) = 2 + a_n + \Big(b_n + \frac{1}{n-1}\Big)|H|^2 - \sqrt{b_n^2|H|^4 + 2a_nb_n|H|^2},$$
$$a_n = 2\sqrt{(n^2-4n+3)b_n},$$
$$b_n = \min\Big\{\frac{n-3}{4n-4}, \frac{2n-5}{n^2+n-2}\Big\}.$$

Remark 1.1 By a computation, if $n > 3$, we have $\varphi(x) > \frac{x}{n-1} + 2$ for $x \geq 0$. Furthermore, if $n \geq 3$, we have $\varphi(x) > \sqrt{2(n-3)}$ for $x \geq 0$. Therefore, Theorem 1.1 substantially improves the hypersurface case of Theorem A.

Theorem 1.2 *Let $F_0 : M^n \to \mathbb{CP}^{\frac{n+q}{2}}$ be an n-dimensional closed submanifold in $\mathbb{CP}^{\frac{n+q}{2}}$. Suppose that the dimension and codimension satisfy $2 \leq q < n-4$. Let $F : M^n \times [0,T) \to \mathbb{CP}^{\frac{n+q}{2}}$ be the mean curvature flow with initial value F_0. If F_0 satisfies*

$$|h|^2 < \frac{1}{n-1}|H|^2 + 2 - \frac{3}{n}, \tag{1.3}$$

then F_t converges to a round point in finite time, or converges to a totally geodesic submanifold $\mathbb{CP}^{\frac{n}{2}}$ as $t \to \infty$.

Notice that Theorem 1.2 improves the $q \geq 2$ case of Theorem A.

Theorem 1.3 *Let $F_0 : M^n \to \mathbb{CP}^{\frac{n+q}{2}}$ be an n-dimensional closed submanifold in $\mathbb{CP}^{\frac{n+q}{2}}$. Suppose that the dimension and codimension satisfy $q \geq n-4 \geq 2$. Let $F : M^n \times [0,T) \to \mathbb{CP}^{\frac{n+q}{2}}$ be the mean curvature flow with initial value F_0. If F_0 satisfies*

$$|h|^2 < \psi(|H|^2), \tag{1.4}$$

then F_t converges to a round point in finite time. Here $\psi(|H|^2)$ is given by

$$\psi(|H|^2) = \frac{9}{n^2-3n-3} + \frac{n^2-3n}{n^3-4n^2+3}|H|^2 - \frac{3\sqrt{|H|^4 + \frac{2}{n}(n-1)(n^2-3)|H|^2 + 9(n-1)^2}}{n^3-4n^2+3}.$$

Remark 1.2 The function ψ satisfies $\psi(x) \geq \frac{x}{n}$ for $x \geq 0$, and the equality holds if and only if $x = 0$. In addition, the function $\psi(x)$ has an asymptote $\frac{x}{n-1} - \frac{3}{n}$ as $x \to +\infty$.

From the convergence results of the mean curvature flow, we obtain a classification theorem for submanifolds in complex projective space.

Theorem 1.4 *Let M^n be an n-dimensional closed submanifold in $\mathbb{CP}^{\frac{n+q}{2}}$. If M satisfies*

$$|h|^2 \leq \begin{cases} \varphi(|H|^2), & q = 1, n \geq 3, \\ \frac{1}{n-1}|H|^2 + 2 - \frac{3}{n}, & 2 \leq q < n-4, \\ \psi(|H|^2), & q \geq n-4 \geq 2, \end{cases} \tag{1.5}$$

then (i) M *is diffeomorphic to* $\mathbb{S}^n$ *or* $\mathbb{CP}^{\frac{n}{2}}$, *or* (ii) M *is congruent to the totally geodesic* $\mathbb{RP}^n$.

2 Notations and Formulas

Let $\mathbb{CP}^m$ be the m-dimensional complex projective space with the Fubini-Study metric g_{FS}. Let J be its complex structure. We denote by $\overline{\nabla}$ the Levi-Civita connection of $(\mathbb{CP}^m, g_{FS})$. Since the Fubini-Study metric is a Kähler metric, we have $\overline{\nabla}J = 0$. The curvature tensor $\overline{R}$ of $\mathbb{CP}^m$ can be written as

$$\begin{aligned}\overline{R}(X,Y,Z,W) = &\langle X,Z\rangle\langle Y,W\rangle - \langle X,W\rangle\langle Y,Z\rangle \\ &+ \langle X,JZ\rangle\langle Y,JW\rangle - \langle X,JW\rangle\langle Y,JZ\rangle \\ &+ 2\langle X,JY\rangle\langle Z,JW\rangle. \end{aligned} \tag{2.1}$$

Let (M^n, g) be a real n-dimensional Riemannian submanifold immersed in $(\mathbb{CP}^m, g_{FS})$. Let q be its codimension, i.e., $n + q = 2m$. At a point $p \in M$, let T_pM and N_pM be the tangent space and normal space, respectively. For a vector in $T_pM \oplus N_pM$, we denote by $(\cdot)^T$ and $(\cdot)^N$ its projections onto T_pM and N_pM, respectively. We use the same symbol ∇ to represent the connections of tangent bundle TM and normal bundle NM. Denote by $\Gamma(E)$ the space of smooth sections of a vector bundle E. For $X, Y \in \Gamma(TM)$, $\xi \in \Gamma(NM)$, the connections ∇ are given by $\nabla_X Y = (\overline{\nabla}_X Y)^T$ and $\nabla_X \xi = (\overline{\nabla}_X \xi)^N$. The second fundamental form of M is defined as $h(X,Y) = (\overline{\nabla}_X Y)^N$.

Throughout this paper, we shall make the following convention on indices:

$$1 \le A, B, C, \cdots \le n+q, \quad 1 \le i, j, k, \cdots \le n, \quad n+1 \le \alpha, \beta, \gamma, \cdots \le n+q.$$

We choose a local orthonormal frame $\{e_i\}$ for the tangent bundle and a local orthonormal frame $\{e_\alpha\}$ for the normal bundle. With the local frame, the components of h are given by $h^\alpha_{ij} = \langle h(e_i, e_j), e_\alpha\rangle$. The mean curvature vector is defined as $H = \sum\limits_\alpha H^\alpha e_\alpha$, where $H^\alpha = \sum\limits_i h^\alpha_{ii}$. Let $\mathring{h} = h - \frac{1}{n}H \otimes g$ be the traceless second fundamental form. We have the relations $|\mathring{h}|^2 = |h|^2 - \frac{1}{n}|H|^2$ and $|\nabla\mathring{h}|^2 = |\nabla h|^2 - \frac{1}{n}|\nabla H|^2$.

We denote by (J_{AB}) the matrix of J with respect to the frame $\{e_A\}$, i.e., $J_{AB} = \langle e_A, Je_B\rangle$. This matrix satisfies $J_{AB} = -J_{BA}$ and $\sum\limits_B J_{AB}J_{BC} = -\delta_{AC}$.

At each point $p \in M$, we define a tensor $P : N_pM \to T_pM$ by

$$P\xi = (J\xi)^T \quad \text{for } \xi \in N_pM.$$

Then we have

$$|P|^2 = \sum_\alpha |Pe_\alpha|^2 \le \sum_\alpha |e_\alpha|^2 = q$$

and

$$|P|^2 = \sum_{\alpha,i} (J_{i\alpha})^2 = \sum_{A,i} (J_{iA})^2 - \sum_{i,j} (J_{ij})^2 = n - \sum_{i,j} (J_{ij})^2.$$

We have the following estimates for the gradient of the second fundamental form.

Lemma 2.1 *For an n-dimensional submanifold in $\mathbb{CP}^{\frac{n+q}{2}}$, we have*

$$|\nabla h|^2 \geq \begin{cases} \dfrac{3}{n+2}|\nabla H|^2 + 2(n-1), & q = 1, \\ \dfrac{3}{n+8}|\nabla H|^2 + 2(n-q)|P|^2, & 2 \leq q < n, \\ \dfrac{3}{n+8}|\nabla H|^2, & q \geq n. \end{cases}$$

Proof Let S be the symmetric part of ∇h, i.e., $S^\alpha_{ijk} = \frac{1}{3}(\nabla_i h^\alpha_{jk} + \nabla_j h^\alpha_{ik} + \nabla_k h^\alpha_{ij})$. Using the same argument as in the proof of [13, Lemma 2.2], we have

$$|S|^2 \geq \frac{3}{n+2} \sum_{\alpha,i} \Big(\sum_k S^\alpha_{ikk} \Big)^2. \tag{2.2}$$

By the Codazzi equation, we have $\sum\limits_k S^\alpha_{ikk} = \nabla_i H^\alpha + 2 \sum\limits_k J_{\alpha k} J_{ki}$. Then we obtain

$$\sum_{\alpha,i} \Big(\sum_k S^\alpha_{ikk} \Big)^2 = |\nabla H|^2 + 4 \sum_{\alpha,i,k} \nabla_i H^\alpha J_{\alpha k} J_{ki} + 4 \sum_{\alpha,i} \Big(\sum_k J_{\alpha k} J_{ki} \Big)^2. \tag{2.3}$$

Using the Codazzi equation, the symmetry of h and the skew-symmetry of $\overline{R}$, we get

$$\begin{aligned} |S|^2 &= \frac{1}{3}|\nabla h|^2 + \frac{2}{3} \sum_{\alpha,i,j,k} \nabla_k h^\alpha_{ij} \nabla_j h^\alpha_{ik} \\ &= \frac{1}{3}|\nabla h|^2 + \frac{2}{3} \sum_{\alpha,i,j,k} \nabla_k h^\alpha_{ij} (\nabla_k h^\alpha_{ij} + \overline{R}_{\alpha ijk}) \\ &= |\nabla h|^2 + \frac{2}{3} \sum_{\alpha,i,j,k} \nabla_k h^\alpha_{ij} \overline{R}_{\alpha ijk} \\ &= |\nabla h|^2 + \frac{2}{3} \sum_{\alpha,i,j,k} (\nabla_i h^\alpha_{jk} + \overline{R}_{\alpha jki}) \overline{R}_{\alpha ijk} \\ &= |\nabla h|^2 + \frac{2}{3} \sum_{\alpha,i,j,k} \overline{R}_{\alpha jki} \overline{R}_{\alpha ijk} \\ &= |\nabla h|^2 - 2 \sum_{\alpha,i,j,k} [J_{\alpha j} J_{ji} J_{\alpha k} J_{ki} + (J_{\alpha i})^2 (J_{jk})^2] \\ &= |\nabla h|^2 - 2 \sum_{\alpha,i} \Big(\sum_k J_{\alpha k} J_{ki} \Big)^2 - 2|P|^2(n - |P|^2). \end{aligned} \tag{2.4}$$

From (2.2)–(2.4), we obtain

$$\begin{aligned} |\nabla h|^2 \geq{}& \frac{1}{n+2} \Big[3|\nabla H|^2 + 12 \sum_{\alpha,i,k} \nabla_i H^\alpha J_{\alpha k} J_{ki} + 2(n+8) \sum_{\alpha,i} \Big(\sum_k J_{\alpha k} J_{ki} \Big)^2 \Big] \\ &+ 2|P|^2(n - |P|^2). \end{aligned} \tag{2.5}$$

If $q=1$, we have $\sum\limits_k J_{\alpha k}J_{ki}=0$ and $|P|^2=1$. Thus (2.5) becomes

$$|\nabla h|^2\geq\frac{3}{n+2}|\nabla H|^2+2(n-1).$$

If $q\geq 2$, from (2.5) we get

$$\begin{aligned}|\nabla h|^2\geq&\frac{2}{n+2}\sum_{\alpha,i}\Big(3\sqrt{\frac{1}{n+8}}\nabla_iH^\alpha+\sqrt{n+8}\sum_k J_{\alpha k}J_{ki}\Big)^2\\&+\frac{3}{n+8}|\nabla H|^2+2|P|^2(n-|P|^2).\end{aligned}$$

From $|P|^2\leq q$ and $|P|^2\leq n$, we complete the proof of Lemma 2.1.

Let $F:M^n\times[0,T)\to\mathbb{CP}^m$ be a mean curvature flow in a complex projective space. For a fixed t, letting $F_t=F(\cdot,t)$, then $F_t:M^n\to\mathbb{CP}^m$ is a Riemannian submanifold in $\mathbb{CP}^m$. We denote by M_t the submanifold at time t. Following [1, 26], we have the evolution equations below.

$$\begin{aligned}\partial_t|h|^2=&\Delta|h|^2-2|\nabla h|^2\\&+2\sum_{\alpha,\beta}\Big(\sum_{i,j}h^\alpha_{ij}h^\beta_{ij}\Big)^2+2\sum_{i,j,\alpha,\beta}\Big(\sum_k(h^\alpha_{ik}h^\beta_{jk}-h^\beta_{ik}h^\alpha_{jk})\Big)^2\\&+2\sum_{\alpha,\beta,i,j,k}h^\alpha_{ij}h^\beta_{ij}\overline{R}_{\alpha k\beta k}+8\sum_{\alpha,\beta,i,j,k}h^\alpha_{ik}h^\beta_{jk}\overline{R}_{\alpha\beta ij}\\&+4\sum_{\alpha,i,j,k,l}(h^\alpha_{ik}h^\alpha_{jl}\overline{R}_{ijkl}-h^\alpha_{ik}h^\alpha_{jk}\overline{R}_{iljl}),\\ \partial_t|H|^2=&\Delta|H|^2-2|\nabla H|^2+2\sum_{i,j}\Big(\sum_\alpha H^\alpha h^\alpha_{ij}\Big)^2+2\sum_{\alpha,\beta,k}H^\alpha H^\beta\overline{R}_{\alpha k\beta k}.\end{aligned}$$

From (2.1), these evolution equations can be written as the following form.

Lemma 2.2 *For mean curvature flow $F:M^n\times[0,T)\to\mathbb{CP}^m$, we have*

(i) $\partial_t|h|^2=\Delta|h|^2-2|\nabla h|^2-2n|h|^2+4|H|^2+2R_1+2S_1$,

(ii) $\partial_t|H|^2=\Delta|H|^2-2|\nabla H|^2+2n|H|^2+2R_2+6S_2$,

(iii) $\partial_t|\mathring{h}|^2=\Delta|\mathring{h}|^2-2|\nabla\mathring{h}|^2-2n|\mathring{h}|^2+2R_1-\frac{2}{n}R_2+2S_1-\frac{6}{n}S_2$,

where

$$\begin{aligned}R_1=&\sum_{\alpha,\beta}\Big(\sum_{i,j}h^\alpha_{ij}h^\beta_{ij}\Big)^2+\sum_{i,j,\alpha,\beta}\Big(\sum_k(h^\alpha_{ik}h^\beta_{jk}-h^\beta_{ik}h^\alpha_{jk})\Big)^2,\\ R_2=&\sum_{i,j}\Big(\sum_\alpha H^\alpha h^\alpha_{ij}\Big)^2,\end{aligned}$$

$$
\begin{aligned}
S_1 = {} & 3\sum_{i,j,k}\Big(\sum_{\alpha} h^{\alpha}_{ij}J_{k\alpha}\Big)^2 + 4\sum_{\alpha,\beta,i,j,k}\mathring{h}^{\alpha}_{ik}\mathring{h}^{\beta}_{jk}(J_{i\alpha}J_{j\beta}-J_{i\beta}J_{j\alpha})\\
& + 6\sum_{\alpha,i,j,k,l}(\mathring{h}^{\alpha}_{ij}\mathring{h}^{\alpha}_{kl}J_{il}J_{jk}-\mathring{h}^{\alpha}_{ik}\mathring{h}^{\alpha}_{jk}J_{il}J_{jl})\\
& + 8\sum_{\alpha,\beta,i,j,k}\mathring{h}^{\alpha}_{ik}\mathring{h}^{\beta}_{jk}J_{\alpha\beta}J_{ij},\\
S_2 = {} & \sum_{k}\Big(\sum_{\alpha}H^{\alpha}J_{k\alpha}\Big)^2.
\end{aligned}
$$

To do computations involving J_{AB}, we present the following well-known property of the skew-symmetric matrix.

