Chin. Ann. Math. Ser. B
44(6), 2023, 913–928
DOI: 10.1007/s11401-023-0051-8

Chinese Annals of Mathematics, Series B

Harmonic Measures and Numerical Computation of Cauchy Problems for Laplace Equations*

Yu CHEN[1] Jin CHENG[2] Shuai LU[3] Masahiro YAMAMOTO[4]

Abstract It is well known that the Cauchy problem for Laplace equations is an ill-posed problem in Hadamard's sense. Small deviations in Cauchy data may lead to large errors in the solutions. It is observed that if a bound is imposed on the solution, there exists a conditional stability estimate. This gives a reasonable way to construct stable algorithms. However, it is impossible to have good results at all points in the domain. Although numerical methods for Cauchy problems for Laplace equations have been widely studied for quite a long time, there are still some unclear points, for example, how to evaluate the numerical solutions, which means whether they can approximate the Cauchy data well and keep the bound of the solution, and at which points the numerical results are reliable? In this paper, the authors will prove the conditional stability estimate which is quantitatively related to harmonic measures. The harmonic measure can be used as an indicate function to pointwisely evaluate the numerical result, which further enables us to find a reliable subdomain where the local convergence rate is higher than a certain order.

Keywords Conditional stability, Cauchy problem, Laplace equation, Indicate function
2000 MR Subject Classification 65N21, 35J05

1 Introduction

The Cauchy problem for the Laplace equation is a classical problem and has a long history (e.g., [2]). The study of Cauchy problem is of fundamental significance both theoretically and practically (see [22]). However, the numerical treatment is usually challenging, caused by the well-known ill-posedness in Hadamard's sense (see [10]). Small changes in Cauchy data may lead to large deviations in the solution due to the instability of the problem.

The stability may be restored by introducing some conditions on the solutions. It is observed that, if a bound is imposed on the solution, then we can prove a conditional stability estimate assuming a priori boundness condition on solutions. This will give a reasonable way

Manuscript received May 23, 2023. Revised August 9, 2023.
[1]School of Mathematics, Shanghai University of Finance and Economics, Shanghai 200433, China.
E-mail: yuchen@sufe.edu.cn
[2]Corresponding author. School of Mathematical Sciences, Fudan University, Shanghai 200433, China.
E-mail: jcheng@fudan.edu.cn
[3]School of Mathematical Sciences, Fudan University, Shanghai 200433, China.
E-mail: slu@fudan.edu.cn
[4]School of Mathematical Sciences, the University of Tokyo, Tokyo 153, Japan.
E-mail: myama@next.odn.ne.jp
*This work was suported by the National Natural Science Foundation of China (Nos. 11971121, 12201386, 12241103) and Grant-in-Aid for Scientific Research (A) 20H00117 of Japan Society for the Promotion of Science.

to construct a stable algorithm for solving the Cauchy problem for the Laplace equation by Tikhonov regularization. The conditional stability estimates imply the convergence rate of the regularized solution (see [7]). However, it is impossible to have reasonably accurate results everywhere over the domain, even if we can approximate the Cauchy data well and keep the bound of the solution. Then there raises the issue at which points the numerical results are reliable, i.e., how to evaluate the numerical solutions. This is crucial for real applications such as remote measurement problems and design problems. Theoretically, by a Carleman estimate, usually a qualitative conditional stability can be obtained (e.g., [13]), whereas a classical quantitative estimate in the three-circle form needs to be adapted to general geometries. In some works involving pointwise estimate, it is not direct to obtain the stability index function (e.g., [22]). In real applications, it would be very useful if a pointwise estimate like

$$|u(x)| \leq C\varepsilon^{\tau(x)}$$

is available, where ε is the observation error in a certain norm, and $\tau(x)$ is a function that is convenient to evaluate, because one can evaluate where the reconstruction is reliable based on the convergence rate $\tau(x)$.

