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The Dirac Equation on Metrics of Eguchi-Hanson Type II with Negative Constant Scalar Curvature*

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Dedicate to Professor Su Buqing for his great achievements in mathematical research and education

Abstract On metrics of Eguchi-Hanson type II with negative constant Ricci curvatures, the authors show that there is no nontrivial Killing spinor. On metrics of Eguchi-Hanson type II with negative constant scalar curvature, they show that there is no nontrivial L^p eigenspinor for $0 < p < 2$ if the eigenvalue has nontrivial real part, and no nontrivial L^2 eigenspinor if either the eigenvalue has trivial real part or the eigenvalue is real, the eigenspinor is isotropic and the parameter η in radial and angular equations for eigenspinors is real. They also solve harmonic spinors and eigenspinors explicitly on metrics of Eguchi-Hanson type II with certain special potentials.

Keywords Metric of Eguchi-Hanson type II, Killing spinor, Eigenspinor
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1 Introduction

Eguchi-Hanson metrics, referred as gravitational instantons, are Ricci flat, anti-self-dual 4-dimensional asymptotically local flat Riemannian metrics arisen in the Euclidean approach of gravitational quantization (cf. [6–7]). The metrics of Eguchi-Hanson type with zero scalar curvature were constructed by LeBrun using the method of algebraic geometry (cf. [14]) and by the third author solving an ordinary differential equation (cf. [17]). These metrics provide counter-examples of Hawking and Pope's generalized positive action conjecture (cf. [11]). Following the idea of [17], the first and the third authors constructed metrics of Eguchi-Hanson type II with negative constant scalar curvature (cf. [5]). They are asymptotically local hyperbolic (ALH for short) and also provide the positive action conjecture for negative cosmological constant.

It is well-known that spinors and the Dirac operator play important roles in geometry (cf. [1, 9, 13] and references therein). They are also used to describe spin-$\frac{1}{2}$ particles in quantum

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field theory. In [4], Chandrasekhar observed that spinors can be separated variables to reduce the Dirac equation into radical and angular ordinary differential equations. This results in significant physical implication that Dirac particles must either disappear into the black hole or escape to infinity (cf. [8, 16]).

It is therefore an interesting question to study the Dirac equation on metrics of Eguchi-Hanson type. In [15], Sucu and Ünal solved the Dirac equation on Eguchi-Hanson and Bianchi VII_0 gravitational instanton metrics by separating variables and obtained the solutions in terms of the product of two hypergeometric functions. In [3], Cai and the third author investigated the parallel spinors and the harmonic spinors on metrics of Eguchi-Hanson type. They found that the space of complex parallel spinors are complex 2-dimensional on Eguchi-Hanson metrics, and the harmonic spinors can be solved explicitly by separating variables on metrics of Eguchi-Hanson type with zero scalar curvature.

Let θ, ϕ, ψ be the Euler angles on the 3-sphere S^3. The Cartan-Maurer one-forms for $SU(2) \cong S^3$ are

$$\begin{aligned} \sigma_1 &= \frac{1}{2}(\sin\psi \mathrm{d}\theta - \sin\theta\cos\psi \mathrm{d}\phi), \quad \sigma_2 = \frac{1}{2}(-\cos\psi \mathrm{d}\theta - \sin\theta \sin\psi \mathrm{d}\phi), \\ \sigma_3 &= \frac{1}{2}(d\psi + \cos\theta \mathrm{d}\phi). \end{aligned}$$

Metrics of Eguchi-Hanson type II are given by

$$g = \frac{\mathrm{d}r^2}{(1+Br^2)f^2} + r^2(\sigma_1^2 + \sigma_2^2 + f^2\sigma_3^2) \tag{1.1}$$

for constant $B > 0$ and function $f > 0$, $\lim\limits_{r\to\infty} f = 1$. We refer f as the potential function.

Recall that in [5], the first and the third authors constructed the following metrics with constant negative scalar curvature $-12B$,

$$g = \frac{\mathrm{d}r^2}{(1+Br^2)\big(1 + \frac{\sqrt{1+Br^2}C}{r^4} + \frac{A}{r^4}\big)} + r^2\Big(\sigma_1^2 + \sigma_2^2 + \Big(1 + \frac{\sqrt{1+Br^2}C}{r^4} + \frac{A}{r^4}\Big)\sigma_3^2\Big) \tag{1.2}$$

with the potential function

$$f = \sqrt{1 + \frac{\sqrt{1+Br^2}C}{r^4} + \frac{A}{r^4}}, \tag{1.3}$$

where constant C is chosen to satisfy

$$C \le \frac{d^3 - 36d + (d^2+12)\sqrt{d^2+12}}{27B^2}$$

or

$$C > \frac{(d^4-4)(3d + \sqrt{3d^2+24})}{18B^2}$$

for certain given natural number $d \geq 2$, and constant A is chosen as

$$A = -r_0^4 - \sqrt{1 + Br_0^2}C \tag{1.4}$$

for the largest positive $r_0 > 0$ of the equation

$$(r^2)^3 + \frac{4 - d^2}{4B}(r^2)^2 + \frac{dC}{4}r^2 - \frac{BC^2}{16} = 0. \tag{1.5}$$

If $A = 0$, the metrics

$$g = \frac{\mathrm{d}r^2}{(1 + Br^2)\big(1 + \frac{\sqrt{1+Br^2}C}{r^4}\big)} + r^2\Big(\sigma_1^2 + \sigma_2^2 + \Big(1 + \frac{\sqrt{1 + Br^2}C}{r^4}\Big)\sigma_3^2\Big) \tag{1.6}$$

have constant Ricci curvature $-3B$. The potential function is

$$f = \sqrt{1 + \frac{\sqrt{1 + Br^2}C}{r^4}}. \tag{1.7}$$

Metrics (1.2), (1.6) for

$$r \geq r_0, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi \leq 2\pi, \quad 0 \leq \psi \leq \frac{4\pi}{d}$$

are geodesically complete and ALH. Topologically, the manifolds are

$$R_{\geq 0} \times SU(2)/Z_d \cong R_{\geq 0} \times S^3/Z_d.$$

In this paper, we investigate Killing spinors and show that they are always trivial on metric (1.6) with negative constant Ricci curvatures. We then study eigenspinors that can be separated variables as follows

$$\Psi = \mathrm{e}^{\mathrm{i}(n+\frac{1}{2})\phi}\begin{pmatrix} \mathrm{e}^{\mathrm{i}\frac{d}{2}(m_1+\frac{1}{2})\psi}\Phi_1(r)J_+(\theta) \\ \mathrm{e}^{\mathrm{i}\frac{d}{2}(m_2+\frac{1}{2})\psi}\Phi_2(r)J_-(\theta) \\ \mathrm{e}^{\mathrm{i}\frac{d}{2}(m_1+\frac{1}{2})\psi}\Phi_3(r)J_+(\theta) \\ \mathrm{e}^{\mathrm{i}\frac{d}{2}(m_2+\frac{1}{2})\psi}\Phi_4(r)J_-(\theta) \end{pmatrix} \tag{1.8}$$

on metric (1.2) with negative constant scalar curvature, where m_1, m_2 and n are integers, and $\Phi_i(r)$ $(i = 1, 2, 3, 4)$, $J_\pm(\theta)$ are referred as radial and angular parts of the eigenspinors, respectively. In particular, we refer them as isotropic eigenspinor if

$$\Phi_1 = \Phi_2, \quad \Phi_3 = \Phi_4. \tag{1.9}$$

We show that there is no nontrivial L^p eigenspinor for $0 < p < 2$ if the eigenvalue has nontrivial real part, and no nontrivial L^2 eigenspinor if either the eigenvalue has trivial real part or the eigenvalue is real, the eigenspinor is isotropic and the parameter η in radial and angular equations for eigenspinors is real.