Proposition 2.1 *Let A be a real skew-symmetric matrix. Then there exists an orthogonal matrix C, such that $C^{-1}AC$ takes the following form*

$$
\begin{pmatrix}
0 & \lambda_1 & & & & & & \\
-\lambda_1 & 0 & & & & & & \\
& & 0 & \lambda_3 & & & & \\
& & -\lambda_3 & 0 & & & & \\
& & & & 0 & \lambda_5 & & \\
& & & & -\lambda_5 & 0 & & \\
& & & & & & \ddots & \\
& & & & & & & \ddots
\end{pmatrix}. \tag{2.6}
$$

We use a notation

$$
\bar{i} = \begin{cases} i+1, & i \text{ is odd},\\ i-1, & i \text{ is even}. \end{cases}
$$

If a matrix (a_{ij}) takes the form of (2.6), then $a_{ij}=0$ for all $j \neq \bar{i}$.

3 Preservation of Curvature Pinching

3.1 The case of $q = 1$

For the mean curvature flow of hypersurfaces in $\mathbb{CP}^m$, the evolution equations in Lemma 2.2 become

$$
\begin{aligned}
\partial_t|\mathring{h}|^2 = {} & \Delta|\mathring{h}|^2 - 2|\nabla\mathring{h}|^2 - 2n|\mathring{h}|^2 + 2|\mathring{h}|^2|h|^2 + 6|\mathring{h}|^2\\
& + 12\sum_{i,j,k,l}(\mathring{h}_{ij}\mathring{h}_{kl}J_{il}J_{jk} - \mathring{h}_{ik}\mathring{h}_{jk}J_{il}J_{jl}),
\end{aligned} \tag{3.1}
$$

$$
\partial_t|H|^2 = \Delta|H|^2 - 2|\nabla H|^2 + 2n|H|^2 + 2|H|^2|h|^2 + 6|H|^2. \tag{3.2}
$$

We choose a orthonormal frame $\{e_i\}$ such that the matrix (J_{ij}) takes the form of (2.6). Denote by $\check{n}$ the largest even integer satisfying $\check{n} \le n$. Then we have

$$\begin{aligned}&\sum_{i,j,k,l}(\mathring{h}_{ij}\mathring{h}_{kl}J_{il}J_{jk}-\mathring{h}_{ik}\mathring{h}_{jk}J_{il}J_{jl})\\&=\sum_{i,k\le\check{n}}(-\mathring{h}_{i\bar{k}}\mathring{h}_{k\bar{i}}J_{i\bar{i}}J_{k\bar{k}}-(\mathring{h}_{\bar{i}k}J_{i\bar{i}})^2)\\&=-\frac{1}{2}\sum_{i,k\le\check{n}}(\mathring{h}_{i\bar{k}}J_{k\bar{k}}+\mathring{h}_{\bar{i}k}J_{i\bar{i}})^2\le 0.\end{aligned}$$

So, we get

$$\partial_t|\mathring{h}|^2 \le \Delta|\mathring{h}|^2 - 2|\nabla\mathring{h}|^2 + 2|\mathring{h}|^2(|h|^2 - n + 3). \tag{3.3}$$

For a real number $\varepsilon \in [0,1]$, we define a function $\varphi_\varepsilon : [0,+\infty) \to \mathbb{R}$ by

$$\varphi_\varepsilon(x) := d_\varepsilon + c_\varepsilon x - \sqrt{b^2x^2 + 2abx + e}, \tag{3.4}$$

where $a = 2\sqrt{(n^2-4n+3)b}$, $b = \min\left\{\frac{n-3}{4n-4}, \frac{2n-5}{n^2+n-2}\right\}$, $c_\varepsilon = b + \frac{1}{n-1+\varepsilon}$, $d_\varepsilon = 2 - 2\varepsilon + a$, $e = \sqrt{\varepsilon}$. We define $\varphi = \varphi_0$.

Let $\mathring{\varphi}_\varepsilon(x) = \varphi_\varepsilon(x) - \frac{x}{n}$. The following lemma will be proved in the Appendix.

Lemma 3.1 *For sufficiently small ε, the function φ_ε satisfies*

(i) $2x\mathring{\varphi}''_\varepsilon(x) + \mathring{\varphi}'_\varepsilon(x) < \frac{2(n-1)}{n(n+2)}$,

(ii) $\mathring{\varphi}_\varepsilon(x)(\varphi_\varepsilon(x) - n + 3) - x\mathring{\varphi}'_\varepsilon(x)(\varphi_\varepsilon(x) + n + 3) < 2(n-1)$,

(iii) $\mathring{\varphi}_\varepsilon(x) - x\mathring{\varphi}'_\varepsilon(x) > 1$.

Suppose that M_0 is an n-dimensional closed hypersurface in $\mathbb{CP}^{\frac{n+1}{2}}$ satisfying $|h|^2 < \varphi(|H|^2)$. Let $F : M^n \times [0,T) \to \mathbb{CP}^{\frac{n+1}{2}}$ be a mean curvature flow with initial value M_0. We will show that the pinching condition is preserved along the flow. For convenience, we denote $\mathring{\varphi}_\varepsilon(|H|^2)$, $\mathring{\varphi}'_\varepsilon(|H|^2)$, $\mathring{\varphi}''_\varepsilon(|H|^2)$ by $\mathring{\varphi}_\varepsilon$, $\mathring{\varphi}'_\varepsilon$, $\mathring{\varphi}''_\varepsilon$, respectively.

Theorem 3.1 *If the initial value M_0 satisfies $|h|^2 < \varphi(|H|^2)$, then there exists a small positive number ε, such that for all $t \in [0,T)$, we have $|h|^2 < \varphi - \varepsilon|H|^2 - \varepsilon$.*

Proof Since M_0 is compact, there exists a small positive number ε_1, such that M_0 satisfies $|\mathring{h}|^2 < \mathring{\varphi}_{\varepsilon_1}$.

From (3.2) and Lemma 3.1(i), we have

$$\begin{aligned}(\partial_t - \Delta)\mathring{\varphi}_{\varepsilon_1} &= -2(\mathring{\varphi}'_{\varepsilon_1} + 2\mathring{\varphi}''_{\varepsilon_1}\cdot|H|^2)|\nabla H|^2 + 2\mathring{\varphi}'_{\varepsilon_1}\cdot|H|^2(|h|^2+n+3)\\&\ge -\frac{4(n-1)}{n(n+2)}|\nabla H|^2 + 2\mathring{\varphi}'_{\varepsilon_1}\cdot|H|^2(|h|^2+n+3).\end{aligned} \tag{3.5}$$

Let $U = |\mathring{h}|^2 - \mathring{\varphi}_{\varepsilon_1}$. We obtain

$$\begin{aligned}\frac{1}{2}(\partial_t - \Delta)U \leq -|\nabla\mathring{h}|^2 + \frac{2(n-1)}{n(n+2)}|\nabla H|^2 \\ + |\mathring{h}|^2(|h|^2 - n + 3) - \mathring{\varphi}'_{\varepsilon_1}\cdot|H|^2(|h|^2+n+3).\end{aligned}$$

By Lemma 2.1, we have

$$-|\nabla\mathring{h}|^2 + \frac{2(n-1)}{n(n+2)}|\nabla H|^2 \leq -2(n-1).$$

Thus, at the points where $U = 0$, we get

$$\frac{1}{2}(\partial_t - \Delta)U \leq -2(n-1) + \mathring{\varphi}_{\varepsilon_1}(\varphi_{\varepsilon_1} - n + 3) - \mathring{\varphi}'_{\varepsilon_1}\cdot|H|^2(\varphi_{\varepsilon_1} + n + 3) < 0.$$

Applying the maximum principle, we obtain $U < 0$ for all $t \in [0, T)$.

Thus, we have $|h|^2 < \varphi_{\varepsilon_1}$ for all $t \in [0, T)$. By choosing ε small enough, we complete the proof of Theorem 3.1.

Let

$$f_\sigma = \frac{|\mathring{h}|^2}{\mathring{\varphi}^{1-\sigma}},$$

where $\sigma \in (0, \varepsilon^2)$ is a positive constant. Then we have the following lemma.

Lemma 3.2 *If M_0 satisfies $|h|^2 < \varphi(|H|^2)$, then there exists a small positive number ε, such that the following inequality holds along the mean curvature flow*:

$$\partial_t f_\sigma \leq \Delta f_\sigma + \frac{2}{\mathring{\varphi}}|\nabla f_\sigma||\nabla\mathring{\varphi}| - \frac{2\varepsilon f_\sigma}{n|\mathring{h}|^2}|\nabla\mathring{h}|^2 + 2\sigma|h|^2 f_\sigma - \frac{\varepsilon}{n}f_\sigma.$$

Proof By a straightforward calculation, we have

$$\begin{aligned}(\partial_t - \Delta)f_\sigma = f_\sigma\Big[\frac{1}{|\mathring{h}|^2}(\partial_t - \Delta)|\mathring{h}|^2 - \frac{1-\sigma}{\mathring{\varphi}}(\partial_t - \Delta)\mathring{\varphi}\Big] \\ + 2(1-\sigma)\frac{\langle\nabla f_\sigma, \nabla\mathring{\varphi}t\rangle}{\mathring{\varphi}} - \sigma(1-\sigma)f_\sigma\frac{|\nabla\mathring{\varphi}|^2}{|\mathring{\varphi}|^2}.\end{aligned}$$

Using (3.3) and (3.5), we have

$$\begin{aligned}(\partial_t - \Delta)f_\sigma \leq 2f_\sigma\Big[-\frac{|\nabla\mathring{h}|^2}{|\mathring{h}|^2} + \frac{2(n-1)}{n(n+2)}\frac{|\nabla H|^2}{\mathring{\varphi}}\Big] \\ + 2f_\sigma\Big[|h|^2 - n + 3 - (1-\sigma)\frac{\mathring{\varphi}'\cdot|H|^2}{\mathring{\varphi}}(|h|^2+n+3)\Big] \\ + \frac{2}{\mathring{\varphi}}|\nabla f_\sigma||\nabla\mathring{\varphi}|.\end{aligned}$$

From Lemma 2.1 and Theorem 3.1, we have

$$\begin{aligned}
&-\frac{|\nabla\mathring{h}|^2}{|\mathring{h}|^2}+\frac{2(n-1)}{n(n+2)}\frac{|\nabla H|^2}{\mathring{\varphi}}\\
\le&-\frac{|\nabla\mathring{h}|^2}{|\mathring{h}|^2}+\frac{|\nabla\mathring{h}|^2-2(n-1)}{\mathring{\varphi}}\\
\le&\frac{|\mathring{h}|^2-\mathring{\varphi}}{|\mathring{h}|^2\mathring{\varphi}}|\nabla\mathring{h}|^2-\frac{2(n-1)}{\mathring{\varphi}}\\
\le&-\varepsilon\frac{|H|^2+1}{|\mathring{h}|^2\mathring{\varphi}}|\nabla\mathring{h}|^2-\frac{2(n-1)}{\mathring{\varphi}}\\
\le&-\frac{\varepsilon}{n|\mathring{h}|^2}|\nabla\mathring{h}|^2-\frac{2(n-1)}{\mathring{\varphi}}.
\end{aligned}$$

From Lemma 3.1(ii) and (iii), we have

$$\begin{aligned}
&|h|^2-n+3-(1-\sigma)\frac{\mathring{\varphi}'\cdot|H|^2}{\mathring{\varphi}}(|h|^2+n+3)\\
=&\frac{1-\sigma}{\mathring{\varphi}}[(\mathring{\varphi}-\mathring{\varphi}'\cdot|H|^2)|h|^2-\mathring{\varphi}'\cdot|H|^2(n+3)]-n+3+\sigma|h|^2\\
\le&\frac{1-\sigma}{\mathring{\varphi}}[(\mathring{\varphi}-\mathring{\varphi}'\cdot|H|^2)(\varphi-\varepsilon|H|^2-\varepsilon)-\mathring{\varphi}'\cdot|H|^2(n+3)]-n+3+\sigma|h|^2\\
=&\frac{1-\sigma}{\mathring{\varphi}}[(\mathring{\varphi}-\mathring{\varphi}'\cdot|H|^2)\varphi-\mathring{\varphi}'\cdot|H|^2(n+3)]-n+3+\sigma|h|^2\\
&-\frac{(1-\sigma)\varepsilon}{\mathring{\varphi}}(\mathring{\varphi}-\mathring{\varphi}'\cdot|H|^2)(|H|^2+1)\\
\le&\frac{1-\sigma}{\mathring{\varphi}}[(n-3)\mathring{\varphi}+2(n-1)]-n+3+\sigma|h|^2-\frac{(1-\sigma)\varepsilon}{\mathring{\varphi}}(|H|^2+1)\\
\le&\sigma|h|^2+\frac{2(n-1)}{\mathring{\varphi}}-\frac{\varepsilon}{2n}.
\end{aligned}$$

This completes the proof of the lemma.

3.2 The case of $2\le q<n-4$

At a fixed point $p\in M$, we always choose the orthonormal frame $\{e_\alpha\}$ for N_pM such that $H=|H|e_{n+1}$. Set

$$|\mathring{h}^\alpha|^2=\sum_{i,j}(\mathring{h}^\alpha_{ij})^2,\quad \rho_1=|\mathring{h}^{n+1}|^2,\quad \rho_2=\sum_{\alpha>n+1}|\mathring{h}^\alpha|^2,$$

then $|\mathring{h}|^2=\rho_1+\rho_2$. Set

$$\theta_1=|Pe_{n+1}|^2,\quad \theta_2=\sum_{\alpha>n+1}|Pe_\alpha|^2,$$

then $|P|^2=\theta_1+\theta_2$, $\theta_1\le 1$ and $\theta_2\le q-1$.

Lemma 3.3

(i) $R_1 \leq |h|^4 - \frac{2}{n}\rho_2|H|^2 + 2\rho_1\rho_2 + \frac{1}{2}\rho_2^2$,

(ii) $R_2 = |H|^2(|h|^2 - \rho_2)$,

(iii) $S_1 \leq \frac{3}{n}S_2 + 3|\mathring{h}|^2 + 8\sqrt{\theta_2\rho_1\rho_2} + 4\theta_2\rho_2$.