Various numerical algorithms have been developed to deal with the Cauchy problems. For example, the methods through Maz'ya iterative algorithm (see [17]) based on weak form (see [14]) and regularized boundary element method (BEM for short) (e.g., [12, 24]), the moment method (see [5]), the methods through solving optimal control problem based on finite element method (see [3–4]), and a more common approach by Tikhonov regularization involving modifications of the operators of the problem (e.g., [19]). Besides the convergence and stability of used methods, relatively less studied is the evaluation of the reconstructed solution.

In this paper, we will discuss the Cauchy problem for the Laplace equation and prove the conditional stability estimate, in which the order function can be given in the form of the harmonic measure. The explicit expression of the order function in stability estimate can be used for estimation of discretized solutions to the Cauchy problem. In [21], general treatments are described for such estimation for discretized Tikhonov regularized solutions, and we discuss more details limited to the Cauchy problem. By such an indicate function, when the reconstruction domain and the part of boundary with Cauchy data are given, we can propose a trustable sub-domain, in which the numerical solutions can have order of convergence rate greater than $\frac{1}{2}$ for example.

This paper is organized as follows: We will formulate the problem and discuss the conditional stability of the problem in Section 2. The numerical scheme and the related analysis are presented in Section 3, and error estimates are proved for discretised regularization scheme in Section 4. In Section 5, some examples are given to illustrate the numerical method. We present some remarks and conclusions finally in Section 6.

2 Conditional Stability

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary $\partial\Omega$. We consider the following Cauchy problem,

$$\begin{aligned} \Delta u &= 0 \quad \text{in } \Omega, \\ u &= f \quad \text{on } \Gamma \subset \partial\Omega, \\ \partial_\nu u &= g \quad \text{on } \Gamma \subset \partial\Omega, \end{aligned}$$

where $\Gamma \subset \partial\Omega$ is an open subset of the boundary $\partial\Omega$, ν denotes the outer normal vector to $\partial\Omega$ and $\partial_\nu u := \nabla u \cdot \nu$. We usually assume that the mathematical models should well describe the real problems, which implies the existence of solutions, but measurement errors may disturb stable construction of approximatng solutions.

In general, the Cauchy problem is unstable and causes difficulties in numerical treatments. For example, we consider the following example (see [11]),

$$\Phi_n(x_1, x_2) = \frac{1}{n}(\sin nx_1)\exp(nx_2),$$

which is harmonic in $x := (x_1, x_2) \in \mathbb{R}^2$. When $\Gamma = \{x_2 = b\}$ with some $b < 0$, the Cauchy data will be small but the solutions increase drastically as x_2 increases. This illustrates that small errors in data may probably be enlarged for the numerical solutions.

Then one turns to seek conditional stability results. If we can prove conditional stability, then we can construct stable algorithms. The conditional stability estimates imply the convergence rate of the regularized solution (see [7]). We understand conditional stability as follows.

Definition 2.1 (Conditional stability [7]) *Let K be a densely defined injective operator from a Banach space X to a Banach space Y, and $\omega : \{\xi \geq 0\} \to \{\xi \geq 0\}$ is a monotone increasing continuous function satisfying $\omega(0) = 0$. Moreover $Z \subset X$ is assumed to be continuously embedded in X and $Q \subset Z$. Then we say that in the operator equation $Kf = g$, the conditional stability holds if, for any given $M > 0$, there exists a constant $C = C(M) > 0$ such that*

$$\|f_1 - f_2\|_X \leq C(M)\omega(\|K(f_1) - K(f_2)\|_Y)$$

for all $f_1, f_2 \in \mathcal{U}_M \cap Q$. Here we set $\mathcal{U}_M = \{f \in Z; \|f\|_Z \leq M\}$.

Here we call ω the modulus of the conditional stability under consideration.

The following harmonic measure will be used to specify stability moduli for the Cauchy problem.