We also solve harmonic spinors explicitly for the potential

$$f=\sqrt{1-\frac{(d^2-4)^2}{16B^2r^4}},\quad d\geq 3 \tag{1.10}$$

and solve isotropic eigenspinors with

$$\lambda=\pm\frac{\mathrm{i}\sqrt{B}}{2},\quad d=3,\quad m_1=0,\quad m_2=-1 \tag{1.11}$$

in terms of hypergeometric functions for the potential

$$f=\sqrt{1-\frac{25}{16B^2r^4}},\quad r\geq\sqrt{\frac{5}{4B}}. \tag{1.12}$$

The paper is organized as follows. In Section 2, we introduce spin connections and prove the nonexistence of the Killing equations on (1.6). In Section 3, we provide the Dirac equation by separating variables and solve the angular equations. In Section 4, we prove the nonexistence of L^p eigenspinor on (1.2). In Section 5, we explicitly solve harmonic spinors for potential (1.10) and isotropic eigenspinors for potential (1.12). In Appendix A, we provide some results for hypergeometric functions.

2 Spin Connection and Killing Equation

In this section, we introduce spin connections on metrics of Eguchi-Hanson type II and show that there is no nontrivial Killing spinor on metrics (1.6). Denote the frame of (1.1),

$$\begin{aligned}
e_1&=\sqrt{1+Br^2}f\partial_r,\\
e_2&=\frac{2}{r}\Big(\sin\psi\partial_\theta-\frac{\cos\psi}{\sin\theta}\partial_\phi+\frac{\cos\psi\cos\theta}{\sin\theta}\partial_\psi\Big),\\
e_3&=\frac{2}{r}\Big(-\cos\psi\partial_\theta-\frac{\sin\psi}{\sin\theta}\partial_\phi+\frac{\sin\psi\cos\theta}{\sin\theta}\partial_\psi\Big),\\
e_4&=\frac{2}{rf}\partial_\psi.
\end{aligned}$$

The connection 1-form $\{\omega^i{}_j\}$ of (1.1) are [5],

$$
\begin{aligned}
\omega^2{}_1 &= \frac{\sqrt{1+Br^2}f}{r}\mathrm{e}^2,\\
\omega^3{}_1 &= \frac{\sqrt{1+Br^2}f}{r}\mathrm{e}^3,\\
\omega^4{}_1 &= \Big(\frac{\sqrt{1+Br^2}f}{r} + \sqrt{1+Br^2}f'\Big)\mathrm{e}^4,\\
\omega^3{}_4 &= \frac{f}{r}\mathrm{e}^2,\\
\omega^4{}_2 &= \frac{f}{r}\mathrm{e}^3,\\
\omega^2{}_3 &= \Big(\frac{2}{rf} - \frac{f}{r}\Big)\mathrm{e}^4.
\end{aligned}
\tag{2.1}
$$

Note that the space of complex spinors is complex 4-dimensional. Let $\Psi = (\Psi_1, \Psi_2, \Psi_3, \Psi_4)^t$ be a complex spinor. The spin connections are given by

$$\nabla_{e_k}\Psi = e_k\Psi + \frac{1}{4}\sum_{i,j=1}^{4} g(\nabla_{e_k}e_i, e_j)e_i\cdot e_j\cdot\Psi.$$

In terms of connection 1-forms (2.1), we obtain

$$
\begin{aligned}
\nabla_{e_1}\Psi &= e_1\Psi,\\
\nabla_{e_2}\Psi &= e_2\Psi + \frac{1}{2}\omega^2{}_1(e_2)e_1\cdot e_2\cdot\Psi + \frac{1}{2}\omega^4{}_3(e_2)e_3\cdot e_4\cdot\Psi,\\
\nabla_{e_3}\Psi &= e_3\Psi + \frac{1}{2}\omega^3{}_1(e_3)e_1\cdot e_3\cdot\Psi + \frac{1}{2}\omega^4{}_2(e_3)e_2\cdot e_4\cdot\Psi,\\
\nabla_{e_4}\Psi &= e_4\Psi + \frac{1}{2}\omega^4{}_1(e_4)e_1\cdot e_4\cdot\Psi + \frac{1}{2}\omega^3{}_2(e_4)e_2\cdot e_3\cdot\Psi.
\end{aligned}
\tag{2.2}
$$

Throughout the paper we fix the Clifford representation

$$
\begin{aligned}
&e_1 \mapsto \begin{pmatrix} & & 1 & \\ & & & 1\\ -1 & & & \\ & -1 & & \end{pmatrix}, \qquad
e_2 \mapsto \begin{pmatrix} & & & \mathrm{i}\\ & & \mathrm{i} & \\ & \mathrm{i} & & \\ \mathrm{i} & & & \end{pmatrix},\\
&e_3 \mapsto \begin{pmatrix} & & & -1\\ & & 1 & \\ & -1 & & \\ 1 & & & \end{pmatrix}, \qquad
e_4 \mapsto \begin{pmatrix} & & \mathrm{i} & \\ & & & -\mathrm{i}\\ \mathrm{i} & & & \\ & -\mathrm{i} & & \end{pmatrix}.
\end{aligned}
\tag{2.3}
$$

It is well-known that the existence of imaginary Killing spinors implies that metrics have negative constant Ricci curvature (cf. [9]). In order that metrics of Eguchi-Hanson type II have negative constant Ricci curvature $-3B$, the Killing equations are

$$\nabla_{e_k}\Psi = \pm\frac{\mathrm{i}\sqrt{B}}{2}e_k\cdot\Psi, \quad k = 1,2,3,4. \tag{2.4}$$

Before we study the Killing equation (2.4), we prove the following proposition.

Proposition 2.1 *The root r_0 of* (1.5) *is simple if $d = 2$ or $A = 0$.*

Proof If $d = 2$, then (1.5) implies that $C \neq 0$. Denote

$$\Delta = \frac{B^2C^4}{1024} + \frac{C^3}{216}.$$

If $\Delta > 0$ or $\Delta < 0$, r_0 must be single. If $\Delta = 0$, we have

$$r_0 = \frac{4}{3\sqrt{B}}.$$

This shows that r_0 is simple. In the case $A = 0$, (1.4) gives that

$$h(r) = r^4 + \sqrt{1 + Br^2}C = r^4 - \frac{\sqrt{1 + Br^2}}{\sqrt{1 + Br_0^2}} r_0^4.$$

Since

$$h'(r_0) = r_0^2\Big(\frac{3 + 2Br_0^2}{1 + Br_0^2}\Big) \neq 0,$$

we know that r_0 is simple.

Theorem 2.1 *The Killing equations* (2.4) *has no nontrivial solution on metrics* (1.6) *for $C \neq 0$.*

Proof Let Ψ be the solution of (2.4). The first equation of (2.2) gives

$$\begin{aligned} \partial_r \begin{pmatrix} \Psi_1 \\ \Psi_3 \end{pmatrix} &= \pm \frac{\mathrm{i}\sqrt{B}}{2\sqrt{1 + Br^2}f} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_3 \end{pmatrix}, \\ \partial_r \begin{pmatrix} \Psi_2 \\ \Psi_4 \end{pmatrix} &= \pm \frac{\mathrm{i}\sqrt{B}}{2\sqrt{1 + Br^2}f} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \Psi_2 \\ \Psi_4 \end{pmatrix}. \end{aligned} \tag{2.5}$$

Then the general solution are

$$\begin{aligned} \begin{pmatrix} \Psi_1 \\ \Psi_3 \end{pmatrix} &= Q_1 \begin{pmatrix} \mathrm{e}^{-L(r)} \\ \mathrm{i}\mathrm{e}^{-L(r)} \end{pmatrix} + Q_3 \begin{pmatrix} \mathrm{e}^{L(r)} \\ -\mathrm{i}\mathrm{e}^{L(r)} \end{pmatrix}, \\ \begin{pmatrix} \Psi_2 \\ \Psi_4 \end{pmatrix} &= Q_2 \begin{pmatrix} \mathrm{e}^{-L(r)} \\ \mathrm{i}\mathrm{e}^{-L(r)} \end{pmatrix} + Q_4 \begin{pmatrix} \mathrm{e}^{L(r)} \\ -\mathrm{i}\mathrm{e}^{L(r)} \end{pmatrix}, \end{aligned} \tag{2.6}$$

where Q_i $(i = 1, 2, 3, 4)$ are functions of θ, ϕ, ψ and

$$L(r) = \pm \int_{r_0}^{r} \frac{\sqrt{B}}{2\sqrt{1 + B\overline{r}^2}f} \mathrm{d}\overline{r}. \tag{2.7}$$

As r_0 is a simple root, $L(r)$ is convergent at $r = r_0$.