Proof The estimates of R_1 and R_2 are similar to that in [1]. We choose an orthonormal frame $\{e_i\}$ for the tangent space, such that $\mathring{h}_{ij}^{n+1} = \mathring{\lambda}_i\delta_{ij}$. Then we have

$$\begin{aligned} R_1 = {} & \rho_1^2 + \frac{2}{n}\rho_1|H|^2 + \frac{1}{n^2}|H|^4 \\ & + 2\sum_{\alpha>n+1}\Big(\sum_i \mathring{\lambda}_i\mathring{h}_{ii}^{\alpha}\Big)^2 + 2\sum_{\substack{\alpha>n+1\\ i\neq j}}((\mathring{\lambda}_i - \mathring{\lambda}_j)\mathring{h}_{ij}^{\alpha})^2 \\ & + \sum_{\alpha,\beta>n+1}\Big(\sum_{i,j}\mathring{h}_{ij}^{\alpha}\mathring{h}_{ij}^{\beta}\Big)^2 + \sum_{\substack{\alpha,\beta>n+1\\ i,j}}\Big(\sum_k(\mathring{h}_{ik}^{\alpha}\mathring{h}_{jk}^{\beta} - \mathring{h}_{jk}^{\alpha}\mathring{h}_{ik}^{\beta})\Big)^2. \end{aligned} \tag{3.6}$$

Using the Cauchy-Schwarz inequality, we get

$$\begin{aligned} \Big(\sum_i \mathring{\lambda}_i\mathring{h}_{ii}^{\alpha}\Big)^2 + \sum_{i\neq j}((\mathring{\lambda}_i - \mathring{\lambda}_j)\mathring{h}_{ij}^{\alpha})^2 & \leq \rho_1\sum_i(\mathring{h}_{ii}^{\alpha})^2 + 2\sum_{i\neq j}(\mathring{\lambda}_i^2 + \mathring{\lambda}_j^2)(\mathring{h}_{ij}^{\alpha})^2 \\ & \leq \rho_1\sum_i(\mathring{h}_{ii}^{\alpha})^2 + 2\rho_1\sum_{i\neq j}(\mathring{h}_{ij}^{\alpha})^2 \\ & \leq 2\rho_1|\mathring{h}^{\alpha}|^2. \end{aligned}$$

It follows from [19, Theorem 1] that

$$\sum_{\alpha,\beta>n+1}\Big(\sum_{i,j}\mathring{h}_{ij}^{\alpha}\mathring{h}_{ij}^{\beta}\Big)^2 + \sum_{\substack{\alpha,\beta>n+1\\ i,j}}\Big(\sum_k(\mathring{h}_{ik}^{\alpha}\mathring{h}_{jk}^{\beta} - \mathring{h}_{jk}^{\alpha}\mathring{h}_{ik}^{\beta})\Big)^2 \leq \frac{3}{2}\rho_2^2.$$

Thus we obtain

$$\begin{aligned} R_1 & \leq \rho_1^2 + \frac{2}{n}\rho_1|H|^2 + \frac{1}{n^2}|H|^4 + 4\rho_1\rho_2 + \frac{3}{2}\rho_2^2 \\ & = |h|^4 - \frac{2}{n}\rho_2|H|^2 + 2\rho_1\rho_2 + \frac{1}{2}\rho_2^2. \end{aligned}$$

R_2 can be written as

$$R_2 = \sum_{i,j}(|H|h_{ij}^{n+1})^2 = |H|^2(|h|^2 - \rho_2). \tag{3.7}$$

Next, we have

$$\begin{aligned}
&\sum_{i,j,k}\Big(\sum_{\alpha}h_{ij}^{\alpha}J_{k\alpha}\Big)^2\\
&=\frac{1}{n}\sum_{k}\Big(\sum_{\alpha}H^{\alpha}J_{k\alpha}\Big)^2+\sum_{i,j,k}\Big(\sum_{\alpha}\mathring{h}_{ij}^{\alpha}J_{k\alpha}\Big)^2\\
&=\frac{S_2}{n}+\sum_{i,j}|P\mathring{h}(e_i,e_j)|^2\\
&=\frac{S_2}{n}+|P\mathring{h}|^2.
\end{aligned}\tag{3.8}$$

Choose an orthonormal frame $\{e_i\}$ such that the matrix (J_{ij}) takes the form of (2.6). Thus

$$\begin{aligned}
&6\sum_{\alpha,i,j,k,l}(\mathring{h}_{ij}^{\alpha}\mathring{h}_{kl}^{\alpha}J_{il}J_{jk}-\mathring{h}_{ik}^{\alpha}\mathring{h}_{jk}^{\alpha}J_{il}J_{jl})+8\sum_{\alpha,\beta,i,j,k}\mathring{h}_{ik}^{\alpha}\mathring{h}_{jk}^{\beta}J_{\alpha\beta}J_{ij}\\
&=6\sum_{\substack{i,k\le\check{n}\\ \alpha}}(-\mathring{h}_{i\bar{k}}^{\alpha}\mathring{h}_{k\bar{i}}^{\alpha}J_{i\bar{i}}J_{k\bar{k}}-(\mathring{h}_{\bar{i}k}^{\alpha}J_{i\bar{i}})^2)+8\sum_{\substack{i\le\check{n}\\ k,\alpha,\beta}}\mathring{h}_{ik}^{\alpha}\mathring{h}_{\bar{i}k}^{\beta}J_{\alpha\beta}J_{i\bar{i}}\\
&=-3\sum_{\substack{i,k\le\check{n}\\ \alpha}}(\mathring{h}_{i\bar{k}}^{\alpha}J_{k\bar{k}}+\mathring{h}_{\bar{i}k}^{\alpha}J_{i\bar{i}})^2-4\sum_{\substack{i,k\le\check{n}\\ \alpha}}\Big[(\mathring{h}_{i\bar{k}}^{\alpha}J_{k\bar{k}}+\mathring{h}_{\bar{i}k}^{\alpha}J_{i\bar{i}})\sum_{\beta}\mathring{h}_{ik}^{\beta}J_{\alpha\beta}\Big]\\
&\le\frac{4}{3}\sum_{i,k,\alpha}\Big(\sum_{\beta}\mathring{h}_{ik}^{\beta}J_{\alpha\beta}\Big)^2\\
&=\frac{4}{3}\sum_{i,k}|(J\mathring{h}(e_i,e_k))^N|^2\\
&=\frac{4}{3}\sum_{i,k}(|J\mathring{h}(e_i,e_k)|^2-|P\mathring{h}(e_i,e_k)|^2)\\
&=\frac{4}{3}(|\mathring{h}|^2-|P\mathring{h}|^2).
\end{aligned}\tag{3.9}$$

For fixed α,β, we choose an orthonormal frame $\{e_i\}$, such that the $n\times n$ matrix $(J_{i\alpha}J_{j\beta}-J_{j\alpha}J_{i\beta})$ takes the form of (2.6). Thus we get

$$\begin{aligned}
&\sum_{i,j,k}\mathring{h}_{ik}^{\alpha}\mathring{h}_{jk}^{\beta}(J_{i\alpha}J_{j\beta}-J_{j\alpha}J_{i\beta})\\
&=\sum_{\substack{i\le\check{n}\\ k}}\mathring{h}_{ik}^{\alpha}\mathring{h}_{\bar{i}k}^{\beta}(J_{i\alpha}J_{\bar{i}\beta}-J_{\bar{i}\alpha}J_{i\beta})\\
&\le\sum_{\substack{i\le\check{n}\\ k}}|\mathring{h}_{ik}^{\alpha}\mathring{h}_{\bar{i}k}^{\beta}|\sqrt{(J_{i\alpha})^2+(J_{\bar{i}\alpha})^2}\sqrt{(J_{i\beta})^2+(J_{\bar{i}\beta})^2}\\
&\le\sum_{\substack{i\le\check{n}\\ k}}|\mathring{h}_{ik}^{\alpha}\mathring{h}_{\bar{i}k}^{\beta}||Pe_{\alpha}||Pe_{\beta}|\\
&\le|\mathring{h}^{\alpha}||\mathring{h}^{\beta}||Pe_{\alpha}||Pe_{\beta}|.
\end{aligned}$$

Then

$$\begin{aligned}&\sum_{\alpha,\beta,i,j,k} \mathring{h}^{\alpha}_{ik}\mathring{h}^{\beta}_{jk}(J_{i\alpha}J_{j\beta}-J_{j\alpha}J_{i\beta})\\ &\le \sum_{\alpha\neq\beta}|Pe_\alpha||Pe_\beta||\mathring{h}^\alpha||\mathring{h}^\beta|\\ &=2|Pe_{n+1}||\mathring{h}^{n+1}|\sum_{\alpha>n+1}|Pe_\alpha||\mathring{h}^\alpha|+\sum_{\substack{\alpha,\beta>n+1\\ \alpha\neq\beta}}|Pe_\alpha||Pe_\beta||\mathring{h}^\alpha||\mathring{h}^\beta|\\ &\le 2\sqrt{\rho_1}\sum_{\alpha>n+1}|Pe_\alpha||\mathring{h}^\alpha|+\Big(\sum_{\alpha>n+1}|Pe_\alpha||\mathring{h}^\alpha|\Big)^2.\end{aligned}$$

Using the Cauchy inequality, we have

$$\Big(\sum_{\alpha>n+1}|Pe_\alpha||\mathring{h}^\alpha|\Big)^2\le\Big(\sum_{\alpha>n+1}|Pe_\alpha|^2\Big)\Big(\sum_{\alpha>n+1}|\mathring{h}^\alpha|^2\Big)=\theta_2\rho_2.$$

So

$$\sum_{\alpha,\beta,i,j,k} \mathring{h}^{\alpha}_{ik}\mathring{h}^{\beta}_{jk}(J_{i\alpha}J_{j\beta}-J_{j\alpha}J_{i\beta})\le 2\sqrt{\theta_2\rho_1\rho_2}+\theta_2\rho_2.$$

Thus we obtain $S_1\le\frac{3}{n}S_2+3|\mathring{h}|^2+8\sqrt{\theta_2\rho_1\rho_2}+4\theta_2\rho_2$.

Let $F: M^n\times[0,T)\to\mathbb{CP}^{\frac{n+q}{2}}$ be a mean curvature flow. Note that $n+q$ must be even. Let $2\le q\le n-6$. Suppose that the initial submanifold M_0 satisfies the pinching condition

$$|\mathring{h}|^2<k|H|^2+l,\quad k=\frac{1}{n(n-1)},\ l=2-\frac{3}{n}.$$

Since M_0 is compact, there exists a small positive number ε, such that M_0 satisfies $|\mathring{h}|^2<(k|H|^2+l)(1-\varepsilon)$.

Now we prove that the pinching condition is preserved.

Theorem 3.2 *If M_0 satisfies $|\mathring{h}|^2<(k|H|^2+l)(1-\varepsilon)$, then this condition holds for all time $t\in[0,T)$.*

Proof Let $k_\varepsilon=k(1-\varepsilon), l_\varepsilon=l(1-\varepsilon)$. We set $U=|\mathring{h}|^2-k_\varepsilon|H|^2-l_\varepsilon$. From the evolution equations we have

$$\begin{aligned}&\frac{1}{2}(\partial_t-\Delta)U\\ &=k_\varepsilon|\nabla H|^2-|\nabla\mathring{h}|^2-n(|\mathring{h}|^2+k_\varepsilon|H|^2)\\ &\quad+R_1-\Big(k_\varepsilon+\frac{1}{n}\Big)R_2+S_1-3\Big(k_\varepsilon+\frac{1}{n}\Big)S_2.\end{aligned}$$

Now we show that $(\partial_t-\Delta)U$ is nonpositive at all points where $|\mathring{h}|^2=k_\varepsilon|H|^2+l_\varepsilon$.

From Lemma 3.3, if $|\mathring{h}|^2 = k_\varepsilon|H|^2 + l_\varepsilon$, we get the following estimates

$$\begin{aligned}
&R_1 - \Big(k_\varepsilon + \frac{1}{n}\Big)R_2\\
&\le (|\mathring{h}|^2 - k_\varepsilon|H|^2)|h|^2 - \Big(\frac{1}{n} - k_\varepsilon\Big)|H|^2\rho_2 + 2\rho_1\rho_2 + \frac{1}{2}\rho_2^2\\
&\le l_\varepsilon|h|^2 - (n-2)k_\varepsilon|H|^2\rho_2 + 2\rho_1\rho_2 + \frac{1}{2}\rho_2^2\\
&= l_\varepsilon\Big[\Big(k_\varepsilon + \frac{1}{n}\Big)|H|^2 + l_\varepsilon\Big] - (n-2)(\rho_1 + \rho_2 - l_\varepsilon)\rho_2 + 2\rho_1\rho_2 + \frac{1}{2}\rho_2^2\\
&= \Big(k_\varepsilon + \frac{1}{n}\Big)l_\varepsilon|H|^2 + l_\varepsilon^2 + (n-2)l_\varepsilon\rho_2 - (n-4)\rho_1\rho_2 - \Big(n - \frac{5}{2}\Big)\rho_2^2
\end{aligned}$$

and

$$\begin{aligned}
&S_1 - 3\Big(k_\varepsilon + \frac{1}{n}\Big)S_2\\
&\le 3|\mathring{h}|^2 + 8\sqrt{\theta_2\rho_1\rho_2} + 4\theta_2\rho_2\\
&= 3k_\varepsilon|H|^2 + 3l_\varepsilon + 8\sqrt{\theta_2\rho_1\rho_2} + 4\theta_2\rho_2.
\end{aligned}$$

Thus, at a point where $U = 0$, we get

$$\begin{aligned}
&\frac{1}{2}(\partial_t - \Delta)U\\
&\le k_\varepsilon|\nabla H|^2 - |\nabla\mathring{h}|^2 - n(2k_\varepsilon|H|^2 + l_\varepsilon)\\
&\quad + \Big(k_\varepsilon + \frac{1}{n}\Big)l_\varepsilon|H|^2 + l_\varepsilon^2 + (n-2)l_\varepsilon\rho_2 - (n-4)\rho_1\rho_2 - \Big(n - \frac{5}{2}\Big)\rho_2^2\\
&\quad + 3k_\varepsilon|H|^2 + 3l_\varepsilon + 8\sqrt{\theta_2\rho_1\rho_2} + 4\theta_2\rho_2\\
&= k_\varepsilon|\nabla H|^2 - |\nabla\mathring{h}|^2 + \Big[(3-2n)k_\varepsilon + \Big(k_\varepsilon + \frac{1}{n}\Big)l_\varepsilon\Big]|H|^2\\
&\quad + (3 - n + l_\varepsilon)l_\varepsilon + (n-2)l_\varepsilon\rho_2 - \Big(n - \frac{5}{2}\Big)\rho_2^2\\
&\quad - (n-4)\rho_1\rho_2 + 8\sqrt{\theta_2\rho_1\rho_2} + 4\theta_2\rho_2.
\end{aligned}\tag{3.10}$$

By Lemma 2.1, we get

$$|\nabla\mathring{h}|^2 - k_\varepsilon|\nabla H|^2 \ge 8\theta_2.$$

By the definitions of $k_\varepsilon, l_\varepsilon$, we have

$$(3-2n)k_\varepsilon + \Big(k_\varepsilon + \frac{1}{n}\Big)l_\varepsilon < 0.$$

By the AM-GM inequality, we have

$$-(n-4)\rho_1\rho_2 + 8\sqrt{\theta_2\rho_1\rho_2} \le \frac{16\theta_2}{n-4} \tag{3.11}$$

and

$$(3 - n + l_\varepsilon)l_\varepsilon + (n-2)l_\varepsilon\rho_2 \le \frac{(n-2)^2 l_\varepsilon}{4(n-3-l_\varepsilon)}\rho_2^2 \le \frac{(n-2)^2 l}{4(n-3-l)}\rho_2^2. \tag{3.12}$$

Hence, we get

$$\frac{1}{2}(\partial_t-\Delta)U\le\Big[\frac{(n-2)^2l}{4(n-3-l)}-\Big(n-\frac{5}{2}\Big)\Big]\rho_2^2+4\theta_2\rho_2+\frac{16\theta_2}{n-4}-8\theta_2.$$

Since $n\ge q+6\ge 8$, we have $\frac{(n-2)^2l}{4(n-3-l)}-\big(n-\frac{5}{2}\big)<0$. Then we get

$$\begin{aligned}&\Big[\frac{(n-2)^2l}{4(n-3-l)}-\Big(n-\frac{5}{2}\Big)\Big]\rho_2^2+4\theta_2\rho_2+\frac{16\theta_2}{n-4}-8\theta_2\\ \le&\ \frac{4}{n-\frac{5}{2}-\frac{(n-2)^2l}{4(n-3-l)}}\theta_2^2+\frac{16\theta_2}{n-4}-8\theta_2\\ \le&\ \Big[\frac{4(q-1)}{n-\frac{5}{2}-\frac{(n-2)^2l}{4(n-3-l)}}+\frac{16}{n-4}-8\Big]\theta_2\\ \le&\ \Big[\frac{4(n-7)}{n-\frac{5}{2}-\frac{(n-2)^2l}{4(n-3-l)}}+\frac{16}{n-4}-8\Big]\theta_2\\ =&\ -8\times\frac{-60+76n-16n^2+n^3}{(n-4)(-18+42n-19n^2+2n^3)}\theta_2\le 0.\end{aligned}\tag{3.13}$$

Then the assertion follows from the maximum principle.