Definition 2.2 (Harmonic measure [8]) *Let $U \subset \mathbb{C}$ be a simply connected domain with piecewise regular boundary, and ℓ be a nonempty open subset of ∂U. We call $\mu(\zeta)$ the harmonic measure for U and ℓ, if*

$$\begin{aligned} \Delta\mu(\zeta) &= 0, \quad \zeta \in U, \\ \mu(\zeta) &= 0, \quad \zeta \in \partial U \backslash \overline{\ell}, \\ \mu(\zeta) &= 1, \quad \zeta \in \overline{\ell}. \end{aligned}$$

For the details of harmonic measure, we refer to [8, 16] for example.

For a holomophic function $w(z)$, $z := x_1 + \sqrt{-1}x_2 \in \mathbb{C}$ with $x_1, x_2 \in \mathbb{R}$, the following is known (e.g., [2, 6]).

Lemma 2.1 *If $w(z)$ is holomorphic in Ω and continuous on $\overline{\Omega}$, and*

$$\begin{aligned} |w(z)| &\le \varepsilon, \quad \forall z \in \Gamma, \\ |w(z)| &\le M_1, \quad \forall z \in \Omega, \end{aligned}$$

where $\varepsilon \le M_1$, then

$$|w(z)| \le M_1 \Big(\frac{\varepsilon}{M_1}\Big)^{\tau(z)},$$

where τ is the harmonic measure for Ω and Γ.

Proof For w, we can construct a subharmonic function F:

$$F(z) = \frac{\ln\big(\frac{|w(z)|}{M_1}\big)}{\ln\big(\frac{\varepsilon}{M_1}\big)}.$$

Then

$$F|_\Omega \ge 0, \quad F|_\Gamma \ge 1.$$

The harmonic measure $\tau(z)$ satisfies $\tau|_\Gamma = 1, \tau|_{\partial\Omega\backslash\Gamma} = 0$ and $\tau(z)$ is harmonic in Ω. Then it holds that

$$\tau(z) \le F(z),$$

which leads to the conclusion that

$$|w(z)| \le M_1^{1-\tau(z)} \varepsilon^{\tau(z)}.$$

The following example illustrates that the estimate for $w(z), z \in \mathbb{C}$ is sharp.

Example 2.1 Suppose that $\Omega = \{z;\, 1 \le |z| \le R\}$ and $\Gamma = \{z;\, |z| = 1\}$. In Lemma 2.1, we consider

$$w(z) = \varepsilon z^n, \quad n \in \mathbb{N}.$$

It holds that

$$\Big|\frac{\omega(z)}{M}\Big| = \Big(\frac{\varepsilon}{M}\Big)^{\tau(z)}, \quad M = |w(z)|_{|z|=R},$$

which means that the conclusion of Lemma 2.1 is the best possible for these $w(z)$.

Theorem 2.1 (Conditional stability) *Let Ω be a simply connected domain in $\mathbb{R}^2$ and let Γ be a non-empty open subset of $\partial\Omega$. Suppose that $u(x)$ satisfies*

$$\begin{aligned}\Delta u(x) &= 0, \quad x \in \Omega,\\ u(x) &= f(x), \quad x \in \Gamma,\\ \partial_\nu u(x) &= g(x), \quad x \in \Gamma.\end{aligned}$$

If $\|u\|_{C^1(\overline{\Omega})} \le M$ with arbitrarily given constant $M > 0$, then we have

$$|u(x)| \le C(M,\Omega)\varepsilon^{\tau(x)} \quad \textit{for } x \in \Omega, \tag{2.1}$$

where

$$\varepsilon = \|f\|_{W^{1,\infty}(\Gamma)} + \|g\|_{L^\infty(\Gamma)}$$

and $\tau(x)$ is the harmonic measure with respect to Γ and Ω.