If $\begin{pmatrix} \Psi_1 \\ \Psi_3 \end{pmatrix}$ is nontrivial, then the fourth equation of (2.2) gives

$$\begin{aligned} \partial_\psi Q_1 &= \frac{\mathrm{i}rfG(r)}{2} Q_1 - \frac{\mathrm{i}rf\mathrm{e}^{2L}\big(F(r) \mp \frac{\mathrm{i}\sqrt{B}}{2}\big)}{2} Q_3, \\ \partial_\psi Q_3 &= -\frac{\mathrm{i}rf\big(F(r) \pm \frac{\mathrm{i}\sqrt{B}}{2}\big)}{2\mathrm{e}^{2L}} Q_1 + \frac{\mathrm{i}rfG(r)}{2} Q_3, \end{aligned} \tag{2.8}$$

where

$$F(r)=\frac{\sqrt{1+Br^2}f}{2r}+\frac{\sqrt{1+Br^2}f'}{2},\quad G(r)=\frac{1}{rf}-\frac{f}{2r}.$$

Since the left hand sides of (2.8) do not depend on r, we must have

$$\partial_\psi Q_i=0,\quad i=1,3.$$

Hence, (2.8) reduces to

$$\frac{G(r)}{\mathrm{e}^{2L}\big(F(r)\mp\frac{\mathrm{i}\sqrt{B}}{2}\big)}=\frac{F(r)\pm\frac{\mathrm{i}\sqrt{B}}{2}}{\mathrm{e}^{2L}G(r)}=\frac{Q_3(\theta,\phi)}{Q_1(\theta,\phi)}. \tag{2.9}$$

Since the third term does not depend on r, the three terms in (2.9) must be nonzero constant. Therefore,

$$0=F(r)^2-G(r)^2-\frac{B}{4}=\frac{BC(4r^2-2\sqrt{1+Br^2})+BC^2r^2}{16r^4(C\sqrt{1+Br^2}+r^4)}.$$

So that $C=0$ which gives contradiction. Similarly, $\begin{pmatrix}\Psi_2\\ \Psi_4\end{pmatrix}$ must also be trivial. Thus the proof of the theorem is complete.

Killing spinors on standard hyperbolic metric are explicitly obtained for arbitrary dimension, e.g. [1]. On metric (1.6) with $C=0$, they can be solved as follows:

$$\Psi=\begin{pmatrix}C_1F_+\\ C_2F_+\\ \mathrm{i}C_1F_-\\ \mathrm{i}C_2F_-\end{pmatrix}+C_3\mathrm{e}^{-\frac{\mathrm{i}\phi}{2}}\begin{pmatrix}\mathrm{e}^{\frac{\mathrm{i}\psi}{2}}\sin\frac{\theta}{2}F_-\\ \mathrm{e}^{-\frac{\mathrm{i}\psi}{2}}\cos\frac{\theta}{2}F_-\\ \mathrm{i}\mathrm{e}^{\frac{\mathrm{i}\psi}{2}}\sin\frac{\theta}{2}F_+\\ \mathrm{i}\mathrm{e}^{-\frac{\mathrm{i}\psi}{2}}\cos\frac{\theta}{2}F_+\end{pmatrix}+C_4\mathrm{e}^{\frac{\mathrm{i}\phi}{2}}\begin{pmatrix}\mathrm{e}^{\frac{\mathrm{i}\psi}{2}}\cos\frac{\theta}{2}F_-\\ -\mathrm{e}^{-\frac{\mathrm{i}\psi}{2}}\sin\frac{\theta}{2}F_-\\ \mathrm{i}\mathrm{e}^{\frac{\mathrm{i}\psi}{2}}\cos\frac{\theta}{2}F_+\\ -\mathrm{i}\mathrm{e}^{-\frac{\mathrm{i}\psi}{2}}\sin\frac{\theta}{2}F_+\end{pmatrix},$$

where C_i $(i=1,2,3,4)$ are complex constant and

$$\begin{aligned}F_+&=\sqrt{\sqrt{1+Br^2}\mp\sqrt{B}r}+\sqrt{\sqrt{1+Br^2}\pm\sqrt{B}r},\\ F_-&=\sqrt{\sqrt{1+Br^2}\mp\sqrt{B}r}-\sqrt{\sqrt{1+Br^2}\pm\sqrt{B}r}.\end{aligned}$$

3 Dirac Equation

In this section, we study the Dirac equation

$$D\Psi=\sum_{k=1}^{4}e_k\cdot\nabla_{e_k}\Psi=\lambda\Psi \tag{3.1}$$

on metrics (1.1). As the manifold is noncompact, λ is a complex number in general.

For noncompact manifolds, there are point spectrum, essential spectrum, discrete spectrum and continue spectrum. Denote ΣM the spinor fields on (1.1) and $L^2(\Sigma M)$ the square integrable spinors. The point spectrum of D is define as

$$\mathrm{Spec}_p(D) := \big\{\lambda \in \mathbb{C} \mid L^2(\Sigma M) \cap \mathrm{Ker}(\lambda - D) \neq \emptyset\big\}.$$

If there exists a sequence $(\varphi_k) \subset L^2(\Sigma M)$, which are orthonormal with canonical L^2 inner product, the essential spectrum of D is define as

$$\mathrm{Spec}_e(D) := \big\{\lambda \in \mathbb{C} \mid \lim_{k\to\infty} ||(D-\lambda)\varphi_k||_{L^2} = 0\big\}.$$

The discrete spectrum of D is defined as

$$\mathrm{Spec}_d(D) := \mathrm{Spec}_p(D) \backslash \mathrm{Spec}_e(D).$$

The continue spectrum of D is defined as

$$\mathrm{Spec}_c(D) := \mathrm{Spec}_e(D) \backslash \mathrm{Spec}_p(D).$$

The spectrum of D is

$$\mathrm{Spec}(D) = \mathrm{Spec}_e(D) \cup \mathrm{Spec}_d(D) = \mathrm{Spec}_p(D) \cup \mathrm{Spec}_c(D).$$

Spectrums of the Dirac operator are studied extensively, and we refer to [9] and references therein for many interesting results. In particular, $\mathrm{Spec}_p(D)$ is empty on the real hyperbolic space (cf. [2]), and either empty or $\{0\}$ on Riemannian symmetric spaces of non-compact type (cf. [10]).