Let

$$f_\sigma=\frac{|\mathring{h}|^2}{(k|H|^2+l)^{1-\sigma}},$$

where $\sigma\in(0,\varepsilon^2)$ is a positive constant. Then we have the following lemma.

Lemma 3.4 *If M_0 satisfies $|\mathring{h}|^2<(k|H|^2+l)(1-\varepsilon)$, then the following inequality holds along the mean curvature flow*:

$$\partial_t f_\sigma\le\Delta f_\sigma+\frac{2k|\nabla f_\sigma||\nabla|H|^2|}{k|H|^2+l}-\frac{2f_\sigma}{3|\mathring{h}|^2}|\nabla\mathring{h}|^2+2f_\sigma(\sigma|h|^2-\varepsilon).$$

Proof By a direct calculation, we have

$$\begin{aligned}&(\partial_t-\Delta)f_\sigma\\ =&\ f_\sigma\Big[\frac{1}{|\mathring{h}|^2}(\partial_t-\Delta)|\mathring{h}|^2-\frac{1-\sigma}{|H|^2+\frac{l}{k}}(\partial_t-\Delta)|H|^2\Big]\\ &+2(1-\sigma)\frac{\langle\nabla f_\sigma,\nabla|H|^2\rangle}{|H|^2+\frac{l}{k}}-\sigma(1-\sigma)f_\sigma\frac{|\nabla|H|^2|^2}{\big(|H|^2+\frac{l}{k}\big)^2}.\end{aligned}$$

From Lemma 3.3, we get

$$\begin{aligned}&(\partial_t-\Delta)|\mathring{h}|^2\\ \le&\ -2|\nabla\mathring{h}|^2+2|\mathring{h}|^2(|h|^2-n+3)-\frac{2}{n}\rho_2|H|^2\\ &+4\rho_1\rho_2+\rho_2^2+16\sqrt{\theta_2\rho_1\rho_2}+8\theta_2\rho_2\\ =&\ -2|\nabla\mathring{h}|^2+2|\mathring{h}|^2(|h|^2-n+3+(n-2)\rho_2)-\frac{2}{n}\rho_2|H|^2\\ &+(5-2n)\rho_2^2-2(n-4)\rho_1\rho_2+16\sqrt{\theta_2\rho_1\rho_2}+8\theta_2\rho_2.\end{aligned}\tag{3.14}$$

Combining (3.11) and (3.13), we get

$$-\Big(n-\frac{5}{2}\Big)\rho_2^2-(n-4)\rho_1\rho_2+8\sqrt{\theta_2\rho_1\rho_2}+4\theta_2\rho_2\le 8\theta_2-\frac{(n-2)^2l}{4(n-3-l)}\rho_2^2.$$

Thus we get

$$\begin{aligned}&\frac{1}{|\mathring{h}|^2}(\partial_t-\Delta)|\mathring{h}|^2\\ &\le\frac{2}{|\mathring{h}|^2}(-|\nabla\mathring{h}|^2+8\theta_2)+2(|h|^2-n+3+(n-2)\rho_2)\\ &\quad-\frac{1}{|\mathring{h}|^2}\Big(\frac{2}{n}\rho_2|H|^2+\frac{(n-2)^2l}{2(n-3-l)}\rho_2^2\Big).\end{aligned}$$

We also have

$$\begin{aligned}&-\frac{1-\sigma}{|H|^2+\frac{l}{k}}(\partial_t-\Delta)|H|^2\\ &\le\frac{2|\nabla H|^2}{|H|^2+\frac{l}{k}}-\frac{2(1-\sigma)|H|^2}{|H|^2+\frac{l}{k}}(|h|^2-\rho_2+n)\\ &\le\frac{2|\nabla H|^2}{|H|^2+\frac{l}{k}}-\Big(2-\frac{2l}{k|H|^2+l}\Big)(|h|^2-\rho_2+n)+2\sigma(|h|^2+n)\\ &=\frac{2|\nabla H|^2}{|H|^2+\frac{l}{k}}+\frac{2l(|h|^2-\rho_2+n)}{k|H|^2+l}-2(|h|^2-\rho_2+n)+2\sigma(|h|^2+n).\end{aligned}$$

It follows from $|\mathring{h}|^2<(k|H|^2+l)(1-\varepsilon)$ that

$$\frac{2l(|h|^2-\rho_2+n)}{k|H|^2+l}\le\frac{2l\big(\frac{1}{n}|H|^2-\rho_2+n\big)}{k|H|^2+l}+2l(1-\varepsilon).$$

Thus we get

$$\begin{aligned}&-\frac{1-\sigma}{|H|^2+\frac{l}{k}}(\partial_t-\Delta)|H|^2\\ &\le\frac{2k|\nabla H|^2}{|\mathring{h}|^2}+\frac{2l\big(\frac{1}{n}|H|^2-\rho_2+n\big)}{k|H|^2+l}+2l-2(|h|^2-\rho_2+n)+2\sigma|h|^2-2\varepsilon.\end{aligned}$$

Therefore,

$$\begin{aligned}(\partial_t-\Delta)f_\sigma&\le f_\sigma\Big[\frac{1}{|\mathring{h}|^2}(\partial_t-\Delta)|\mathring{h}|^2-\frac{1-\sigma}{|H|^2+\frac{l}{k}}(\partial_t-\Delta)|H|^2\Big]+\frac{2|\nabla f_\sigma||\nabla|H|^2|}{|H|^2+\frac{l}{k}}\\ &\le\frac{2f_\sigma}{|\mathring{h}|^2}[-|\nabla\mathring{h}|^2+8\theta_2+k|\nabla H|^2]+\frac{2|\nabla f_\sigma||\nabla|H|^2|}{|H|^2+\frac{l}{k}}\\ &\quad+2f_\sigma[-2n+3+(n-1)\rho_2+l+\sigma|h|^2-\varepsilon]\\ &\quad+2f_\sigma\Big[-\frac{1}{|\mathring{h}|^2}\Big(\frac{1}{n}\rho_2|H|^2+\frac{(n-2)^2l}{4(n-3-l)}\rho_2^2\Big)+\frac{l\big(\frac{1}{n}|H|^2-\rho_2+n\big)}{k|H|^2+l}\Big]\\ &\le\frac{2f_\sigma}{|\mathring{h}|^2}[-|\nabla\mathring{h}|^2+8\theta_2+k|\nabla H|^2]+\frac{2|\nabla f_\sigma||\nabla|H|^2|}{|H|^2+\frac{l}{k}}\\ &\quad+2f_\sigma[(n-1)(\rho_2-l)+\sigma|h|^2-\varepsilon]\\ &\quad+\frac{2f_\sigma}{k|H|^2+l}\Big[-\frac{1}{n}\rho_2|H|^2-\frac{(n-2)^2l}{4(n-3-l)}\rho_2^2+l\Big(\frac{1}{n}|H|^2-\rho_2+n\Big)\Big].\end{aligned}$$

By Lemma 2.1 and the condition $n \geq q+6 \geq 8$, we have

$$-|\nabla \mathring{h}|^2 + 8\theta_2 + k|\nabla H|^2 \leq -\frac{1}{3}|\nabla \mathring{h}|^2.$$

Using (3.12), we get

$$\begin{aligned}
&-\frac{1}{n}\rho_2|H|^2 - \frac{(n-2)^2 l}{4(n-3-l)}\rho_2^2 + l\Big(\frac{1}{n}|H|^2 - \rho_2 + n\Big)\\
&\leq -\frac{1}{n}\rho_2|H|^2 - (3-n+l)l - (n-2)l\rho_2 + l\Big(\frac{1}{n}|H|^2 - \rho_2 + n\Big)\\
&= (n-1)(l-\rho_2)(k|H|^2 + l).
\end{aligned}$$

Therefore, we complete the proof of this lemma.

3.3 The case of $q \geq n-4 \geq 2$

We choose an orthonormal frame $\{e_\alpha\}$ for the normal space such that $H = |H|e_{n+1}$. Let $\rho_1 = |\mathring{h}^{n+1}|^2$, $\rho_2 = \sum\limits_{\alpha>n+1} |\mathring{h}^\alpha|^2$. We have the following lemma.

Lemma 3.5

(i) $R_1 \leq |h|^4 - \frac{2}{n}\rho_2|H|^2 + 2|\mathring{h}|^2\rho_2 - \frac{3}{2}\rho_2^2$,

(ii) $R_2 = |H|^2(|h|^2 - \rho_2)$,

(iii) $S_1 \leq \frac{3}{n}S_2 + (2n+3)|\mathring{h}|^2$.

Proof Using the same proof as Lemma 3.3, we obtain (i) and (ii).

Now we re-estimate S_1. From (3.8) and (3.9), we have

$$S_1 \leq \frac{3}{n}S_2 + 3|\mathring{h}|^2 + 4\sum_{\alpha,\beta,i,j,k} \mathring{h}_{ik}^\alpha \mathring{h}_{jk}^\beta (J_{i\alpha}J_{j\beta} - J_{i\beta}J_{j\alpha}).$$

With a local orthonormal frame, let v be a vector given by $v_k = \mathring{h}_{ik}^\alpha J_{i\alpha}$. We define two tensors D and E by

$$D_{ijk} = \mathring{h}_{ij}^\alpha J_{k\alpha} + \frac{\mathring{h}_{ik}^\alpha J_{j\alpha} + \mathring{h}_{jk}^\alpha J_{i\alpha}}{n+\eta}, \quad E_{ijk} = -\frac{2\delta_{ij}v_k}{(n+\eta)\eta} + \frac{\delta_{ik}v_j + \delta_{jk}v_i}{\eta},$$

where $\eta = \sqrt{n^2+n-2}$. Then we have $\langle D-E, E\rangle = 0$. This implies $|D|^2 \geq |E|^2$. By the definitions of D and E, we get

$$\begin{aligned}
|D|^2 &= \frac{5n-2-4\eta}{(n-2)^2}(n\mathring{h}_{ij}^\alpha \mathring{h}_{ij}^\beta J_{k\alpha}J_{k\beta} + 2\mathring{h}_{ik}^\alpha \mathring{h}_{jk}^\beta J_{i\beta}J_{j\alpha}),\\
|E|^2 &= \frac{2(5n-2-4\eta)}{(n-2)^2}\mathring{h}_{ik}^\alpha \mathring{h}_{jk}^\beta J_{i\alpha}J_{j\beta}.
\end{aligned}$$

Thus we obtain

$$\sum_{\alpha,\beta,i,j,k} \mathring{h}_{ik}^\alpha \mathring{h}_{jk}^\beta (J_{i\alpha}J_{j\beta} - J_{i\beta}J_{j\alpha}) \leq \frac{n}{2}\sum_{i,j,k}\Big(\sum_\alpha \mathring{h}_{ij}^\alpha J_{k\alpha}\Big)^2 \leq \frac{n}{2}|\mathring{h}|^2.$$

Define a function $\psi : [0, +\infty) \to \mathbb{R}$ by

$$\psi(x) := \frac{9}{n^2 - 3n - 3} + \frac{n(n-3)}{n^3 - 4n^2 + 3}x - \frac{3\sqrt{x^2 + \frac{2}{n}(n-1)(n^2-3)x + 9(n-1)^2}}{n^3 - 4n^2 + 3}. \tag{3.15}$$

Let $\mathring{\psi}(x) = \psi(x) - \frac{x}{n}$. From Lemma 7.3 in the Appendix, the function $\mathring{\psi}$ has the following properties.

Lemma 3.6 *The function* $\mathring{\psi}$ *satisfies*

(i) $0 \le \mathring{\psi}'(x) < \frac{1}{n(n-1)}$, $0 \le \mathring{\psi}(x) \le \frac{x}{n(n-1)}$,

(ii) $\max\limits_{x \ge 0}(2x\mathring{\psi}''(x) + \mathring{\psi}'(x)) < \frac{2(n-4)}{n(n+8)}$,

(iii) $3\mathring{\psi}(x) + (\mathring{\psi}(x) - x\mathring{\psi}'(x))\big(\mathring{\psi}(x) + \frac{x}{n} + n\big) \le 0$, *and the equality holds if and only if* $x = 0$,

(iv) $0 \le x\mathring{\psi}'(x) - \mathring{\psi}(x) < 2$.

For convenience, we denote $\mathring{\psi}(|H|^2)$, $\mathring{\psi}'(|H|^2)$, $\mathring{\psi}''(|H|^2)$ by $\mathring{\psi}$, $\mathring{\psi}'$, $\mathring{\psi}''$, respectively. Let $F : M^n \times [0, T) \to \mathbb{CP}^{\frac{n+q}{2}}$ $(q \ge n - 4 \ge 2)$ be a mean curvature flow whose initial value M_0 satisfies $|h|^2 < \psi(|H|^2)$. Since M_0 is compact, there exists a small positive number ε, such that M_0 satisfies $|\mathring{h}|^2 < \mathring{\psi} - \varepsilon|H|^2 - \varepsilon$.

Theorem 3.3 *If the initial value* M_0 *satisfies* $|\mathring{h}|^2 < \mathring{\psi} - \varepsilon|H|^2 - \varepsilon$, *then this pinching condition holds for all* $t \in [0, T)$.