Proof We define an analytic function $w(z)$ as

$$w(z) = \frac{\partial u}{\partial \overline{z}} = \frac{1}{2}\Big(\frac{\partial u}{\partial x_1} + \mathrm{i}\frac{\partial u}{\partial x_2}\Big),$$

which is holomorphic in Ω.

Since $\|u\|_{C^1(\overline{\Omega})} \le M$, one has $\|w\|_{L^\infty(\Omega)} \le CM$. Since

$$\|w(z)\|_{L^\infty(\Gamma)} \le C\varepsilon,$$

Lemma 2.1 yields

$$|w(z)| \le CM^{1-\tau(z)}\varepsilon^{\tau(z)}.$$

For $x \in \Omega$, let L denote a path connecting x and $x_0 \in \Gamma$. Then

$$\begin{aligned}|u(x)| &= \Big|u(x_0) + \int_L \frac{\partial u}{\partial s}(s)\mathrm{d}\ell_s\Big|\\ &\le |u(x_0)| + \int_L |\nabla u|\mathrm{d}\ell_s \le |u(x_0)| + C\int_L \varepsilon^{\tau(s)}\mathrm{d}\ell_s, \quad x_0 \in \Gamma.\end{aligned}$$

We claim that there exists a path L from some $x_0 \in \Gamma$ to x along which τ monotonously decreases. Otherwise, the point x must be enclosed by a closed contour on which $\nabla\tau(x) = 0$, which implies that $\nabla\tau(x) \equiv 0$ in Ω, leading to a contradiction. Therefore, we have

$$|u(x)| \le |u(x_0)| + C\int_L \varepsilon^{\tau(x)}\mathrm{d}\ell_s \le C\varepsilon^{\tau(x)},$$

which leads to (2.1).

In real applications, we often measure the observation errors by $L^2(\Omega)$-based norms. In that case, we can prove the following corollary.

Corollary 2.1 *Let Ω be a simply connected domain in $\mathbb{R}^2$ with piecewise smooth boundary and let Γ be an open subset of $\partial\Omega$. Suppose that $u(x)$ satisfies*

$$\begin{aligned} \Delta u(x) &= 0, \quad x \in \Omega, \\ u(x) &= f(x), \quad x \in \Gamma, \\ \partial_\nu u(x) &= g(x), \quad x \in \Gamma. \end{aligned}$$

If $\|u\|_{H^2(\partial\Omega)} \le M$, then

$$|u(x)| \le C(M, \Omega, \Gamma)\widetilde{\varepsilon}^{\tau(x)}, \quad x \in \Omega,$$

where $\widetilde{\varepsilon} = \varepsilon^{\tau_0}$, $\tau_0 > 0$, $\varepsilon = \|f\|_{H^1(\Gamma)} + \|g\|_{L^2(\Gamma)}$ and $\tau(x)$ denotes the harmonic measure with respect to Γ and Ω.

Proof Since $\|u\|_{H^1(\Gamma)} \le \varepsilon$ and $\|u\|_{H^2(\Gamma)} \le M$, by Sobolev interpolation (e.g., [13]), we obtain

$$\|u\|_{H^s(\Gamma)} \le C(\Gamma)\|u\|_{H^1(\Gamma)}^{\theta}\|u\|_{H^2(\Gamma)}^{1-\theta} = C(\Gamma, M)\varepsilon^{\theta}$$

with $s = \theta + 2(1-\theta) > \frac{3}{2}$ when $0 < \theta < \frac{1}{2}$.

The Sobolev embedding (e.g., [1]) implies

$$\|u\|_{W^{1,\infty}(\Gamma)} \le C\|u\|_{H^s(\Gamma)} \le C(M, \Gamma)\varepsilon^{\tau_0} =: C(M, \Gamma)\widetilde{\varepsilon}, \quad 0 < \tau_0 < \frac{1}{2}.$$

Then, by applying Theorem 2.1 we have

$$|u(x)| \le C(\Gamma, \Omega, M)\widetilde{\varepsilon}^{\tau(x)}, \quad x \in \Omega.$$

We remark that even if $u(x)$ is less regular in Ω, one can also have similar estimate in a subset of Ω whose regularity can be ensured due to the interior regularity of elliptic equations (e.g., [9]), which is standard and will not be shown here.