For metrics (1.2), with respect to Clifford representation (2.3), the Dirac equation (3.1) are

$$\begin{aligned}
\Big(\sqrt{1+Br^2}f\partial_r - \mathrm{i}\frac{2}{rf}\partial_\psi - G_-(r)\Big)\Psi_1 - \frac{2}{r}\mathrm{e}^{\mathrm{i}\psi}\Big(\partial_\theta - \frac{\mathrm{i}}{\sin\theta}\partial_\phi + \mathrm{i}\cot\theta\partial_\psi\Big)\Psi_2 &= -\lambda\Psi_3,\\
\Big(\sqrt{1+Br^2}f\partial_r + \mathrm{i}\frac{2}{rf}\partial_\psi - G_-(r)\Big)\Psi_2 + \frac{2}{r}\mathrm{e}^{-\mathrm{i}\psi}\Big(\partial_\theta + \frac{\mathrm{i}}{\sin\theta}\partial_\phi - \mathrm{i}\cot\theta\partial_\psi\Big)\Psi_1 &= -\lambda\Psi_4,\\
\Big(\sqrt{1+Br^2}f\partial_r + \mathrm{i}\frac{2}{rf}\partial_\psi + G_+(r)\Big)\Psi_3 + \frac{2}{r}\mathrm{e}^{\mathrm{i}\psi}\Big(\partial_\theta - \frac{\mathrm{i}}{\sin\theta}\partial_\phi + \mathrm{i}\cot\theta\partial_\psi\Big)\Psi_4 &= \lambda\Psi_1,\\
\Big(\sqrt{1+Br^2}f\partial_r - \mathrm{i}\frac{2}{rf}\partial_\psi + G_+(r)\Big)\Psi_4 - \frac{2}{r}\mathrm{e}^{-\mathrm{i}\psi}\Big(\partial_\theta + \frac{\mathrm{i}}{\sin\theta}\partial_\phi - \mathrm{i}\cot\theta\partial_\psi\Big)\Psi_3 &= \lambda\Psi_2,
\end{aligned} \tag{3.2}$$

where

$$G_\pm(r) = \frac{1}{rf} + \frac{f}{2r} \pm \Big(\frac{3\sqrt{1+Br^2}f}{2r} + \frac{\sqrt{1+Br^2}f'}{2}\Big).$$

Let

$$\Phi_i(r) = \frac{1}{r\sqrt{(1+\sqrt{1+Br^2})f}} X_i(r), \quad i = 1,2,3,4. \tag{3.3}$$

Substituting (1.8) into (3.2) and absorbing functions involving r to the left hand side and involving θ, ϕ, ψ to the right hand side, we find that both sides must be equal to certain complex number η in order that the equality holds. It yields the radial equations

$$\begin{aligned}
\Big(\sqrt{1+Br^2}f\frac{\mathrm{d}}{\mathrm{d}r} + \frac{d\big(m_1+\frac{1}{2}\big)-1}{rf}\Big)X_1 + \frac{2\eta}{r}X_2 + \lambda X_3 = 0,\\
\Big(\sqrt{1+Br^2}f\frac{\mathrm{d}}{\mathrm{d}r} - \frac{d\big(m_2+\frac{1}{2}\big)+1}{rf}\Big)X_2 + \frac{2\eta}{r}X_1 + \lambda X_4 = 0,\\
\Big(\sqrt{1+Br^2}f\frac{\mathrm{d}}{\mathrm{d}r} - \frac{d\big(m_1+\frac{1}{2}\big)-1}{rf} + \frac{f}{r}\Big)X_3 - \frac{2\eta}{r}X_4 + \lambda X_1 = 0,\\
\Big(\sqrt{1+Br^2}f\frac{\mathrm{d}}{\mathrm{d}r} + \frac{d\big(m_2+\frac{1}{2}\big)+1}{rf} + \frac{f}{r}\Big)X_4 - \frac{2\eta}{r}X_3 + \lambda X_2 = 0,
\end{aligned} \tag{3.4}$$

and the angular equations

$$\begin{aligned}
\frac{1}{J_-(\theta)}\Big(\partial_\theta - \Big(n+\frac{1}{2}\Big)\csc\theta + \frac{d}{2}\Big(m_1+\frac{1}{2}\Big)\cot\theta\Big)J_+(\theta) = \eta \mathrm{e}^{\mathrm{i}\left(\frac{d}{2}(m_2-m_1)+1\right)\psi},\\
\frac{1}{J_+(\theta)}\Big(-\partial_\theta - \Big(n+\frac{1}{2}\Big)\csc\theta + \frac{d}{2}\Big(m_2+\frac{1}{2}\Big)\cot\theta\Big)J_-(\theta) = \eta \mathrm{e}^{-\mathrm{i}\left(\frac{d}{2}(m_2-m_1)+1\right)\psi}.
\end{aligned} \tag{3.5}$$

Moreover, as the right hand sides of (3.5) do not depend on θ, it must be a constant. Therefore, it is either

$$\eta = 0, \quad m_1,\, m_2 \text{ arbitary}$$

or

$$\eta \text{ arbitary}, \quad \frac{d}{2}(m_2-m_1)+1 = 0 \Longleftrightarrow d=2, \quad m_1 = m_2+1. \tag{3.6}$$

Thus the angular equations reduce to, for $d \geq 2$,

$$\begin{aligned}
\Big(\partial_\theta - \Big(n+\frac{1}{2}\Big)\csc\theta + \frac{d}{2}\Big(m_1+\frac{1}{2}\Big)\cot\theta\Big)J_+(\theta) = 0,\\
\Big(-\partial_\theta - \Big(n+\frac{1}{2}\Big)\csc\theta + \frac{d}{2}\Big(m_2+\frac{1}{2}\Big)\cot\theta\Big)J_-(\theta) = 0,
\end{aligned} \tag{3.7}$$

or, for $d = 2$,

$$\begin{aligned}
\Big(\partial_\theta - \Big(n+\frac{1}{2}\Big)\csc\theta + \Big(m+\frac{1}{2}\Big)\cot\theta\Big)J_+ = \eta J_-,\\
\Big(-\partial_\theta - \Big(n+\frac{1}{2}\Big)\csc\theta + \Big(m-\frac{1}{2}\Big)\cot\theta\Big)J_- = \eta J_+.
\end{aligned} \tag{3.8}$$

Proposition 3.1 *The solutions of* (3.7) *are*

$$\begin{aligned} J_+ &= \Big(\sin\frac{\theta}{2}\Big)^{n+\frac{1}{2}-\frac{d}{2}(m_1+\frac{1}{2})}\Big(\cos\frac{\theta}{2}\Big)^{-n-\frac{1}{2}-\frac{d}{2}(m_1+\frac{1}{2})}, \\ J_- &= \Big(\sin\frac{\theta}{2}\Big)^{-n-\frac{1}{2}+\frac{d}{2}(m_2+\frac{1}{2})}\Big(\cos\frac{\theta}{2}\Big)^{n+\frac{1}{2}+\frac{d}{2}(m_2+\frac{1}{2})}, \end{aligned} \tag{3.9}$$

which are regular if

$$m_1 \le -\frac{2}{d}\Big|n+\frac{1}{2}\Big| - \frac{1}{2}, \quad m_2 \ge \frac{2}{d}\Big|n+\frac{1}{2}\Big| - \frac{1}{2}. \tag{3.10}$$

Proof Note that (3.7) reduces to

$$\begin{aligned} \partial_\theta J_+ &= \Big(\Big(n+\frac{1}{2}\Big)\csc\theta - \frac{d}{2}\Big(m_1+\frac{1}{2}\Big)\cot\theta\Big)J_+, \\ \partial_\theta J_- &= \Big(-\Big(n+\frac{1}{2}\Big)\csc\theta + \frac{d}{2}\Big(m_2+\frac{1}{2}\Big)\cot\theta\Big)J_-. \end{aligned}$$

Solving them we obtain (3.9). The regularity follows if the power indices of $\sin\frac{\theta}{2}$ and $\cos\frac{\theta}{2}$ are nonnegative.

The solutions of (3.8) are solved by Sucu and Ünal [15] in terms of hypergeometric function.

4 Nonexistence of Eigenspinor

In this section, we investigate nonexistence of eigenspinor Ψ on L^p spaces implicated by radial equations (3.4).