Proof From Lemma 3.5, we get

$$\begin{aligned}(\partial_t - \Delta)|\mathring{h}|^2 &= -2|\nabla\mathring{h}|^2 - 2n|\mathring{h}|^2 + 2R_1 - \frac{2}{n}R_2 + 2S_1 - \frac{6}{n}S_2 \\ &\le -2|\nabla\mathring{h}|^2 + 2|\mathring{h}|^2|h|^2 - \frac{2}{n}\rho_2|H|^2 + 4|\mathring{h}|^2\rho_2 + 2(n+3)|\mathring{h}|^2.\end{aligned} \tag{3.16}$$

We have the following evolution equation of $\mathring{\psi}$.

$$\begin{aligned}(\partial_t - \Delta)\mathring{\psi} &= -2\mathring{\psi}' \cdot |\nabla H|^2 - \mathring{\psi}'' \cdot |\nabla|H|^2|^2 + 2\mathring{\psi}'(n|H|^2 + R_2 + 3S_2) \\ &\ge -2(\mathring{\psi}' + 2\mathring{\psi}'' \cdot |H|^2)|\nabla H|^2 + 2\mathring{\psi}' \cdot |H|^2(|h|^2 + n - \rho_2).\end{aligned} \tag{3.17}$$

Let $U = |\mathring{h}|^2 - \mathring{\psi} + \varepsilon|H|^2 + \varepsilon$. We obtain

$$\begin{aligned}\frac{1}{2}(\partial_t - \Delta)U \le{}& (\mathring{\psi}' - \varepsilon + 2\mathring{\psi}'' \cdot |H|^2)|\nabla H|^2 - |\nabla\mathring{h}|^2 \\ &+ |\mathring{h}|^2\Big(|\mathring{h}|^2 + \frac{1}{n}|H|^2 + n + 3\Big) - (\mathring{\psi}' - \varepsilon)|H|^2\Big(|\mathring{h}|^2 + \frac{1}{n}|H|^2 + n\Big) \\ &+ \rho_2\Big(-\frac{1}{n}|H|^2 + 2|\mathring{h}|^2 + (\mathring{\psi}' - \varepsilon)|H|^2\Big).\end{aligned}$$

By Lemmas 2.1 and 3.6(ii), the first line of the RHS of the formula above is nonpositive. From

$|\mathring{h}|^2 = U + \mathring{\psi} - \varepsilon|H|^2 - \varepsilon$, we obtain

$$\begin{aligned}
&\frac{1}{2}(\partial_t - \Delta)U \\
&\le U\Big(U + 2\mathring{\psi} + n + 3 - 2\varepsilon + \Big(\frac{1}{n} - \mathring{\psi}' - \varepsilon\Big)|H|^2 + 2\rho_2\Big) \\
&\quad + \mathring{\psi}\Big(\mathring{\psi} + \frac{1}{n}|H|^2 + n + 3\Big) - \mathring{\psi}' \cdot |H|^2\Big(\mathring{\psi} + \frac{1}{n}|H|^2 + n\Big) \\
&\quad + \varepsilon\Big[-n - 3 + \varepsilon + \mathring{\psi}' \cdot |H|^2 - 2\mathring{\psi} \\
&\quad + |H|^2\Big(\mathring{\psi}' \cdot |H|^2 - \mathring{\psi} - 3 - \frac{1}{n} + \varepsilon\Big)\Big] \\
&\quad + \rho_2\Big(-\frac{1}{n}|H|^2 + 2\mathring{\psi} + \mathring{\psi}'|H|^2 - \varepsilon(3|H|^2 + 2)\Big).
\end{aligned} \tag{3.18}$$

This together with Lemma 3.6 implies

$$\frac{1}{2}(\partial_t - \Delta)U < U\Big(U + 2\mathring{\psi} + n + 3 - 2\varepsilon + \Big(\frac{1}{n} - \mathring{\psi}' - \varepsilon\Big)|H|^2 + 2\rho_2\Big). \tag{3.19}$$

Then the assertion follows from the maximum principle.

Now we prove that the mean curvature flow has finite maximal existence time in this case.

Lemma 3.7 *If the initial value M_0 satisfies $|\mathring{h}|^2 < \mathring{\psi}$, then T is finite.*

Proof Let $U = |\mathring{h}|^2 - \mathring{\psi}$. Then $U < 0$ holds for all $t \in [0, T)$. From (3.19), we have

$$\begin{aligned}
(\partial_t - \Delta)U &\le 2U\Big(U + 2\mathring{\psi} + n + 3 + \Big(\frac{1}{n} - \mathring{\psi}'\Big)|H|^2 + 2\rho_2\Big) \\
&\le 2U(U + 2\mathring{\psi}) \\
&= 2U(2|\mathring{h}|^2 - U) \le -2U^2.
\end{aligned}$$

From the maximum principle, U will blow up in finite time. Therefore, T must be finite.

Let

$$f_\sigma = \frac{|\mathring{h}|^2}{\mathring{\psi}^{1-\sigma}},$$

where $\sigma \in (0, \varepsilon^2)$ is a positive constant. Then we have the following lemma.

Lemma 3.8 *If M_0 satisfies $|\mathring{h}|^2 < \mathring{\psi}$, then there exists a small positive constant ε, such that the following inequality holds along the mean curvature flow*:

$$\partial_t f_\sigma \le \Delta f_\sigma + \frac{2}{\mathring{\psi}}|\nabla f_\sigma||\nabla\mathring{\psi}| - \frac{2\varepsilon f_\sigma}{|\mathring{h}|^2}|\nabla\mathring{h}|^2 + 2\sigma|h|^2 f_\sigma + 4n f_\sigma.$$

Proof By a straightforward calculation, we have

$$\begin{aligned}
(\partial_t - \Delta)f_\sigma &= f_\sigma\Big[\frac{1}{|\mathring{h}|^2}(\partial_t - \Delta)|\mathring{h}|^2 - \frac{1-\sigma}{\mathring{\psi}}(\partial_t - \Delta)\mathring{\psi}\Big] \\
&\quad + 2(1-\sigma)\frac{\langle\nabla f_\sigma, \nabla\mathring{\psi}\rangle}{\mathring{\psi}} - \sigma(1-\sigma)f_\sigma\frac{|\nabla\mathring{\psi}|^2}{|\mathring{\psi}|^2}.
\end{aligned}$$

Using (3.16) and (3.17), we have

$$\begin{aligned}
&(\partial_t - \Delta) f_\sigma \\
&\le 2f_\sigma\Big[-\frac{|\nabla\mathring{h}|^2}{|\mathring{h}|^2} + (1-\sigma)\frac{\mathring{\psi}' + 2|H|^2\mathring{\psi}''}{\mathring{\psi}}|\nabla H|^2\Big] \\
&\quad + 2f_\sigma\Big[|h|^2 + n + 3 - (1-\sigma)\frac{\mathring{\psi}'\cdot|H|^2}{\mathring{\psi}}(|h|^2+n)\Big] \\
&\quad + 2f_\sigma\rho_2\Big[-\frac{|H|^2}{n|\mathring{h}|^2} + 2 + (1-\sigma)\frac{\mathring{\psi}'\cdot|H|^2}{\mathring{\psi}}\Big] + \frac{2}{\mathring{\psi}}|\nabla f_\sigma||\nabla\mathring{\psi}|.
\end{aligned}$$

From Lemmas 2.1 and 3.6(ii), we have

$$\begin{aligned}
&-\frac{|\nabla\mathring{h}|^2}{|\mathring{h}|^2} + (1-\sigma)\frac{\mathring{\psi}' + 2|H|^2\mathring{\psi}''}{\mathring{\psi}}|\nabla H|^2 \\
&\le -\frac{|\nabla\mathring{h}|^2}{|\mathring{h}|^2} + (1-\sigma)\frac{(1-\varepsilon)\frac{2(n-4)}{n(n+8)}}{\mathring{\psi}}|\nabla H|^2 \\
&\le -\frac{|\nabla\mathring{h}|^2}{|\mathring{h}|^2} + \frac{(1-\varepsilon)|\nabla\mathring{h}|^2}{|\mathring{h}|^2} = -\frac{\varepsilon|\nabla\mathring{h}|^2}{|\mathring{h}|^2}.
\end{aligned}$$

By Lemma 3.6(iv), we get

$$\begin{aligned}
&|h|^2 + n + 3 - (1-\sigma)\frac{\mathring{\psi}'\cdot|H|^2}{\mathring{\psi}}(|h|^2+n) \\
&\le (1-\sigma)\frac{\mathring{\psi} - \mathring{\psi}'\cdot|H|^2}{\mathring{\psi}}|h|^2 + n + 3 + \sigma|h|^2 \\
&\le n + 3 + \sigma|h|^2.
\end{aligned}$$

By Lemma 3.6(i), we have

$$-\frac{|H|^2}{n|\mathring{h}|^2} + 2 + (1-\sigma)\frac{\mathring{\psi}'\cdot|H|^2}{\mathring{\psi}} \le \frac{-\frac{1}{n}|H|^2 + 2\mathring{\psi} + \mathring{\psi}'\cdot|H|^2}{\mathring{\psi}} \le 0.$$

This completes the proof of the lemma.

4 An Estimate for Traceless Second Fundamental Form

Let $F: M^n \times [0,T) \to \mathbb{CP}^{\frac{n+q}{2}}$ be a mean curvature flow. Suppose that the initial value M_0 satisfies

$$|\mathring{h}|^2 < W, \quad W = \begin{cases} \mathring{\varphi}, & q = 1, \\ \dfrac{|H|^2}{n(n-1)} + 2 - \dfrac{3}{n}, & 2 \le q < n-4, \\ \mathring{\psi}, & q \ge n-4 \ge 2. \end{cases} \tag{4.1}$$

From the definition of W, we have $W < \frac{|H|^2}{n(n-1)} + n$. By the conclusions of the previous section, there exists a sufficiently small positive number ε, such that for all $t \in [0,T)$, the following pinching condition holds

$$|\mathring{h}|^2 < W - \varepsilon|H|^2 - \varepsilon.$$

We investigate the auxiliary function $f_\sigma = \frac{|\mathring{h}|^2}{W^{1-\sigma}}$. In this section, we will show that f_σ decays exponentially.

Lemma 4.1 *There exist positive constants ε and C_1 depending on M_0, such that*

$$\begin{aligned}\partial_t f_\sigma \leq \Delta f_\sigma &+ \frac{2C_1}{|\mathring{h}|}|\nabla f_\sigma||\nabla\mathring{h}| - \frac{\varepsilon f_\sigma}{n|\mathring{h}|^2}|\nabla\mathring{h}|^2 \\ &- \frac{2(1-\chi)\varepsilon f_\sigma}{n|\mathring{h}|^2}|P|^2 + 2\sigma f_\sigma|h|^2 + \Big(5n\chi - \frac{\varepsilon}{n}\Big)f_\sigma,\end{aligned}$$

where

$$\chi = \begin{cases} 0, & q = 1 \text{ or } 2 \leq q < n-4, \\ 1, & q \geq n-4 \geq 2. \end{cases}$$

Proof Combining the conclusions of Lemmas 3.2, 3.4 and 3.8, we have the following inequality with some suitable small ε :

$$\partial_t f_\sigma \leq \Delta f_\sigma + \frac{2}{W}|\nabla f_\sigma||\nabla W| - \frac{2\varepsilon f_\sigma}{n|\mathring{h}|^2}|\nabla\mathring{h}|^2 + 2\sigma f_\sigma|h|^2 + \Big(5n\chi - \frac{\varepsilon}{n}\Big)f_\sigma.$$

From Lemma 2.1, we have $|\nabla\mathring{h}|^2 \geq 2(1-\chi)|P|^2$.

By the definition of W, there exists a constant B_1 such that $|\nabla W| \leq B_1|\nabla|H|^2|$ and $|H| \leq B_1\sqrt{W}$. Letting C_1 be a constant such that $2B_1^2|\nabla H| \leq C_1|\nabla\mathring{h}|$, we have

$$\frac{|\nabla W|}{W} \leq \frac{B_1|\nabla|H|^2|}{W} \leq \frac{2B_1^2|\nabla H|}{\sqrt{W}} \leq \frac{C_1|\nabla\mathring{h}|}{|\mathring{h}|}. \tag{4.2}$$

Thus we complete the proof of this lemma.

We need the following estimate for the Laplacian of $|\mathring{h}|^2$.

Lemma 4.2

$$\Delta|\mathring{h}|^2 \geq 2\langle\mathring{h}, \nabla^2 H\rangle + 2(\varepsilon|h|^2 - 2n^2)|\mathring{h}|^2 - 6|\mathring{h}||H||P|^2.$$

Proof We have the following identity

$$\Delta|\mathring{h}|^2 = 2\langle\mathring{h}, \nabla^2 H\rangle + 2|\nabla\mathring{h}|^2 - 2R_1 + 2R_3 + 2n|\mathring{h}|^2 - 2S_1 + \frac{6}{n}S_2 + 6S_3,$$

where $R_3 = H^\alpha h^\alpha_{ik} h^\beta_{ij} h^\beta_{jk}$, $S_3 = \mathring{h}^\alpha_{ij} H^\beta J_{i\alpha} J_{j\beta}$. From Lemma 3.5(iii) and $S_3 \geq -|\mathring{h}||H||P|^2$, we get

$$\Delta|\mathring{h}|^2 \geq 2\langle\mathring{h}, \nabla^2 H\rangle + 2(R_3 - R_1) - 2(n+3)|\mathring{h}|^2 - 6|\mathring{h}||H||P|^2.$$

We choose a local orthonormal frame, such that $H = |H|e_{n+1}$ and $\mathring{h}^1 = \mathrm{diag}(\mathring{\lambda}_1, \cdots, \mathring{\lambda}_n)$. Let $\rho_1 = |\mathring{h}^{n+1}|^2$, $\rho_2 = \sum\limits_{\alpha>n+1} |\mathring{h}^\alpha|^2$. Expanding R_3, we get

$$R_3 = \frac{1}{n^2}|H|^4 + \frac{3\rho_1+\rho_2}{n}|H|^2 + |H|\sum_{\alpha,i}\mathring{\lambda}_i(\mathring{h}^\alpha_{ii})^2 + \frac{|H|}{2}\sum_{\substack{\alpha>n+1\\ i\neq j}}(\mathring{\lambda}_i+\mathring{\lambda}_j)(\mathring{h}^\alpha_{ij})^2.$$

By [27, Lemma 2.6] or [30, Proposition 1.6], we have

$$\begin{aligned}\sum_{\alpha,i}\mathring{\lambda}_i(\mathring{h}^\alpha_{ii})^2 &\geq -\frac{n-2}{\sqrt{n(n-1)}}\sqrt{\rho_1}\Big(|\mathring{h}|^2 - \sum_{\substack{\alpha>n+1\\ i\neq j}}(\mathring{h}^\alpha_{ij})^2\Big)\\ &\geq -\frac{n-2}{\sqrt{n(n-1)}}\Big(\frac{1}{2}(\rho_1+|\mathring{h}|^2)|\mathring{h}| - \sqrt{\rho_1}\sum_{\substack{\alpha>n+1\\ i\neq j}}(\mathring{h}^\alpha_{ij})^2\Big)\\ &= -\frac{n-2}{\sqrt{n(n-1)}}\Big(|\mathring{h}|^3 - \frac{1}{2}|\mathring{h}|\rho_2 - \sqrt{\rho_1}\sum_{\substack{\alpha>n+1\\ i\neq j}}(\mathring{h}^\alpha_{ij})^2\Big).\end{aligned}$$

We also have

$$\sum_{\substack{\alpha>n+1\\ i\neq j}}(\mathring{\lambda}_i+\mathring{\lambda}_j)(\mathring{h}^\alpha_{ij})^2 \geq \sum_{\substack{\alpha>n+1\\ i\neq j}} -\sqrt{2(\mathring{\lambda}_i^2+\mathring{\lambda}_j^2)}(\mathring{h}^\alpha_{ij})^2 \geq -\sqrt{2\rho_1}\sum_{\substack{\alpha>n+1\\ i\neq j}}(\mathring{h}^\alpha_{ij})^2.$$

Note that $\frac{n-2}{\sqrt{n(n-1)}} > \frac{\sqrt{2}}{2}$ if $n \geq 6$, and $\sum\limits_{\alpha>n+1, i\neq j}(\mathring{h}^\alpha_{ij})^2 = 0$ if $q = 1$. We get

$$R_3 \geq \frac{1}{n^2}|H|^4 + \frac{3\rho_1+\rho_2}{n}|H|^2 - \frac{n-2}{\sqrt{n(n-1)}}|H|\Big(|\mathring{h}|^3 - \frac{1}{2}|\mathring{h}|\rho_2\Big). \tag{4.3}$$

From (4.3) and Lemma 3.3(i), we get

$$R_3 - R_1 \geq |\mathring{h}|^2\Big(\frac{1}{n}|H|^2 - |\mathring{h}|^2 - \frac{(n-2)|H||\mathring{h}|}{\sqrt{n(n-1)}}\Big) + \rho_2|\mathring{h}|\Big(\frac{(n-2)|H|}{2\sqrt{n(n-1)}} - 2|\mathring{h}|\Big) + \frac{3}{2}\rho_2^2.$$

Since $|\mathring{h}|^2 < W - \varepsilon|H|^2$ and $W < \frac{|H|^2}{n(n-1)} + n$, we have

$$\begin{aligned}&\frac{1}{n}|H|^2 - |\mathring{h}|^2 - \frac{(n-2)|H||\mathring{h}|}{\sqrt{n(n-1)}}\\ &\geq \frac{1}{n}|H|^2 - \Big(\frac{|H|^2}{n(n-1)} + n - \varepsilon|H|^2\Big) - (n-2)\Big(\frac{|H|^2}{n(n-1)} + n\Big)\\ &= \varepsilon|H|^2 - (n-1)n\\ &> \varepsilon|h|^2 - n^2.\end{aligned}$$

If $q = 1$, then $\rho_2 = 0$. If $q \geq 2$ and $n \geq 6$, we have

$$\frac{(n-2)|H|}{2\sqrt{n(n-1)}} - 2|\mathring{h}| \geq \frac{(n-2)|H|}{2\sqrt{n(n-1)}} - 2\Big(\sqrt{\frac{|H|^2}{n(n-1)}} + \sqrt{n}\Big) \geq -2\sqrt{n}.$$

Thus we obtain

$$R_3 - R_1 \geq |\mathring{h}|^2(\varepsilon|h|^2 - n^2) - 2\sqrt{n}\rho_2|\mathring{h}| + \frac{3}{2}\rho_2^2 \geq |\mathring{h}|^2\Big(\varepsilon|h|^2 - n^2 - \frac{2n}{3}\Big).$$

Hence we complete the proof of this lemma.