3 Numerical Method

The solution $u(x)$ to a Laplace equation in the domain Ω can be represented by Green's function as

$$u(x) = \int_{\partial\Omega} \frac{\partial G(x, \xi)}{\partial \nu} b(\xi) \mathrm{d}s_\xi,$$

where $b(x) = u(x)|_{\partial\Omega}$ is the boundary value function. The Green function G satisfies

$$\Delta_x G(x, y) = \delta(x - y), \quad x, y \in \Omega, \quad G(x, y) = 0, \quad x \in \partial\Omega, \; y \in \Omega.$$

Let $H(x,y)=\partial_\nu G(x,y)$ denote the Poisson kernel. Then

$$\begin{aligned}\int_{\partial\Omega} H(x,\xi)b(\xi)\mathrm{d}s_\xi &= f(x), \quad x\in\Gamma,\\ \int_{\partial\Omega} \nu_\Gamma\cdot\nabla H(x,\xi)b(\xi)\mathrm{d}s_\xi &= g(x), \quad x\in\Gamma.\end{aligned}$$

Thus, in solving the Cauchy problem, $u|_\Omega$ can be determined once the boundary value function $u|_{\partial\Omega}=b(x)$ is recovered from the integral functions.

For technical reasons, in computation we will reconstruct the harmonic function on a slightly larger domain than Ω by Runge's approximation, in order to meet the regularity requirements in Theorem 2.1. We take $\widetilde{\Omega}$ such that $\overline{\Omega}\subset\widetilde{\Omega}$. By Runge's approximation (see [20]), one can approximate the harmonic function $u(x)$ on Ω by a harmonic function $\widetilde{u}(x)$ in $\widetilde{\Omega}$, which will be illustrated in the following section. Let $\widetilde{G}(x,y)$ be the Green function corresponding to $\widetilde{\Omega}$, and denote $\widetilde{H}(x,y)=\partial_{\widetilde{\nu}}\widetilde{G}(x,y)$, where $\widetilde{\nu}$ is the outer normal vector to $\partial\widetilde{\Omega}$. Correspondingly, we denote

$$\begin{aligned}\widetilde{f}(x) &:= \widetilde{u}|_\Gamma = \int_{\partial\widetilde{\Omega}} \widetilde{H}(x,\xi)\widetilde{b}(\xi)\mathrm{d}s(\xi), \quad x\in\Gamma,\\ \widetilde{g}(x) &:= \partial_{\nu_\Gamma}\widetilde{u}|_\Gamma = \int_{\partial\widetilde{\Omega}} \nu_\Gamma\cdot\nabla\widetilde{H}(x,\xi)\widetilde{b}(\xi)\mathrm{d}s(\xi), \quad x\in\Gamma.\end{aligned}$$

Then we solve $\widetilde{b}$ from the measurements $f^\varepsilon, g^\varepsilon$ with noises, and reconstruct $\widetilde{u}(x)|_\Omega$.

Due to the ill-posedness of the problem, the Tikhonov regularization is introduced to weaken the instability induced by the observation error. According to the conditional stability discussed in Section 2, the regularized cost functional is defined as

$$J(\widetilde{b}) := \|\widetilde{f}(\widetilde{b})-f^\varepsilon\|^2_{L^2(\Gamma)} + \|\widetilde{g}(\widetilde{b})-g^\varepsilon\|^2_{L^2(\Gamma)} + \alpha\|\widetilde{b}\|^2_{L^2(\partial\widetilde{\Omega})}.$$

Then we obtain the solution which minimizes the cost functional.