Theorem 4.1 *On metrics* (1.2) *with constant negative scalar curvature, there is no nontrivial L^p eigenspinor taking the form* (1.8) *with eigenvalue $\Re(\lambda)\neq 0$, where $0<p\le\frac{6}{3+\varepsilon}$, $\varepsilon>\frac{2|\Re(\lambda)|}{\sqrt{B}}$.*

Proof Denote

$$X = (X_1, X_2, X_3, X_4)^t.$$

From (3.4), we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}r}|X|^2 = &-\frac{2(\eta+\overline{\eta})}{r\sqrt{1+Br^2}f}(X_1\overline{X_2}+\overline{X_1}X_2 - X_3\overline{X_4}-\overline{X_3}X_4) \\ &-\frac{2}{r\sqrt{1+Br^2}}(|X_3|^2+|X_4|^2) \\ &-\frac{2d(m_1+\frac{1}{2})-2}{r\sqrt{1+Br^2}f^2}(|X_1|^2-|X_3|^2) \\ &+\frac{2d(m_2+\frac{1}{2})+2}{r\sqrt{1+Br^2}f^2}(|X_2|^2-|X_4|^2) \\ &-\frac{(\lambda+\overline{\lambda})}{\sqrt{1+Br^2}f}(X_1\overline{X_2}+\overline{X_1}X_2 - X_3\overline{X_4}-\overline{X_3}X_4). \end{aligned} \tag{4.1}$$

Therefore there exists a large $r_1 > r_0$ with $|X|(r_1) \neq 0$, and for $r \geq r_1$, $\varepsilon > \frac{2|\Re(\lambda)|}{\sqrt{B}}$,

$$\frac{\mathrm{d}}{\mathrm{d}r}|X|^2 \geq -\frac{\varepsilon}{r}|X|^2.$$

Integrating it from r_1 to r, we obtain

$$|X|(r) > C_1 r^{-\frac{\varepsilon}{2}}, \quad r > r_1 \tag{4.2}$$

for some positive constant C_1. Denote by $\mathrm{d}\mu$ the volume element of (1.1) and

$$\mathcal{D} = \Big\{0 \leq \theta \leq \pi,\ 0 \leq \phi \leq 2\pi,\ 0 \leq \psi \leq \frac{4\pi}{d}\Big\}.$$

Integrating (4.2) , we obtain

$$\begin{aligned}
&\int_{\mathcal{D}}\int_{\{r\geq r_1\}} |\Psi|^p \mathrm{d}\mu \\
=&\int_{\mathcal{D}} \sigma_1\sigma_2\sigma_3 \int_{\{r\geq r_1\}} \frac{r^3|\Psi|^p}{\sqrt{1+Br^2}}\mathrm{d}r \\
\geq& C' \frac{2\pi^2}{d} \int_{\{r\geq r_1\}} \frac{r^2|X|^p}{\big(r\sqrt{(1+\sqrt{1+Br^2})f}\big)^p}\mathrm{d}r \\
>& C'' \int_{\{r\geq r_1\}} r^{2-\left(\frac{3}{2}+\frac{\varepsilon}{2}\right)p}\mathrm{d}r > \infty
\end{aligned}$$

for $0 < p \leq \frac{6}{3+\varepsilon}$. Thus the proof of the theorem is complete.

Theorem 4.2 *On metrics* (1.2) *with constant negative scalar curvature, there is no nontrivial L^2 eigenspinor taking the form* (1.8) *with eigenvalue* $\Re(\lambda) = 0$.

Proof By (4.1), there exists a large $r_1 > r_0$ with $|X|(r_1) \neq 0$, and for $r \geq r_1$,

$$\frac{\mathrm{d}}{\mathrm{d}r}|X|^2 \geq -\frac{2C_1}{r^2}|X|^2 \tag{4.3}$$

with some positive constant C_1. Integrating it from r_1 to r, we obtain

$$|X| > C_2 \mathrm{e}^{\frac{C_1}{r}} > C_2, \quad r > r_1.$$

Therefore

$$\int_{\mathcal{D}}\int_{\{r\geq r_1\}} |\Psi|^2 \mathrm{d}\mu > C' \int_{\{r\geq r_1\}} \frac{1}{r}\mathrm{d}r > \infty.$$

The proof of the theorem is complete.

Theorem 4.3 *On metrics* (1.1) *with constant negative scalar curvature, there is no nontrivial L^2 isotropic eigenspinor taking the form* (1.8) *satisfying* (1.9), *and with real eigenvalue* λ, *real* η.

Proof For isotropic eigenspinors, we can assume

$$X_1 = X_2 = X_+, \quad X_3 = X_4 = X_-.$$

Then (3.4) gives that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}r}X_+ &= -\frac{2\eta}{r\sqrt{1+Br^2}f}X_+ - \frac{\lambda}{\sqrt{1+Br^2}f}X_-, \\ \frac{\mathrm{d}}{\mathrm{d}r}X_- &= \frac{\lambda}{\sqrt{1+Br^2}f}X_+ + \Big(\frac{2\eta}{r\sqrt{1+Br^2}f} - \frac{1}{\sqrt{1+Br^2}}\Big)X_-. \end{aligned} \tag{4.4}$$

With η, λ real numbers, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}r}(X_+\overline{X}_- - \overline{X}_+X_-) = -\frac{1}{r\sqrt{1+Br^2}}(X_+\overline{X}_- - \overline{X}_+X_-).$$

This implies that either

$$X_+\overline{X}_- - \overline{X}_+X_- \equiv 0 \quad \text{or} \quad X_+\overline{X}_- - \overline{X}_+X_- \neq 0, \quad r \geq r_0.$$

In the first case (4.1) gives that there exists a large r_1 such that, for $r \geq r_1$,

$$\frac{\mathrm{d}}{\mathrm{d}r}|X|^2 \geq -\frac{2C_1}{r^2}|X|^2 \tag{4.5}$$

for some positive constant C_1. In the second case

$$(X_+\overline{X}_- - \overline{X}_+X_-)(r) = (X_+\overline{X}_- - \overline{X}_+X_-)(r_0)\frac{(1+\sqrt{1+Br^2})r_0}{\big(1+\sqrt{1+Br_0^2}\big)r}.$$

This gives that

$$|X_+\overline{X}_- - \overline{X}_+X_-| \geq |X_+\overline{X}_- - \overline{X}_+X_-|(r_0)\frac{r_0\sqrt{B}}{\big(1+\sqrt{1+Br_0^2}\big)} > 0.$$

Therefore

$$|X_+|^2 + |X_-|^2 \geq |X_+\overline{X}_- - \overline{X}_+X_-| > C_2 > 0. \tag{4.6}$$

Similar to the proof of Theorem 4.2, we know that Ψ is not L^2.

5 Exact Solution

In this section, we study exact solutions of the harmonic spinors and eigenspinors for certain special potentials.

Theorem 5.1 *On metrics* (1.1) *with constant negative scalar curvature where the potential* f *is given by* (1.10), *harmonic spinors* Ψ *taking form* (1.8) *can be solved explicitly for* $d \geq 3$ *as follows*

$$\begin{aligned}
\Phi_1 &= \frac{\mathrm{e}^{\left(d\left(m_1+\frac{1}{2}\right)-1\right)H_1} + H_2^{\left(-d\left(m_1+\frac{1}{2}\right)+1\right)}}{\sqrt{1+\sqrt{1+Br^2}}\left(r^4 - \frac{(d^2-4)^2}{16B^2}\right)^{\frac{1}{4}}},\\
\Phi_2 &= \frac{\mathrm{e}^{\left(-d\left(m_2-\frac{1}{2}\right)-1\right)H_1} + H_2^{\left(d\left(m_2+\frac{1}{2}\right)+1\right)}}{\sqrt{1+\sqrt{1+Br^2}}\left(r^4 - \frac{(d^2-4)^2}{16B^2}\right)^{\frac{1}{4}}},\\
\Phi_3 &= \frac{\sqrt{1+\sqrt{1+Br^2}}\left(\mathrm{e}^{\left(-d\left(m_1+\frac{1}{2}\right)+1\right)H_1} + H_2^{\left(d\left(m_1+\frac{1}{2}\right)-1\right)}\right)}{r\left(r^4 - \frac{(d^2-4)^2}{16B^2}\right)^{\frac{1}{4}}},\\
\Phi_4 &= \frac{\sqrt{1+\sqrt{1+Br^2}}\left(\mathrm{e}^{\left(d\left(m_2-\frac{1}{2}\right)+1\right)H_1} + H_2^{\left(-d\left(m_2-\frac{1}{2}\right)-1\right)}\right)}{r\left(r^4 - \frac{(d^2-4)^2}{16B^2}\right)^{\frac{1}{4}}},
\end{aligned} \tag{5.1}$$

where

$$H_1(r) = \frac{\arcsin\left(\frac{\sqrt{d^2-8}}{\sqrt{4Br^2+d^2-4}}\right)}{4\sqrt{d^2-8}},$$

$$H_2(r) = \left(\frac{2\sqrt{1+Br^2}-d}{2\sqrt{1+Br^2}+2}\right)^{\frac{1}{2d}}$$

and $J_\pm$ *are given by* (3.9). *Moreover, the spinors* Ψ *are singular at* $r_0 = \sqrt{\frac{d^2-4}{4B}}$.