From (4.2) and Lemma 4.2, we have

$$\begin{aligned}
\Delta f_\sigma &= f_\sigma\Big(\frac{\Delta|\mathring{h}|^2}{|\mathring{h}|^2} - (1-\sigma)\frac{\Delta W}{W}\Big) - 2(1-\sigma)\frac{\langle\nabla f_\sigma, \nabla W\rangle}{W} + \sigma(1-\sigma)f_\sigma\frac{|\nabla W|^2}{W^2} \\
&\geq \frac{f_\sigma\Delta|\mathring{h}|^2}{|\mathring{h}|^2} - (1-\sigma)\frac{f_\sigma\Delta W}{W} - \frac{2C_1|\nabla f_\sigma||\nabla\mathring{h}|}{|\mathring{h}|} \\
&\geq \frac{2\langle\mathring{h}, \nabla^2 H\rangle}{W^{1-\sigma}} + 2f_\sigma(\varepsilon|h|^2 - 2n^2) - \frac{6f_\sigma|H||P|^2}{|\mathring{h}|} \\
&\quad - (1-\sigma)\frac{f_\sigma\Delta W}{W} - \frac{2C_1|\nabla f_\sigma||\nabla\mathring{h}|}{|\mathring{h}|}.
\end{aligned}$$

Multiplying both sides of the above inequality by f_σ^{p-1}, we get

$$\begin{aligned}
2\varepsilon f_\sigma^p|h|^2 &\leq f_\sigma^{p-1}\Delta f_\sigma + (1-\sigma)\frac{f_\sigma^p\Delta W}{W} \\
&\quad - \frac{2f_\sigma^{p-1}\langle\mathring{h}, \nabla^2 H\rangle}{W^{1-\sigma}} + \frac{2C_1 f_\sigma^{p-1}|\nabla f_\sigma||\nabla\mathring{h}|}{|\mathring{h}|} + 4n^2 f_\sigma^p + \frac{6f_\sigma^p|H||P|^2}{|\mathring{h}|}.
\end{aligned} \tag{4.4}$$

Then integrate both sides of (4.4) over M_t. By the divergence theorem, we have

$$\int_{M_t} f_\sigma^{p-1}\Delta f_\sigma \mathrm{d}\mu_t = -(p-1)\int_{M_t} f_\sigma^{p-2}|\nabla f_\sigma|^2\mathrm{d}\mu_t.$$

From (4.2), we have

$$\begin{aligned}
&\int_{M_t}\frac{f_\sigma^p}{W}\Delta W\mathrm{d}\mu_t \\
&= -\int_{M_t}\Big\langle\nabla\Big(\frac{f_\sigma^p}{W}\Big), \nabla W\Big\rangle\mathrm{d}\mu_t \\
&= \int_{M_t}\Big(-\frac{pf_\sigma^{p-1}}{W}\langle\nabla f_\sigma, \nabla W\rangle + \frac{f_\sigma^p}{W^2}|\nabla W|^2\Big)\mathrm{d}\mu_t \\
&\leq \int_{M_t}\Big(\frac{C_1 p f_\sigma^{p-1}}{|\mathring{h}|}|\nabla f_\sigma||\nabla\mathring{h}| + \frac{C_1^2 f_\sigma^p}{|\mathring{h}|^2}|\nabla\mathring{h}|^2\Big)\mathrm{d}\mu_t.
\end{aligned}$$

We also have

$$
\begin{aligned}
&-\int_{M_t}\frac{f_\sigma^{p-1}\langle\mathring{h},\nabla^2H\rangle}{W^{1-\sigma}}\mathrm{d}\mu_t\\
&=\int_{M_t}\nabla_i\Big(\frac{f_\sigma^{p-1}}{W^{1-\sigma}}\mathring{h}_{ij}^{\alpha}\Big)\nabla_jH^{\alpha}\mathrm{d}\mu_t\\
&=\int_{M_t}\Big[\frac{(p-1)f_\sigma^{p-2}}{W^{1-\sigma}}\mathring{h}_{ij}^{\alpha}\nabla_if_\sigma-\frac{(1-\sigma)f_\sigma^{p-1}}{W^{2-\sigma}}\mathring{h}_{ij}^{\alpha}\nabla_iW+\frac{f_\sigma^{p-1}}{W^{1-\sigma}}\nabla_i\mathring{h}_{ij}^{\alpha}\Big]\nabla_jH^{\alpha}\mathrm{d}\mu_t\\
&\le\int_{M_t}\Big[\frac{(p-1)f_\sigma^{p-2}}{W^{1-\sigma}}|\mathring{h}||\nabla f_\sigma|+\frac{f_\sigma^{p-1}}{W^{2-\sigma}}|\mathring{h}||\nabla W|+\frac{f_\sigma^{p-1}}{W^{1-\sigma}}n|\nabla\mathring{h}|\Big]|\nabla H|\mathrm{d}\mu_t\\
&\le\int_{M_t}\Big[\frac{(p-1)f_\sigma^{p-1}}{|\mathring{h}|}|\nabla f_\sigma|+\frac{C_1f_\sigma^{p-1}}{W^{1-\sigma}}|\nabla\mathring{h}|+\frac{f_\sigma^{p-1}}{W^{1-\sigma}}n|\nabla\mathring{h}|\Big]n|\nabla\mathring{h}|\mathrm{d}\mu_t\\
&\le\int_{M_t}\Big[\frac{(p-1)f_\sigma^{p-1}}{|\mathring{h}|}|\nabla f_\sigma||\nabla\mathring{h}|+\frac{(C_1+n)f_\sigma^{p}}{|\mathring{h}|^2}|\nabla\mathring{h}|^2\Big]n\ \mathrm{d}\mu_t.
\end{aligned}
$$

Putting the above inequalities together, we get

$$
\begin{aligned}
&\int_{M_t}f_\sigma^p|h|^2\mathrm{d}\mu_t\\
&\le C_2\int_{M_t}\Big[\frac{pf_\sigma^{p-1}}{|\mathring{h}|}|\nabla f_\sigma||\nabla\mathring{h}|+\frac{f_\sigma^p}{|\mathring{h}|^2}|\nabla\mathring{h}|^2+f_\sigma^p+\frac{f_\sigma^p|H||P|^2}{|\mathring{h}|}\Big]\mathrm{d}\mu_t,
\end{aligned}\tag{4.5}
$$

where C_2 is a positive constant independent of t.

Combining Lemma 4.1 and (4.5), we obtain

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{M_t}f_\sigma^p\mathrm{d}\mu_t\\
&=p\int_{M_t}f_\sigma^{p-1}\frac{\partial f_\sigma}{\partial t}\mathrm{d}\mu_t-\int_{M_t}f_\sigma^p|H|^2\mathrm{d}\mu_t\\
&\le p\int_{M_t}f_\sigma^{p-1}\Big(\Delta f_\sigma+\frac{2C_1}{|\mathring{h}|}|\nabla f_\sigma||\nabla\mathring{h}|-\frac{\varepsilon f_\sigma}{n|\mathring{h}|^2}|\nabla\mathring{h}|^2\\
&\quad-\frac{2(1-\chi)\varepsilon f_\sigma}{n|\mathring{h}|^2}|P|^2+2\sigma f_\sigma|h|^2+\Big(5n\chi-\frac{\varepsilon}{n}\Big)f_\sigma\Big)\mathrm{d}\mu_t-\int_{M_t}f_\sigma^p|H|^2\mathrm{d}\mu_t\\
&\le p\int_{M_t}f_\sigma^{p-2}\Big[-(p-1)|\nabla f_\sigma|^2+(2C_1+2\sigma C_2p)\frac{f_\sigma}{|\mathring{h}|}|\nabla f_\sigma||\nabla\mathring{h}|\\
&\quad-\Big(\frac{\varepsilon}{n}-2\sigma C_2\Big)\frac{f_\sigma^2}{|\mathring{h}|^2}|\nabla\mathring{h}|^2\Big]\mathrm{d}\mu_t\\
&\quad+p\int_{M_t}f_\sigma^p\Big[-\frac{2(1-\chi)\varepsilon|P|^2}{n|\mathring{h}|^2}+\frac{2\sigma C_2|H||P|^2}{|\mathring{h}|}\\
&\quad+2\sigma C_2+5n\chi-\frac{\varepsilon}{n}-\frac{|H|^2}{p}\Big]\mathrm{d}\mu_t.
\end{aligned}\tag{4.6}
$$

Now we show that the L^p-norm of f_σ decays exponentially.

Lemma 4.3 *There exist positive constants C_5, p_0, σ_0 depending on M_0, such that for all $p\ge p_0$ and $\sigma\le\frac{\sigma_0}{\sqrt{p}}$, we have*

$$\Big(\int_{M_t}f_\sigma^p\mathrm{d}\mu_t\Big)^{\frac{1}{p}}<C_5\mathrm{e}^{-\varepsilon t}.$$

Proof The expression in the first square bracket of the right hand side of (4.6) is a quadratic polynomial. With p_0 large enough and σ_0 small enough, its discriminant satisfies

$$(2C_1 + 2\sigma C_2 p)^2 - 4(p-1)\Big(\frac{\varepsilon}{n} - 2\sigma C_2\Big) < 0.$$

Thus this quadratic polynomial is nonpositive.

Then we consider the expression in the second square bracket of the right hand side of (4.6). We have

$$\begin{aligned}
&-\frac{2(1-\chi)\varepsilon}{n|\mathring{h}|^2}|P|^2 + \frac{2\sigma C_2|H||P|^2}{|\mathring{h}|} + 2\sigma C_2 + 5n\chi - \frac{\varepsilon}{n} - \frac{1}{p}|H|^2\\
\le{}& \frac{2(\chi-1)\varepsilon}{n|\mathring{h}|^2}|P|^2 + \frac{p\sigma^2 C_2^2|P|^4}{|\mathring{h}|^2} + 2\sigma C_2 + 5n\chi - \frac{\varepsilon}{n}\\
\le{}& \Big(\frac{2}{n}(\chi-1)\varepsilon + np\sigma^2 C_2^2\Big)\frac{|P|^2}{|\mathring{h}|^2} + 5n\chi - \Big(\frac{\varepsilon}{n} - 2\sigma C_2\Big)\\
\le{}& \frac{2}{n}\chi\varepsilon\frac{|P|^2}{|\mathring{h}|^2} + 5n\chi - \frac{\varepsilon}{2n}\\
\le{}& \frac{2\chi\varepsilon}{|\mathring{h}|^2} + 5n\chi - \frac{\varepsilon}{2n}.
\end{aligned}$$

Thus, if $q = 1$ or $2 \le q < n-4$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{M_t} f_\sigma^p \mathrm{d}\mu_t \le -\frac{p\varepsilon}{2n}\int_{M_t} f_\sigma^p \mathrm{d}\mu_t.$$

So, we get

$$\int_{M_t} f_\sigma^p \mathrm{d}\mu_t \le \mathrm{e}^{-\frac{p\varepsilon t}{2n}}\int_{M_0} f_\sigma^p \mathrm{d}\mu_0.$$

If $q \ge n-4 \ge 2$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{M_t} f_\sigma^p \mathrm{d}\mu_t \le p\int_{M_t}\Big(\frac{2\varepsilon}{|\mathring{h}|^2} + 5n\Big) f_\sigma^p \mathrm{d}\mu_t.$$

Since $W > \varepsilon$, we have

$$\frac{2p\varepsilon}{|\mathring{h}|^2} f_\sigma^p = \frac{2p\varepsilon}{W^{1-\sigma}} f_\sigma^{p-1} \le 2p f_\sigma^{p-1} \le 2 + 2(p-1) f_\sigma^p.$$

Then we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{M_t} f_\sigma^p \mathrm{d}\mu_t &\le 2\mathrm{Vol}(M_t) + (2p - 2 + 5np)\int_{M_t} f_\sigma^p \mathrm{d}\mu_t\\
&\le 2\mathrm{Vol}(M_0) + 6np\int_{M_t} f_\sigma^p \mathrm{d}\mu_t.
\end{aligned}$$

Hence,

$$\int_{M_t} f_\sigma^p \mathrm{d}\mu_t \le \mathrm{e}^{6npt}\Big(\int_{M_0} f_\sigma^p \mathrm{d}\mu_0 + \frac{\mathrm{Vol}(M_0)}{3np}\Big) - \frac{\mathrm{Vol}(M_0)}{3np}.$$

Noting that T is finite if $q \geq n-4 \geq 2$, we obtain the conclusion.

Let $g_\sigma = f_\sigma \mathrm{e}^{\frac{\varepsilon t}{2}}$. By the Sobolev inequality on submanifolds (see [12]) and a Stampacchia iteration procedure, we obtain that g_σ is uniformly bounded for all t (see [13] or [17] for the details). Then we obtain the following theorem.

Theorem 4.1 *If M_0 satisfies the condition* (4.1), *then there exist positive constants ε, σ and C_0 depending only on M_0, such that for all $t \in [0, T)$ we have*

$$|\mathring{h}|^2 \leq C_0(|H|^2+1)^{1-\sigma}\mathrm{e}^{-\frac{\varepsilon t}{2}}.$$

5 A Gradient Estimate

In the following, we derive an estimate for $|\nabla H|^2$ along the mean curvature flow.

Firstly, with the same method as in [2], we get the following evolution equation for $|\nabla H|^2$.

$$\partial_t|\nabla H|^2 = \Delta|\nabla H|^2 - 2|\nabla^2 H|^2 + h*h*\nabla h*\nabla h + \overline{R}*\nabla H*\nabla H + \overline{R}*h*h*\nabla H + \overline{\nabla}\,\overline{R}*H*\nabla H. \tag{5.1}$$

Here we use Hamilton's $*$ notation. For tensors T and S, $T*S$ means any linear combination of tensors formed by contraction on T and S by g. Then we obtain the following lemma.

Lemma 5.1 *There exists a constant $C_6 > 1$ depending only on n, such that*

$$\partial_t|\nabla H|^2 \leq \Delta|\nabla H|^2 + C_6[(|H|^2+1)|\nabla h|^2 + |h|^2|\nabla H|].$$

Secondly, we need the following estimates.