To discretize the problem, suppose that $\widetilde{b}(x)$ can be approximated by

$$\widetilde{b}_n(x)=\sum_{i=1}^n b_i\varphi_i(x),$$

where, for $i=1,\cdots,n$, $\varphi_i(x)$ are basis functions defined on $\partial\widetilde{\Omega}$, and b_i are the corresponding components. The test space $V_n=\mathrm{span}\{\varphi_j\}_{j=1}^n$ is chosen such that $\bigcup\limits_{j=n+1}^{\infty} V_j$ is dense in $L^2(\partial\widetilde{\Omega})$. Let

$$w_i=\int_{\partial\widetilde{\Omega}} \frac{\partial\widetilde{G}}{\partial\nu}\varphi_i\mathrm{d}S, \quad i=1,\cdots,n$$

satisfy

$$\begin{cases}\Delta w_i=0, & x\in\widetilde{\Omega},\\ w_i|_{\partial\widetilde{\Omega}}=\varphi_i.\end{cases}$$

Then the harmonic function in $\widetilde{\Omega}$ can be approximated by

$$\widetilde{u}_n(x) = \sum_{i=1}^{n} b_i w_i(x).$$

In particular, let

$$w_i = \int_{\gamma_i} \frac{\partial G}{\partial \nu} \mathrm{d}s, \quad \bigcup_i \gamma_i = \partial\widetilde{\Omega}, \quad \gamma_i \bigcap_{i\neq j} \gamma_j = \emptyset.$$

The solution can be expressed by

$$\widetilde{u}_n(x) = \sum_{i=1}^{n} b_i \int_{\gamma_i} \frac{\partial \widetilde{G}}{\partial \nu}(x,s)\mathrm{d}s =: \sum_{i=1}^{n} b_i w_i(x),$$

then w_i satisfies

$$\begin{cases} \Delta w_i(x) = 0, & x \in \widetilde{\Omega}, \\ w_i = \chi(\Gamma_i), & x \in \partial\widetilde{\Omega}. \end{cases} \tag{3.1}$$

Notice that for $i = 1, \cdots, n$, the base solution w_i involves the singular integral and one can approximate it by solving (3.1) numerically, which are denoted by $w_i^h(x)$. Here we use h to mark the discrete precision in calculating $w_i(x)$. In this work, we choose the finite difference method (FDM for short) to numerically compute $w_i^h(x)$ with the grid length h. The 2nd order center difference discretization will be adopted.

Let $\mathcal{F}_{n,h}(\widetilde{\Omega}) = \operatorname*{span}_{1\leq i\leq n} w_i^h(x)$ be the space spanned by the base solutions. Then our problem is to find

$$u^* = \operatorname*{arg\,min}_{\widetilde{u}_n^h \in \mathcal{F}_{n,h}(\widetilde{\Omega})} \{\|\widetilde{f}_n^h - f^\delta\|^2_{H^1(\Gamma)} + \|\widetilde{g}_n^h - g^\delta\|^2_{L^2(\Gamma)} + \alpha\|\widetilde{u}_n^h\|^2_{H^2(\partial\Omega)}\}, \tag{3.2}$$

or alternatively,

$$\widetilde{b}_n^* = \operatorname*{arg\,min}_{\widetilde{b}_n \in V_n} \{\|\widetilde{f}_n^h(\widetilde{b}_n) - f^\delta\|^2_{H^1(\Gamma)} + \|\widetilde{g}_n^h(\widetilde{b}_n) - g^\delta\|^2_{L^2(\Gamma)} + \alpha\|\mathcal{H}\widetilde{b}_n\|^2_{H^2(\partial\Omega)}\}, \tag{3.3}$$

where $\widetilde{f}_n^h = \widetilde{u}_n^h|_\Gamma$, and $\widetilde{g}_n^h = \nu_\Gamma \cdot \nabla_h \widetilde{u}_n^h|_\Gamma$, ∇_h being the numerical gradient.