Proof Since $d \geq 3$, we have $\eta = 0$. Thus the radial equations (3.4) with $\lambda = 0$ reduces to

$$\begin{aligned}
\left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{d\left(m_1+\frac{1}{2}\right)-1}{r\sqrt{1+Br^2}f^2}\right)X_1 &= 0,\\
\left(\frac{\mathrm{d}}{\mathrm{d}r} - \frac{d\left(m_2+\frac{1}{2}\right)+1}{r\sqrt{1+Br^2}f^2}\right)X_2 &= 0,\\
\left(\frac{\mathrm{d}}{\mathrm{d}r} - \frac{d\left(m_1+\frac{1}{2}\right)-1}{r\sqrt{1+Br^2}f^2} + \frac{1}{r\sqrt{1+Br^2}}\right)X_3 &= 0,\\
\left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{d\left(m_2+\frac{1}{2}\right)+1}{r\sqrt{1+Br^2}f^2} + \frac{1}{r\sqrt{1+Br^2}}\right)X_4 &= 0.
\end{aligned} \tag{5.2}$$

Solving these ODEs, we obtain (5.1).

In the following we assume the potential function f of metric (1.1) is given by (1.12). We will solve the isotropic eigenspinors under condition (1.11). Let

$$X_\pm = \left(\left(\frac{\sqrt{1+Br^2}+\frac{3}{2}}{\sqrt{1+Br^2}-\frac{3}{2}}\right)^{\frac{1}{12}}\mathrm{e}^{-\frac{1}{2}\arctan(2\sqrt{1+Br^2})}\right)U_\pm. \tag{5.3}$$

Then the radial equations (3.4) become

$$\frac{\mathrm{d}}{\mathrm{d}r}\begin{pmatrix} U_+ \\ U_- \end{pmatrix} = \frac{1}{r\sqrt{1+Br^2}f}\begin{pmatrix} 0 & \mp\frac{\mathrm{i}\sqrt{B}r}{2} \\ \pm\frac{\mathrm{i}\sqrt{B}r}{2} & \frac{1-f^2}{f} \end{pmatrix}\begin{pmatrix} U_+ \\ U_- \end{pmatrix}. \tag{5.4}$$

Let

$$x = \sqrt{1+Br^2}, \quad x \geq \frac{3}{2}, \tag{5.5}$$

then

$$0 < \arctan\frac{1}{2x} \leq \arctan\frac{1}{3}.$$

Throughout the section, we fix branches of the following complex logarithmic functions in terms of integers k_1, k_2 and k_3,

$$\begin{aligned} \ln(2x+\mathrm{i}) &= \ln\sqrt{4x^2+1} + \mathrm{i}\arctan\frac{1}{2x} + 2\mathrm{i}k_1\pi, \\ \ln(2x-\mathrm{i}) &= \ln\sqrt{4x^2+1} - \mathrm{i}\arctan\frac{1}{2x} + 2\mathrm{i}k_2\pi, \\ \ln x &= \ln|x| + 2\mathrm{i}k_3\pi. \end{aligned} \tag{5.6}$$

Then the following functions of complex exponentiation appeared in the section are given as

$$\begin{aligned} (2x+\mathrm{i})^{a+b\mathrm{i}} &= \mathrm{e}^{a\ln\sqrt{4x^2+1} - b\arctan\frac{1}{2x} - 2bk_1\pi}\mathrm{e}^{\mathrm{i}\left(b\ln\sqrt{4x^2+1} + a\arctan\frac{1}{2x} + 2ak_1\pi\right)}, \\ (2x-\mathrm{i})^{a+b\mathrm{i}} &= \mathrm{e}^{a\ln\sqrt{4x^2+1} + b\arctan\frac{1}{2x} - 2bk_2\pi}\mathrm{e}^{\mathrm{i}\left(b\ln\sqrt{4x^2+1} - a\arctan\frac{1}{2x} + 2ak_2\pi\right)}, \\ (2x-\mathrm{i})^{-\frac{1}{3}-\frac{\mathrm{i}}{4}}(2x+\mathrm{i})^{\frac{1}{6}+\frac{\mathrm{i}}{4}} &= (4x^2+1)^{-\frac{1}{6}}\mathrm{e}^{\frac{(k_2-k_1)\pi}{2}}\mathrm{e}^{\mathrm{i}\left(\frac{1}{2}\arctan\frac{1}{2x} + \frac{k_1\pi}{3} - \frac{2k_2\pi}{3}\right)}, \\ x^{\frac{1}{3}+\frac{\mathrm{i}}{2}} &= \sqrt[3]{x}\mathrm{e}^{-k_3\pi}\mathrm{e}^{\mathrm{i}\left(\frac{1}{2}\ln|x| + \frac{2k_3\pi}{3}\right)}, \end{aligned}$$

where a, b are real numbers.

Using (5.5), we can reduce (5.4) to the following equation

$$U_+''(x) + \frac{48x^4+16x^3-80x^2-16x+7}{(x+1)(2x-3)(2x+3)(4x^2+1)}U_+'(x) - \frac{4(x^2-1)}{(4x^2+1)(4x^2-9)}U_+(x) = 0. \tag{5.7}$$

Theorem 5.2 *On metrics* (1.1) *with constant negative scalar curvature where the potential f is given by* (1.12), *isotropic eigenspinors* Ψ *taking the form* (1.8) *satisfying* (1.9) *and* (1.11) *can be solved explicitly as follows*

$$\begin{aligned}\Phi_+ =& \Big(\frac{\big(x-\frac{3}{2}\big)^{\frac{1}{3}}x^{\frac{1}{3}+\frac{\mathrm{i}}{2}}\mathrm{e}^{-\frac{1}{2}\arctan 2x}}{\sqrt{1+x}\big(x+\frac{3}{2}\big)^{\frac{1}{6}}(2x+\mathrm{i})^{\frac{1}{4}}(2x-\mathrm{i})^{\frac{3}{4}+\frac{\mathrm{i}}{2}}}\Big)\Big(F\Big(-\frac{1}{6}-\frac{\mathrm{i}}{4},\frac{1}{3}-\frac{\mathrm{i}}{4};\frac{2}{3};\frac{4x^2-9}{40x^2}\Big)\\ &+\Big(\frac{1}{16}-\frac{3\mathrm{i}}{16}\Big)\Big(1+\frac{3}{2x}\Big)F\Big(\frac{5}{6}-\frac{\mathrm{i}}{4},\frac{1}{3}-\frac{\mathrm{i}}{4};\frac{5}{3};\frac{4x^2-9}{40x^2}\Big)\Big),\\ \Phi_- =& h_1h_2F\Big(-\frac{1}{6}-\frac{\mathrm{i}}{4},\frac{1}{3}-\frac{\mathrm{i}}{4};\frac{2}{3};\frac{4x^2-9}{40x^2}\Big)\\ &+\Big(\Big(\frac{1}{16}-\frac{3\mathrm{i}}{16}\Big)\Big(1+\frac{3}{2x}\Big)h_1h_2+\Big(x-\frac{3}{2}\Big)^{\frac{5}{6}}h_2h_3\Big)F\Big(\frac{5}{6}-\frac{\mathrm{i}}{4},\frac{1}{3}-\frac{\mathrm{i}}{4};\frac{5}{3};\frac{4x^2-9}{40x^2}\Big)\\ &+\Big(-\frac{57}{5120}-\frac{81\mathrm{i}}{5120}\Big)\Big(\frac{1}{x^3}+\frac{3}{2x^4}\Big)\Big(x-\frac{3}{2}\Big)^{\frac{5}{6}}h_2F\Big(\frac{11}{6}-\frac{\mathrm{i}}{4},\frac{4}{3}-\frac{\mathrm{i}}{4};\frac{8}{3};\frac{4x^2-9}{40x^2}\Big),\end{aligned}\tag{5.8}$$