Lemma 5.2 *Along the mean curvature flow, we have*

(i) $\partial_t|H|^4 \geq \Delta|H|^4 - 12n|H|^2|\nabla h|^2 + \frac{4}{n}|H|^6$,

(ii) $\partial_t|\mathring{h}|^2 \leq \Delta|\mathring{h}|^2 - \frac{1}{3}|\nabla h|^2 + C_7(|H|^2+1)|\mathring{h}|^2$,

(iii) $\partial_t(|H|^2|\mathring{h}|^2) \leq \Delta(|H|^2|\mathring{h}|^2) - \frac{1}{6}|H|^2|\nabla h|^2 + C_9|\nabla h|^2 + C_8(|H|^2+1)^2|\mathring{h}|^2$,

where C_7, C_8, C_9 are sufficiently large constants.

Proof (i) From Lemma 2.2(ii) we derive that

$$\partial_t|H|^4 = \Delta|H|^4 - 4|H|^2|\nabla H|^2 - 2|\nabla|H|^2|^2 + 4|H|^2(R_2+3S_2) + 4n|H|^4.$$

From Lemma 3.5, we get $R_2 + 3S_2 \geq R_2 \geq \frac{1}{n}|H|^4$. Then using the Cauchy-Schwarz inequality, we have $4|H|^2|\nabla H|^2 + 2|\nabla|H|^2|^2 \leq 12|H|^2|\nabla H|^2 \leq 12n|H|^2|\nabla h|^2$.

(ii) We have

$$\partial_t|\mathring{h}|^2 = \Delta|\mathring{h}|^2 - 2|\nabla\mathring{h}|^2 - 2n|\mathring{h}|^2 + 2R_1 - \frac{2}{n}R_2 + 2S_1 - \frac{6}{n}S_2.$$

From Lemma 2.1, we get $2|\nabla\mathring{h}|^2 \geq \frac{1}{3}|\nabla h|^2$. Then applying Lemma 3.5, we obtain inequality (ii).

(iii) It follows from the evolution equations that

$$\begin{aligned}\partial_t(|H|^2|\mathring{h}|^2) = \Delta(|H|^2|\mathring{h}|^2) + 2|H|^2\Big(R_1 - \frac{R_2}{n} + S_1 - \frac{3S_2}{n}\Big) + 2|\mathring{h}|^2(R_2 + 3S_2)\\ - 2|H|^2|\nabla\mathring{h}|^2 - 2|\mathring{h}|^2|\nabla H|^2 - 2\langle\nabla|H|^2, \nabla|\mathring{h}|^2\rangle.\end{aligned}$$

From Lemma 2.1, we get $-2|H|^2|\nabla\mathring{h}|^2 \leq -\frac{1}{3}|H|^2|\nabla h|^2$.

From Lemma 3.5 and the pinching condition $|\mathring{h}|^2 < W$, we have

$$2|H|^2\Big(R_1 - \frac{R_2}{n} + S_1 - \frac{3S_2}{n}\Big) + 2|\mathring{h}|^2(R_2 + 3S_2) < C_8(|H|^2+1)^2|\mathring{h}|^2.$$

Using Theorem 4.1, we have

$$-2\langle\nabla|H|^2, \nabla|\mathring{h}|^2\rangle \leq 8|H||\nabla H||\mathring{h}||\nabla\mathring{h}| \leq 8n\sqrt{C_0}|H|(|H|^2+1)^{\frac{1-\sigma}{2}}|\nabla h|^2.$$

By Young's inequality, there exists a positive constant C_9, such that

$$-2\langle\nabla|H|^2, \nabla|\mathring{h}|^2\rangle \leq (C_9 + \frac{1}{6}|H|^2)|\nabla h|^2.$$

Now we prove a gradient estimate for the mean curvature.

Theorem 5.1 *If M_0 satisfies the condition* (4.1)*, then for all $\eta \in \big(0, \frac{\varepsilon^{\frac{1}{2}}}{4n\pi}\big)$, there exists a positive number Ψ depending on η and M_0, such that*

$$|\nabla H|^2 < [(\eta|H|)^4 + \Psi^2]e^{-\frac{\varepsilon t}{6}}.$$

Proof Define a scalar

$$f = |\nabla H|^2 e^{\frac{\varepsilon t}{6}} + (B_1 + B_2|H|^2)|\mathring{h}|^2 e^{\frac{5\varepsilon t}{12}} - (\eta|H|)^4,$$

where B_1, B_2 are two positive constants.

From Lemmas 5.1–5.2, we obtain

$$\begin{aligned}&(\partial_t - \Delta)f\\ \leq &\Big[C_6 e^{\frac{\varepsilon t}{6}} - \frac{B_2}{6}e^{\frac{5\varepsilon t}{12}} + 12n\eta^4\Big]|H|^2|\nabla h|^2\\ &+ \Big[C_6 e^{\frac{\varepsilon t}{6}} - \Big(\frac{B_1}{3} - B_2C_9\Big)e^{\frac{5\varepsilon t}{12}}\Big]|\nabla h|^2\\ &+ e^{\frac{5\varepsilon t}{12}}\Big[B_1C_7(|H|^2+1) + B_2C_8(|H|^2+1)^2 + \frac{5\varepsilon}{12}(B_1 + B_2|H|^2)\Big]|\mathring{h}|^2\\ &+ \Big(C_6|h|^2|\nabla H| + \frac{\varepsilon}{6}|\nabla H|^2\Big)e^{\frac{\varepsilon t}{6}} - \frac{4\eta^4}{n}|H|^6.\end{aligned}$$

We choose the constants B_1 and B_2, such that $\frac{B_2}{6} > C_6 + 1$ and $\frac{B_1}{3} - B_2C_9 > C_6 + 1$. Thus we get

$$(\partial_t - \Delta)f \le -\mathrm{e}^{\frac{5\varepsilon t}{12}}|\nabla h|^2 + B_3\mathrm{e}^{\frac{5\varepsilon t}{12}}(|H|^2+1)^2|\mathring{h}|^2 + \Big(C_6|h|^2|\nabla H| + \frac{\varepsilon}{6}|\nabla H|^2\Big)\mathrm{e}^{\frac{\varepsilon t}{6}} - \frac{4\eta^4}{n}|H|^6,$$

where B_3 is a positive constant depending on B_1 and B_2. Then we have

$$\begin{aligned} C_6|h|^2|\nabla H|\mathrm{e}^{\frac{\varepsilon t}{6}} &\le \frac{nC_6^2}{2}|h|^4\mathrm{e}^{-\frac{\varepsilon t}{12}} + \frac{1}{2n}|\nabla H|^2\mathrm{e}^{\frac{5\varepsilon t}{12}} \\ &\le \frac{nC_6^2}{2}|h|^4\mathrm{e}^{-\frac{\varepsilon t}{12}} + |\nabla h|^2\mathrm{e}^{\frac{5\varepsilon t}{12}} - \frac{\varepsilon}{6}|\nabla H|^2\mathrm{e}^{\frac{\varepsilon t}{6}}. \end{aligned} \tag{5.2}$$

By (5.2) and Theorem 4.1, we get

$$(\partial_t - \Delta)f \le \mathrm{e}^{-\frac{\varepsilon t}{12}}\Big[B_3C_0(|H|^2+1)^{3-\sigma} + \frac{nC_6^2}{2}|h|^4 - \frac{4\eta^4}{n}|H|^6\Big]. \tag{5.3}$$

Using the pinching condition $|h|^2 < W + \frac{|H|^2}{n}$, we derive that the expression in the bracket of (5.3) has a upper bound Ψ_2. Then we have $(\partial_t - \Delta)f \le \Psi_2\mathrm{e}^{-\frac{\varepsilon t}{12}}$. It follows from the maximum principle that f is bounded. This completes the proof of Theorem 5.1.

6 Convergence

We will follow Hamilton's idea in [11] to use the Myers theorem.

Theorem 6.1 (Myers) *Let Γ be a geodesic of length l on M. If the Ricci curvature satisfies* $\mathrm{Ric}(X) \ge (n-1)\frac{\pi^2}{l^2}$, *for each unit vector $X \in T_xM$, at any point $x \in \Gamma$, then Γ has conjugate points.*

Now we show that the mean curvature flow converges to a point or a totally geodesic submanifold.

Theorem 6.2 *If M_0 satisfies condition* (4.1) *and T is finite, then F_t converges to a round point as $t \to T$.*

Proof If T is finite, we have $\max\limits_{M_t}|h|^2 \to \infty$ as $t \to T$. Let $|H|_{\min} = \min\limits_{M_t}|H|$, $|H|_{\max} = \max\limits_{M_t}|H|$. From the preserved pinching condition, we get $|H|_{\max} \to \infty$ as $t \to T$.

By Theorem 5.1, for any $\eta \in \big(0, \frac{\varepsilon^{\frac{1}{2}}}{4n\pi}\big)$, there exists a positive number $\Psi > 1$, such that $|\nabla H| < (\eta|H|)^2 + \Psi$. Since $|H|_{\max} \to \infty$ as $t \to T$, there exists a time τ depending on η, such that for $t > \tau$, we have $|H|^2_{\max} > \frac{\Psi}{\eta^2}$. Then we get $|\nabla H| < 2\eta^2|H|^2_{\max}$.

From [36, Lemma 4.1], the sectional curvature K of M_t satisfies

$$K \ge \frac{1}{2}\Big(2 + \frac{1}{n-1}|H|^2 - |h|^2\Big). \tag{6.1}$$

Using the pinching condition $|\mathring{h}| < W - \varepsilon|H|^2$, we get $K > \frac{1}{2}(\varepsilon|H|^2 - n)$.

Let x be a point on M_t where $|H|$ achieves its maximum. Consider all the geodesics of length $l = (4\eta|H|_{\max})^{-1}$ starting from x. Since $|\nabla|H|^2| < 4\eta^2|H|^3_{\max}$, we have $|H|^2 > |H|^2_{\max} - 4\eta^2|H|^3_{\max} \cdot l = (1-\eta)|H|^2_{\max}$ along such a geodesic. Then we get $K > \frac{1}{2}(\varepsilon(1-\eta)|H|^2_{\max} - n) > \frac{\varepsilon}{4}|H|^2_{\max} > \frac{\pi^2}{l^2}$ on such a geodesic. From Myers' theorem, these geodesics can reach any point of M. Therefore, we obtain $|H|^2_{\min} > (1-\eta)|H|^2_{\max}$ and $\operatorname{diam} M_t \leq 2l$. Hence, we have $\operatorname{diam} M_t \to 0$ and $\frac{|H|_{\min}}{|H|_{\max}} \to 1$ as $t \to T$.

Now we dilate the metric of the ambient space such that the submanifold maintains its volume along the flow. Using the same method as in [22], we can prove that the rescaled mean curvature flow converges to a round sphere as the reparameterized time tends to infinity.

Theorem 6.3 *If M_0 satisfies condition* (4.1) *and $T = \infty$, then F_t converges to a totally geodesic submanifold $\mathbb{CP}^{\frac{n}{2}}$ as $t \to \infty$.*

Proof Firstly, we prove $|H|^2 < C\mathrm{e}^{-\frac{\varepsilon t}{12}}$ by contradiction. We assume that $|H|^2_{\max}\mathrm{e}^{\frac{\varepsilon t}{12}}$ becomes unbounded as $t \to \infty$. By Theorem 5.1, for any small positive number η, there exists a positive number Ψ, such that $|\nabla H| < [(\eta|H|)^2 + \Psi]\mathrm{e}^{-\frac{\varepsilon t}{12}}$. Let τ be a time such that $|H|^2_{\max}(\tau) \cdot \mathrm{e}^{\frac{\varepsilon\tau}{12}} > \frac{\Psi}{\eta^2}$. Then we have $|\nabla H| < 2\eta^2|H|^2_{\max}$ on M_τ. From (6.1) and Theorem 4.1, we derive that if τ is large enough, the sectional curvature of M_τ satisfies $K > \frac{1}{2n^2}|H|^2$.

Using Myers' theorem as in the proof of the previous theorem, we obtain $|H|^2_{\min}(\tau) > (1-\eta)|H|^2_{\max}(\tau)$. This together with $|H|^2_{\max}(\tau)\mathrm{e}^{\frac{\varepsilon\tau}{12}} > \frac{\Psi}{\eta^2}$ and Theorem 5.1 yields $|\nabla H|^2 < (\eta|H|)^4 + \frac{\eta^4}{(1-\eta)^2}|H|^4_{\min}(\tau)$ for all $t > \tau$. From the evolution equation of $|H|^2$, we have

$$(\partial_t - \Delta)|H|^2 \geq -2|\nabla H|^2 + \frac{2}{n}|H|^4 > \frac{1}{n}|H|^4 - \frac{1}{2n}|H|^4_{\min}(\tau) \quad \text{for } t > \tau.$$

Using the maximum principle, we see that $|H|^2$ blows up in finite time. This contradicts the infinity of T. Therefore, we obtain $|H|^2 < C\mathrm{e}^{-\frac{\varepsilon t}{12}}$.

From Theorem 4.1, we have $|h|^2 = |\mathring{h}|^2 + \frac{1}{n}|H|^2 \leq C\mathrm{e}^{-\frac{\varepsilon t}{12}}$. Since $|h| \to 0$ as $t \to \infty$, M_t converges to a totally geodesic submanifold M_∞ as $t \to \infty$.

In the case of $q \geq n-4$, since $|h|^2 < \psi(|H|^2)$ is preserved, thus the flow can not converge to a totally geodesic submanifold. So the dimension and codimension of M_∞ satisfies $q < n-4$. From the fact that the totally geodesic submanifolds of $\mathbb{CP}^m$ are totally real submanifolds $\mathbb{RP}^n$ and Kähler submanifolds $\mathbb{CP}^{\frac{n}{2}}$ (see [6]), we see that M_∞ must be $\mathbb{CP}^{\frac{n}{2}}$.

Combining the results of Theorems 6.2–6.3, we complete the proof of Theorems 1.1–1.3.

At last, we prove the classification theorem for submanifolds in $\mathbb{CP}^m$ under the weakly pinching condition $|\mathring{h}|^2 \leq W$.

Proof of Theorem 1.4 Let M evolve along the mean curvature flow. Using the strong maximum principle, we obtain either $|\mathring{h}|^2 < W$ for some $t > 0$, or $|\mathring{h}|^2 = W$ holds for all t.

If $|\mathring{h}|^2 < W$ for some $t > 0$, then M_t converges to a round point or a totally geodesic submanifold $\mathbb{CP}^{\frac{n}{2}}$.

If $|\mathring{h}|^2 = W$ holds for all t, then $(\partial_t - \Delta)(|\mathring{h}|^2 - W) = 0$. So W must be $\mathring{\psi}$ and

$$\frac{1}{2}(\partial_t - \Delta)(|\mathring{h}|^2 - W) = \mathring{\psi}\Big(\mathring{\psi} + \frac{1}{n}|H|^2 + n + 3\Big) - \mathring{\psi}'|H|^2\Big(\mathring{\psi} + \frac{1}{n}|H|^2 + n\Big) = 0.$$

From Lemma 3.6(iii), we get $|H| = 0$. Thus $|h|^2 = \psi(0) = 0$. Therefore, M_t is a totally geodesic submanifold for each t.