Then, $\mathcal{H}\widetilde{b}_n = \int_{\partial\widetilde{\Omega}} \widetilde{H}(x,\xi)\widetilde{b}_n(\xi)\mathrm{d}s$. According to the a priori choice strategy of the regularization parameter (see [7]), α is taken as $\alpha \sim \delta^2$.

We assume that the measurements are taken at $x_j \in \Gamma, j = 1, \cdots, m$ and let

$$A_{ji} := \widetilde{f}_i^h(x_j) = w_i^h(x_j), \quad B_{ji} := \widetilde{g}_i^h(x_j) = \partial_\nu w_i^h(x_j), \quad C_k := \|w_k^h(x)\|_{H^2(\partial\Omega)}.$$

Then we reach the fully discrete form of the regularization cost functional:

$$\widehat{J}(\boldsymbol{b}) := \sum_{j=1}^{m} \Big(\sum_{i=1}^{n} A_{ij} b_i - f_j^\varepsilon\Big)\sigma_j + \sum_{j=1}^{m} \Big(\sum_{i=1}^{n} B_{ij} b_i - g_j^\varepsilon\Big)\sigma_j + \alpha^2 \sum_{k=1}^{N} (C_k b_k)^2,$$

where σ_j denotes the jth curve length element on Γ and $\boldsymbol{b} = (b_1, \cdots, b_n) \in \mathbb{R}^n$. The minimization is a standard linear algebra problem.

4 Error Analysis

In the following, we assume that the exact solution u has enough regularity in Ω. Otherwise one can utilize the interior regularity and pay attention to the reconstruction on any subset whose closure is contained in Ω. The main result on pointwise evaluation of the reconstructed solution is as follows.

Theorem 4.1 (Evaluation on Ω) *Suppose that $\overline{\Omega} \subset \widetilde{\Omega}$ and u_0 is harmonic in Ω, and $\|u_0\|_{H^2(\partial\Omega)} \leq M$. Denote $f_0 = u_0|_\Gamma$ and $g_0 = \partial_\nu u_0|_\Gamma$. Let available data f^ε, g^ε satisfy*

$$\|f^\varepsilon - f_0\|_{H^1(\Gamma)} + \|g^\varepsilon - g_0\|_{L^2(\Gamma)} \leq \varepsilon.$$

Following the scheme presented in Section 3, let u^* be the following minimizer:

$$u^* = \underset{\widetilde{u}_n^h \in \mathcal{F}_n^h(\widetilde{\Omega})}{\arg\min} \{\|\widetilde{f}_n^h - f^\varepsilon\|^2_{H^1(\Gamma)} + \|\widetilde{g}_n^h - g^\varepsilon\|^2_{L^2(\Gamma)} + \alpha\|\widetilde{u}_n^h\|^2_{H^2(\partial\Omega)}\}. \tag{4.1}$$

Then we have the estimate for the Cauchy problem:

$$|u^*(x) - u_0(x)| \leq C(M, \Omega, \Gamma)\varepsilon^{\tau(x)}, \quad x \in \Omega, \tag{4.2}$$

provided that $\alpha \sim \varepsilon^2$, n are sufficiently large and h is sufficiently small. Here $\tau(x)$ is the harmonic measure with characteristic boundary Γ.

Lemma 4.1 *Suppose that $\widetilde{u}_n^h$ is the FDM approximations of a harmonic function $\widetilde{u}$ in $\widetilde{\Omega}$ with the scheme given in Section 3. We set $\widetilde{f} = \widetilde{u}|_\Gamma$, $\widetilde{g} = \partial_\nu \widetilde{u}|_\Gamma$, $\widetilde{f}_n^h = \widetilde{u}_n^h|_\Gamma$, $\widetilde{g}_n^h = \nu \cdot \nabla_h \widetilde{u}_n^h|_\Gamma$ where ∇_h is the gradient approximated by the 1st order difference. Then*

$$\|\widetilde{f}_n^h - \widetilde{f}\|_{H^1(\Gamma)} \leq C_1\delta(n) + C_2 h,$$
$$\|\widetilde{g}_n^h - \widetilde{g}\|_{L^2(\Gamma)} \leq C_3\delta(n) + C_4 h,$$

where $\delta(n) \to 0$ as $n \to \infty$ and C_1, C_2, C_3, C_4 are constants depending on $\Gamma, \widetilde{\Omega}$ and $\widetilde{u}$.