where F is hypergeometric function (cf. Appendix A), *x is given by* (5.5) *satisfying* (5.6),

$$\begin{aligned}h_1 &= \frac{2}{3}\Big(x-\frac{3}{2}\Big)^{-\frac{1}{6}}+\Big(-\frac{2}{3}-\frac{\mathrm{i}}{2}\Big)\frac{\big(x-\frac{3}{2}\big)^{\frac{5}{6}}}{2x-\mathrm{i}}+\Big(\frac{1}{3}+\frac{\mathrm{i}}{2}\Big)\frac{\big(x-\frac{3}{2}\big)^{\frac{5}{6}}}{2x+\mathrm{i}},\\ h_2 &= \pm\frac{\mathrm{i}\big(x+\frac{3}{2}\big)^{\frac{1}{3}}(2x+\mathrm{i})^{\frac{1}{4}}x^{\frac{1}{3}+\frac{\mathrm{i}}{2}}\mathrm{e}^{-\frac{1}{2}\arctan 2x}}{\sqrt{B}r\sqrt{1+x}(2x-\mathrm{i})^{\frac{1}{4}+\frac{\mathrm{i}}{2}}},\\ h_3 &= \frac{\frac{-3+9\mathrm{i}}{8}x^2+\frac{9+33\mathrm{i}}{96}x+\frac{5+15\mathrm{i}}{64}}{(4x^2+1)x^2}\end{aligned}$$

and

$$J_+ = \Big(\sin\frac{\theta}{2}\Big)^{n-\frac{1}{4}}\Big(\cos\frac{\theta}{2}\Big)^{-n-\frac{5}{4}},\quad J_- = \Big(\sin\frac{\theta}{2}\Big)^{-n-\frac{5}{4}}\Big(\cos\frac{\theta}{2}\Big)^{n-\frac{1}{4}}.\tag{5.9}$$

Moreover, the eigenspinors Ψ are singular at $r=\sqrt{\frac{5}{4B}}$, x_1x_2-plane and x_3x_4-plane.

Proof Using the homotopic transformation of dependent variable

$$U_+ = \Big(x-\frac{3}{2}\Big)^{\frac{2}{3}}(2x-\mathrm{i})^{-\frac{1}{3}-\frac{\mathrm{i}}{4}}(2x+\mathrm{i})^{\frac{1}{6}+\frac{\mathrm{i}}{4}}u\tag{5.10}$$

and the transformation of independent variable

$$y=\frac{40x^2}{9(4x^2+1)},\quad y\geq 1,$$

we reduce (5.7) to

$$u''(y)+\frac{I_1}{I}u'(y)+\frac{I_2}{24(10-9y)I}u(y)=0,\tag{5.11}$$

where

$$
\begin{aligned}
I &= 6y(y-1)(10-9y)(3\sqrt{y}+2\sqrt{10-9y}),\\
I_1 &= (60+180\mathrm{i})\sqrt{y}-60\sqrt{10-9y}+(263+27\mathrm{i})\sqrt{10-9y}y+(-180-27\mathrm{i})\sqrt{10-9y}y^2\\
&\quad+(237-342\mathrm{i})\sqrt{10-9y}y^{\frac{3}{2}}+(-270+162\mathrm{i})y^{\frac{5}{2}},\\
I_2 &= 300(64-57\mathrm{i})\sqrt{y}+100(59-27\mathrm{i})\sqrt{10-9y}+90(-91+3\mathrm{i})\sqrt{10-9y}y\\
&\quad+1350(-25+21\mathrm{i})y^{\frac{3}{2}}+648(4+3\mathrm{i})\sqrt{10-9y}y^2+243(61-48\mathrm{i})y^{\frac{5}{2}}.
\end{aligned}
$$

Now we seek the solutions of (5.11) which have the following form

$$u = w_1 + s(y)w_2 \tag{5.12}$$

with

$$w_1 = y^{-\alpha}F\Big(\alpha, 1+\alpha-\gamma; 1+\alpha+\beta-\gamma; 1-\frac{1}{y}\Big)$$

and

$$w_2 = y^{-(\alpha+1)}F\Big(\alpha+1, 1+\alpha-\gamma; 2+\alpha+\beta-\gamma; 1-\frac{1}{y}\Big),$$

where α, β and γ are complex number to be determined in the following.

By (A.3), we know that w_1 satisfies

$$w_1'' + \frac{(\alpha+\beta+1)y-\gamma}{y(y-1)}w_1' + \frac{\alpha\beta}{y(y-1)}w_1 = 0, \tag{5.13}$$

and w_2 satisfies

$$w_2'' + \Big(\frac{(\alpha+\beta+3)-(\gamma+1)}{y(y-1)}\Big)w_2' + \Big(\frac{\alpha+\beta+\alpha\beta+1}{y(y-1)}\Big)w_2 = 0. \tag{5.14}$$

By (A.4), we get

$$\frac{\mathrm{d}}{\mathrm{d}y}w_1 = -\frac{\alpha\beta}{1+\alpha+\beta-\gamma}w_2. \tag{5.15}$$

Substituting (5.15) into (5.13), we obtain

$$w_1 = \frac{y(y-1)}{1+\alpha+\beta-\gamma}\frac{\mathrm{d}}{\mathrm{d}y}w_2 + \frac{(\alpha+\beta+1)y-\gamma}{1+\alpha+\beta-\gamma}w_2. \tag{5.16}$$

Substituting (5.12) into (5.11), and using (5.15) and (5.16), we obtain

$$
\begin{aligned}
&w_2'' + \frac{1}{s}\Big(2s' + ps - \frac{\alpha\beta}{1+\alpha+\beta-\gamma} + \frac{y(y-1)q}{1+\alpha+\beta-\gamma}\Big)w_2'\\
&+\frac{1}{s}\Big(s'' + p\Big(s' - \frac{\alpha\beta}{1+\alpha+\beta-\gamma}\Big) + q\Big(s + \frac{(\alpha+\beta+1)y-\gamma}{1+\alpha+\beta-\gamma}\Big)\Big)w_2 = 0.
\end{aligned}
\tag{5.17}
$$

Since w_2 satisfies (5.14) and (5.17), we obtain

$$2s' + \Big(p - \frac{(\alpha+\beta+3)-(\gamma+1)}{y(y-1)}\Big)s - \frac{\alpha\beta}{1+\alpha+\beta-\gamma} + \frac{y(y-1)q}{1+\alpha+\beta-\gamma} = 0 \tag{5.18}$$

and

$$s'' + ps' + \Big(q - \frac{\alpha+\beta+\alpha\beta+1}{y(y-1)}\Big)s + \frac{(\alpha+\beta+1)y-\gamma}{1+\alpha+\beta-\gamma}q - \frac{\alpha\beta}{1+\alpha+\beta-\gamma}p = 0. \tag{5.19}$$

Therefore, we obtain

$$s(y) = \frac{1}{1+\alpha+\beta-\gamma}\frac{M_1}{M_2}, \tag{5.20}$$

where

$$\begin{aligned} M_1 &= \frac{2\big(\frac{\alpha\beta I_1}{I} - \frac{((\alpha+\beta+1)y-\gamma)I_2}{24(10-9y)I}\big) + \frac{\mathrm{d}}{\mathrm{d}y}\big(\frac{y(y-1)I_2}{24(10-9y)I}\big)}{\frac{I_1}{I} + \frac{(\alpha+\beta+3)y-(\gamma+1)}{y(y-1)}} + \frac{1}{2}\Big(\frac{y(y-1)I_2}{24(10-9y)I} - \alpha\beta\Big), \\ M_2 &= \frac{2\big(\frac{I_2}{24(10-9y)I} - \frac{\alpha\beta+\alpha+\beta+1}{y(y-1)}\big) - \frac{\mathrm{d}}{\mathrm{d}y}\big(\frac{I_1}{I} - \frac{(\alpha+\beta+3)y-(\gamma+1)}{y(y-1)}\big)}{\frac{I_1}{I} + \frac{(\alpha+\beta+3)y-(\gamma+1)}{y(y-1)}} \\ &\quad - \frac{I_1}{2I} + \frac{(\alpha+\beta+3)y-(\gamma+1)}{2y(y-1)}. \end{aligned}$$