7 Appendix

For an odd integer $n \geq 3$, and a real number $\varepsilon \in [0, 1]$, we define a function $\varphi_\varepsilon : [0, +\infty) \to \mathbb{R}$ by

$$\varphi_\varepsilon(x) := d_\varepsilon + c_\varepsilon x - \sqrt{b^2x^2 + 2abx + e},$$

where

$$a = 2\sqrt{(n^2 - 4n + 3)b}, \quad b = \min\Big\{\frac{n-3}{4n-4}, \frac{2n-5}{n^2+n-2}\Big\},$$
$$c_\varepsilon = b + \frac{1}{n - 1 + \varepsilon}, \quad d_\varepsilon = 2 - 2\varepsilon + a, \quad e = \sqrt{\varepsilon}.$$

We set $\varphi = \varphi_0$.

Lemma 7.1 *The function φ satisfies*

(i) $\frac{x}{n-1} + 2 \leq \varphi(x) < \frac{x}{n-1} + n$,

(ii) $\varphi(x) > \sqrt{2(n-3)}$.

Proof If $n = 3$, then $a = b = 0$. We have $\varphi(x) = 2 + \frac{x}{n-1}$. It is easy to verify the inequalities above.

If $n \geq 5$, by direct computations, we get

$$\varphi'(x) = c_0 - \frac{bx + a}{\sqrt{x^2 + \frac{2ax}{b}}},$$
$$\varphi''(x) = \frac{a^2}{b\big(x^2 + \frac{2ax}{b}\big)^{\frac{3}{2}}}.$$

Since $\big(\varphi(x) - \frac{x}{n-1}\big)'' = \varphi''(x) > 0$ and $\lim\limits_{x\to\infty} \varphi'(x) = \frac{1}{n-1}$, we have $\varphi'(x) < \frac{1}{n-1}$. Hence we get

$$2 = \lim_{x\to\infty}\Big(\varphi(x) - \frac{x}{n-1}\Big) < \varphi(x) - \frac{x}{n-1} \leq \varphi(0) = d_0 < n.$$

If $n \geq 5$, we figure out that

$$\min_{x\geq 0}\varphi(x) = \varphi\Big(\frac{ac_0}{b\sqrt{c_0^2 - b^2}} - \frac{a}{b}\Big) = d_0 - \frac{ac_0}{b} + \frac{a}{b}\sqrt{c_0^2 - b^2}.$$

If $n = 5$, we have $\min_{x\geq 0} \varphi(x) = 4\sqrt{2} - 2$. If $n \geq 7$, we have

$$\min_{x\geq 0} \varphi(x) = 2 + 2\sqrt{\frac{n-3}{2n-5}}(\sqrt{5n-8} - \sqrt{n+2}) > \sqrt{2(n-3)}.$$

Letting $\mathring{\varphi}_\varepsilon(x) = \varphi_\varepsilon(x) - \frac{x}{n}$, we have the following lemma.

Lemma 7.2 *There exists a positive constant ε_1 depending on n, such that for all $\varepsilon \in (0, \varepsilon_1)$, the function φ_ε satisfies the following inequalities:*

(i) $2x\mathring{\varphi}''_\varepsilon(x) + \mathring{\varphi}'_\varepsilon(x) < \frac{2(n-1)}{n(n+2)}$,

(ii) $\mathring{\varphi}_\varepsilon(x)(\varphi_\varepsilon(x) - n + 3) - x\mathring{\varphi}'_\varepsilon(x)(\varphi_\varepsilon(x) + n + 3) < 2(n-1)$,

(iii) $\mathring{\varphi}_\varepsilon(x) - x\mathring{\varphi}'_\varepsilon(x) > 1$.

Proof If $n = 3$, then $a = b = 0$. We have $\varphi_\varepsilon(x) = 2 - 2\varepsilon + \frac{x}{2+\varepsilon} - \sqrt[4]{\varepsilon}$. It is easy to verify these inequalities.

If $n \geq 5$, by direct computations, we get

$$\begin{aligned}
\mathring{\varphi}'_\varepsilon(x) &= c_\varepsilon - \frac{1}{n} - \frac{b^2x + ab}{\sqrt{b^2x^2 + 2abx + e}}, \\
\mathring{\varphi}''_\varepsilon(x) &= \frac{b^2(a^2 - e)}{(b^2x^2 + 2abx + e)^{\frac{3}{2}}}, \\
\mathring{\varphi}'''_\varepsilon(x) &= -\frac{3b^3(a^2 - e)(bx + a)}{(b^2x^2 + 2abx + e)^{\frac{5}{2}}}.
\end{aligned}$$

Then we have

$$\begin{aligned}
&2x\mathring{\varphi}''_\varepsilon(x) + \mathring{\varphi}'_\varepsilon(x) \\
&= c_\varepsilon - \frac{1}{n} - \frac{b^3x^2(bx + 3a) + eb(3bx + a)}{(b^2x^2 + 2abx + e)^{\frac{3}{2}}} \\
&< c_\varepsilon - \frac{1}{n} < c_0 - \frac{1}{n} \\
&\leq \frac{2(n-1)}{n(n+2)}.
\end{aligned}$$

This proves inequality (i).

Setting $f(x) = \mathring{\varphi}_\varepsilon(x)(\varphi_\varepsilon(x) - n + 3) - x\mathring{\varphi}'_\varepsilon(x)(\varphi_\varepsilon(x) + n + 3)$, we get

$$\begin{aligned}
f(x) = {}& e + d_\varepsilon(d_\varepsilon + 3 - n) + (2 + ab + c_\varepsilon(d_\varepsilon - 2n))x \\
& - \frac{b(ac_\varepsilon + b(d_\varepsilon - 2n))x^2 + (3ab(d_\varepsilon + 1 - n) + c_\varepsilon e)x + e(2d_\varepsilon + 3 - n)}{\sqrt{b^2x^2 + 2abx + e}}.
\end{aligned}$$

Noticing that $2 + ab + c_\varepsilon(d_\varepsilon - 2n) = ac_\varepsilon + b(d_\varepsilon - 2n)$, we figure out

$$\lim_{x\to+\infty} \Big[(2 + ab + c_\varepsilon(d_\varepsilon - 2n))x - \frac{b(ac_\varepsilon + b(d_\varepsilon - 2n))x^2}{\sqrt{b^2x^2 + 2abx + e}}\Big] = \frac{a^2c_\varepsilon}{b} + a(d_\varepsilon - 2n).$$

Hence we obtain

$$
\begin{aligned}
&\lim_{x\to+\infty} f(x)\\
&= d_\varepsilon(d_\varepsilon+3-n)+\frac{a^2c_\varepsilon}{b}+a(d_\varepsilon-2n)-3a(d_\varepsilon+1-n)+e\Big(1-\frac{c_\varepsilon}{b}\Big)\\
&= 2(n-1)+2\varepsilon\Big(n-9+2\varepsilon+\frac{4+2\varepsilon}{n-1+\varepsilon}\Big)+\sqrt{\varepsilon}\Big(1-\frac{c_\varepsilon}{b}\Big)\\
&< 2(n-1).
\end{aligned}
$$

Calculating the derivative of $f(x)$, we get

$$
\begin{aligned}
f'(x) = 2+ab+c_\varepsilon(d_\varepsilon-2n)-(b^2x^2+2abx+e)^{-\frac{3}{2}}\\
\times [(3a^2b^2(d_\varepsilon+1-n)-3be(b-ac_\varepsilon+bn))x\\
+b^2(ac_\varepsilon+b(d_\varepsilon-2n))x^2(bx+3a)\\
+c_\varepsilon e^2+abe(d_\varepsilon-2n)].
\end{aligned}
$$

Thus we have

$$\lim_{x\to+\infty} f'(x)=2+ab+c_\varepsilon(d_\varepsilon-2n)-ac_\varepsilon-b(d_\varepsilon-2n)=0.$$

Furthermore, we get

$$f''(x)=[b(2b+bd_\varepsilon-ac_\varepsilon)x^2+(ab(1-n+d_\varepsilon)-c_\varepsilon e)x-(n+1)e]\frac{3b^2(a^2-e)}{(b^2x^2+2abx+e)^{\frac{5}{2}}}.$$

From $b\le \frac{n-3}{4n-4}$, we obtain

$$2b+bd_\varepsilon-ac_\varepsilon=(4-2\varepsilon)b-\frac{2\sqrt{(n^2-4n+3)b}}{n-1+\varepsilon}<0$$

and

$$1-n+d_\varepsilon=3-n+2\sqrt{(n^2-4n+3)b}-2\varepsilon<0.$$

So, we get $f''(x)<0$. Then we have $f'(x)>0$. From this we deduce that

$$f(x)<\lim_{x\to+\infty} f(x)<2(n-1).$$

Thus, inequality (ii) is proved.

We have

$$
\begin{aligned}
&\mathring{\varphi}_\varepsilon(x)-x\mathring{\varphi}'_\varepsilon(x)\\
&=d_\varepsilon-\frac{abx+e}{\sqrt{b^2x^2+2abx+e}}\\
&>d_\varepsilon-\frac{abx}{\sqrt{b^2x^2}}-\frac{e}{\sqrt{e}}\\
&=2-2\varepsilon-\sqrt[4]{\varepsilon}.
\end{aligned}
$$

This implies inequality (iii).

For each integer $n \geq 6$, we define the function $\psi : [0, +\infty) \to \mathbb{R}$ by

$$\psi(x) := \nu + \kappa x - \sqrt{\lambda^2 x^2 + 2\lambda\mu x + \nu^2},$$

where $\kappa = \lambda + \frac{1}{n-1}, \lambda = \frac{3}{n^3-4n^2+3}, \mu = \nu + \frac{3}{n}, \nu = \frac{9}{n^2-3n-3}$.

Lemma 7.3 *The function* ψ *satisfies the following inequalities*:

(i) $\frac{1}{n} \leq \psi'(x) < \frac{1}{n-1}$, $\frac{x}{n} \leq \psi(x) \leq \frac{x}{n-1}$, *and the equalities hold if and only if* $x = 0$,

(ii) $\max\limits_{x\geq 0}(2x\psi''(x) + \psi'(x)) < \frac{3}{n+8}$,

(iii) $3\psi(x) - \frac{3x}{n} + (\psi(x) - x\psi'(x))(\psi(x) + n) \leq 0$, *and the equality holds if and only if* $x = 0$,

(iv) $0 \leq x\psi'(x) - \psi(x) < 2$.

Proof Taking derivatives, we get

$$\begin{aligned}
\psi'(x) &= \kappa - \frac{\lambda^2 x + \lambda\mu}{\sqrt{\lambda^2 x^2 + 2\lambda\mu x + \nu^2}},\\
\psi''(x) &= \frac{\lambda^2(\mu^2 - \nu^2)}{(\lambda^2 x^2 + 2\lambda\mu x + \nu^2)^{\frac{3}{2}}},\\
\psi'''(x) &= -\frac{3\lambda^3(\mu^2 - \nu^2)(\lambda x + \mu)}{(\lambda^2 x^2 + 2\lambda\mu x + \nu^2)^{\frac{5}{2}}}.
\end{aligned}$$

(i) We have $\psi'(0) = \kappa - \frac{\lambda\mu}{\nu} = \frac{1}{n}$ and $\lim\limits_{x\to+\infty} \psi'(x) = \kappa - \lambda = \frac{1}{n-1}$. Since $\psi''(x) > 0$, we get $\frac{1}{n} < \psi'(x) < \frac{1}{n-1}$ for $x > 0$. Thus we obtain $\frac{x}{n} < \psi(x) < \frac{x}{n-1}$ for $x > 0$.

(ii) Letting $g(x) = 2x\psi''(x) + \psi'(x)$, we have

$$g(x) = \kappa - \frac{\lambda^4 x^3 + 3\lambda^3\mu x^2 + 3\lambda^2\nu^2 x + \lambda\mu\nu^2}{(\lambda^2 x^2 + 2\lambda\mu x + \nu^2)^{\frac{3}{2}}}$$

and

$$\begin{aligned}
g'(x) &= 2x\psi'''(x) + 3\psi''(x)\\
&= \frac{3\lambda^2(\mu^2 - \nu^2)(\nu^2 - \lambda^2 x^2)}{(\lambda^2 x^2 + 2\lambda\mu x + \nu^2)^{\frac{5}{2}}}.
\end{aligned}$$

Thus we get

$$\begin{aligned}
\max_{x\geq 0} g(x) &= g\Big(\frac{\nu}{\lambda}\Big)\\
&= \kappa - \lambda\sqrt{\frac{2\nu}{\mu + \nu}}\\
&= \frac{n(n-3) - 3\sqrt{\frac{6n}{n^2+3n-3}}}{n^3 - 4n^2 + 3}\\
&< \frac{3}{n+8}.
\end{aligned}$$

(iii) We have

$$\begin{aligned}&3\psi(x)-\frac{3x}{n}+(\psi(x)-x\psi'(x))(\psi(x)+n)\\=&\Big[\nu(n+3+2\nu)+\Big(\lambda\mu+3\kappa+\kappa\nu-\frac{3}{n}\Big)x\Big]\\&-\frac{\lambda(\kappa\mu+3\lambda+\lambda\nu)x^2+(\kappa\nu^2+\lambda\mu(n+6+3\nu))x+\nu^2(n+3+2\nu)}{\sqrt{\lambda^2x^2+2\lambda\mu x+\nu^2}}.\end{aligned}$$

Set

$$\begin{aligned}h(x):=&\Big[\nu(n+3+2\nu)+\Big(\lambda\mu+3\kappa+\kappa\nu-\frac{3}{n}\Big)x\Big]^2(\lambda^2x^2+2\lambda\mu x+\nu^2)\\&-[\lambda(\kappa\mu+3\lambda+\lambda\nu)x^2+(\kappa\nu^2+\lambda\mu(n+6+3\nu))x+\nu^2(n+3+2\nu)]^2.\end{aligned}$$

Now we need to prove $h(x)\le 0$. Since $\kappa=\lambda+\frac{1}{n-1},\mu=\nu+\frac{3}{n}$, we have

$$\lambda\mu+3\kappa+\kappa\nu-\frac{3}{n}=\kappa\mu+3\lambda+\lambda\nu.$$

Putting $A=\kappa\mu+3\lambda+\lambda\nu, B=n+3+2\nu, C=\kappa\nu^2+\lambda\mu(n+6+3\nu)$, we get

$$\begin{aligned}h(x)&=(\nu B+Ax)^2(\lambda^2x^2+2\lambda\mu x+\nu^2)-(\lambda Ax^2+Cx+\nu^2B)^2\\&=2\lambda A(\mu A+\lambda\nu B-C)x^3\\&\quad+[\nu^2(A-\lambda B)^2+4\lambda\mu\nu AB-C^2]x^2\\&\quad+2\nu^2B(\nu A+\lambda\mu B-C)x.\end{aligned}$$

By the definitions of κ,λ,μ,ν, we have

$$\mu A+\lambda\nu B=\nu A+\lambda\mu B=C$$

and

$$\nu^2(A-\lambda B)^2+4\lambda\mu\nu AB-C^2=-\frac{81(n^3-12n+9)^2}{n^2(n-1)^2(n^2-3n-3)^4}<0.$$

Thus, inequality (iii) is proved.

(iv) It follows from inequalities (i) and (iii) that $\psi(x)-x\psi'(x)\le 0$. By direct computations, we get

$$\begin{aligned}&x\psi'(x)-\psi(x)\\&=-\nu+\frac{\lambda\mu x+\nu^2}{\sqrt{\lambda^2x^2+2\lambda\mu x+\nu^2}}\\&\le-\nu+\frac{\lambda\mu x}{\sqrt{\lambda^2x^2}}+\frac{\nu^2}{\sqrt{\nu^2}}\\&=\mu<2.\end{aligned}$$

Declarations

Conflicts of interest The authors declare no conflicts of interest.

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