Substituting (5.20) into (5.18), with the help of Mathematica, we find that

$$\alpha = -\frac{1}{6} - \frac{\mathrm{i}}{4}, \quad \beta = \frac{1}{3} + \frac{\mathrm{i}}{4}, \quad \gamma = \frac{1}{2}. \tag{5.21}$$

Substituting (5.21) into (5.20), we obtain

$$s(y) = \frac{1-3\mathrm{i}}{16}(\sqrt{y}\sqrt{10-9y} + y). \tag{5.22}$$

Thus, we obtain, for $r \geq \sqrt{\frac{5}{4B}}$ or $x \geq \frac{3}{2}$,

$$\begin{aligned} U_+ = \Big(\Big(x-\frac{3}{2}\Big)^{\frac{2}{3}}(2x-\mathrm{i})^{-\frac{1}{2}-\frac{\mathrm{i}}{2}}x^{\frac{1}{3}+\frac{\mathrm{i}}{2}}\Big)\Big(F\Big(-\frac{1}{6}-\frac{\mathrm{i}}{4}, \frac{1}{3}-\frac{\mathrm{i}}{4}; \frac{2}{3}; \frac{4x^2-9}{40x^2}\Big) \\ + \Big(\frac{1}{16}-\frac{3\mathrm{i}}{16}\Big)\Big(1+\frac{3}{2x}\Big)F\Big(\frac{5}{6}-\frac{\mathrm{i}}{4}, \frac{1}{3}-\frac{\mathrm{i}}{4}; \frac{5}{3}; \frac{4x^2-9}{40x^2}\Big)\Big) \end{aligned} \tag{5.23}$$

and

$$U_- = \pm\frac{2\mathrm{i}\sqrt{B^2r^4 - \frac{25}{16}}}{\sqrt{B}r}\frac{\mathrm{d}U_+}{\mathrm{d}x}.$$

Finally, we obtain

$$\Phi_\pm = \frac{\sqrt{2B}\mathrm{e}^{-\frac{1}{2}\arctan 2x}}{\sqrt{1+x}\big(x-\frac{3}{2}\big)^{\frac{1}{3}}\big(x+\frac{3}{2}\big)^{\frac{1}{6}}(4x^2+1)^{\frac{1}{4}}}U_\pm.$$

(5.9) for $J_{\pm}(\theta)$ can be derived from (3.9).

Since h_1 is singular at $r = \sqrt{\frac{5}{4B}}$, we find that Φ_- is singular at $r = \sqrt{\frac{5}{4B}}$. On the other hand, the polar coordinate $\{r, \theta, \phi, \psi\}$ could be transfer to Cartesian coordinate $\{x_1, x_2, x_3, x_4\}$ by, cf. [17],

$$\begin{aligned} x_1 &= r\cos\frac{\theta}{2}\cos\frac{\psi+\phi}{2}, \quad x_2 = r\cos\frac{\theta}{2}\sin\frac{\psi+\phi}{2}, \\ x_3 &= r\sin\frac{\theta}{2}\cos\frac{\psi-\phi}{2}, \quad x_4 = r\sin\frac{\theta}{2}\sin\frac{\psi-\phi}{2}. \end{aligned}$$

It is straightforward that

$$\cos\frac{\theta}{2} = 0 \Longleftrightarrow x_1 = x_2 = 0, \quad \sin\frac{\theta}{2} = 0 \Longleftrightarrow x_3 = x_4 = 0.$$

So $J_{\pm}(\theta)$ are singular at x_1x_2-plane and x_3x_4-plane. Thus the proof of the theorem is complete. Q.E.D.

A Hypergeometric Function

In this appendix, we provide a short introduction to hypergeometric function (cf. [12] for details). The hypergeometric function is defined as

$$F(\alpha, \beta; \gamma; z) = \sum_{n=0}^{\infty} \frac{(\alpha, n)(\beta, n)}{(\gamma, n)} \frac{z^n}{n!}, \quad |z| < 1, \tag{A.1}$$

where α, β, γ are arbitrary complex number and γ is neither zero nor a negative integer, (,) is the Pochhammer symbol. Such a series in (A.1) are absolutely convergent if

$$\Re(\gamma - \alpha - \beta) > 0$$

and F satisfies the following hypergeometric equation

$$u''(z) + \frac{(\alpha+\beta+1)z - \gamma}{z(z-1)}u'(z) + \frac{\alpha\beta}{z(z-1)}u(z) = 0. \tag{A.2}$$

There are 24 different hypergeometric functions solving (A.2) with different domain of convergence respectively. For example, if $\gamma - \alpha - \beta$ is not integer, (A.2) has a solution

$$u = z^{-\alpha} F\Big(\alpha, 1+\alpha-\gamma; 1+\alpha+\beta-\gamma; 1-\frac{1}{z}\Big), \quad \Re(z) \geq \frac{1}{2}. \tag{A.3}$$

The following proposition is not found in some references, and we provide the proof here.

Proposition A.1 *Denote* $w = z^{-\alpha} F\big(\alpha, \beta; \gamma; 1 - \frac{1}{z}\big)$*. Then* w *satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}z} w = -\frac{\alpha(\gamma - \beta)}{\gamma} z^{-(\alpha+1)} F\Big(\alpha + 1, \beta; \gamma + 1; 1 - \frac{1}{z}\Big). \tag{A.4}$$

Proof Since [12],

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}z} F(\alpha, \beta; \gamma; z) &= \frac{\alpha\beta}{\gamma} F(\alpha + 1, \beta + 1; \gamma + 1; z), \\
\frac{\beta}{z} F\Big(\alpha + 1, \beta + 1; \gamma + 1; 1 - \frac{1}{z}\Big) &= \frac{\gamma}{z - 1} F\Big(\alpha + 1, \beta; \gamma; 1 - \frac{1}{z}\Big) \\
&\quad - \frac{\gamma}{z - 1} F\Big(\alpha, \beta; \gamma; 1 - \frac{1}{z}\Big), \\
\frac{(\gamma - \beta)(z - 1)}{\gamma} F\Big(\alpha + 1, \beta; \gamma + 1; 1 - \frac{1}{z}\Big) &= z F\Big(\alpha, \beta; \gamma; 1 - \frac{1}{z}\Big) \\
&\quad - F\Big(\alpha + 1, \beta; \gamma; 1 - \frac{1}{z}\Big),
\end{aligned}$$

we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}z} w &= -\frac{\alpha}{\gamma z^{\alpha+1}} \Big(\gamma F\Big(\alpha, \beta; \gamma; 1 - \frac{1}{z}\Big) - \frac{\beta}{z} F\Big(\alpha + 1, \beta + 1; \gamma + 1; 1 - \frac{1}{z}\Big)\Big) \\
&= -\frac{\alpha}{(z - 1) z^{\alpha+1}} \Big(z F\Big(\alpha, \beta; \gamma; 1 - \frac{1}{z}\Big) - F\Big(\alpha + 1, \beta; \gamma; 1 - \frac{1}{z}\Big)\Big) \\
&= -\frac{\alpha(\gamma - \beta)}{\gamma} z^{-(\alpha+1)} F\Big(\alpha + 1, \beta; \gamma + 1; 1 - \frac{1}{z}\Big).
\end{aligned}$$

Thus the proof of the proposition is complete.

Declarations

Conflicts of interest The authors declare no conflicts of interest.

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