Proof Define $\widetilde{u}_n(x) = \int_{\partial\widetilde{\Omega}} \partial_\nu \widetilde{G}(x,s)\widetilde{b}_n(s)\mathrm{d}s$ for $x \in \overline{\Omega}$. Then

$$|\widetilde{u}(x) - \widetilde{u}_n^h(x)| \leq |\widetilde{u}(x) - \widetilde{u}_n(x)| + |\widetilde{u}_n(x) - \widetilde{u}_n^h(x)|.$$

According to the interior regularity of the Laplace equation, since the above $\widetilde{u}_n$ and $\widetilde{u} = \int_{\partial\widetilde{\Omega}} \partial_\nu \widetilde{G}(x,s)\widetilde{b}(s)\mathrm{d}s$ are both harmonic in $\widetilde{\Omega}$, we see

$$|\widetilde{u}(x) - \widetilde{u}_n(x)| \leq \|\widetilde{u} - \widetilde{u}_n\|_{C(\overline{\Omega})} \leq C(\Omega, \widetilde{\Omega})\|\widetilde{b} - \widetilde{b}_n\|_{L^2(\partial\widetilde{\Omega})}.$$

Due to the density of the space of test functions, we have $|\widetilde{u}(x) - \widetilde{u}_n(x)| \leq C\delta(n)$. For the second part, a standard error estimate for the second order central difference method (e.g., [18]) implies

$$|\widetilde{u}_n(x) - \widetilde{u}_n^h(x)| \leq C\|\widetilde{u}_n\|_{C^4(\overline{\Omega})}h^2 \leq C(\Omega, \widetilde{\Omega})\|\widetilde{u}_n\|_{L^2(\widetilde{\Omega})}h^2 \leq C(\Omega, \widetilde{\Omega}, \widetilde{u}_n)h^2$$

for $x \in \Omega$. Combining the precision of the 1st order difference for the gradient, the estimate for $\widetilde{f}_n^h$ and $\widetilde{g}_n^h$ can be obtained.

The above estimate means that the error can be decomposed by the boundary discrete part and the FDM discrete part, and converges as $n \to \infty$ and $h \to 0$.

Lemma 4.2 *Under the assumption of Theorem* 4.1*, by the scheme presented in Section* 3*, let u^* be constructed as the minimizer*:

$$u^* = \underset{\widetilde{u}_n^h \in \mathcal{F}_n^h(\widetilde{\Omega})}{\arg\min} \{\|\widetilde{f}_n^h - f^\varepsilon\|_{H^1(\Gamma)}^2 + \|\widetilde{g}_n^h - g^\varepsilon\|_{L^2(\Gamma)}^2 + \alpha\|\widetilde{u}_n^h\|_{H^2(\partial\Omega)}^2\}, \tag{4.3}$$

where α is taken as $\alpha \sim \varepsilon^2 + \delta(n)^2 + h^2$. Then we have

$$\|f^* - f_0\|_{H^1(\Gamma)} + \|g^* - g_0\|_{L^2(\Gamma)} \le C(M, \Omega, \widetilde{\Omega})(\varepsilon + \delta(n) + h),$$

where $f^* = u^*|_\Gamma$ and $g^* = \partial_\nu u^*|_\Gamma$.

Proof First, by Runge's approximation (e.g., [20, 23]), there exists a harmonic function $\widetilde{u}$ in $\widetilde{\Omega}$ such that

$$\|u_0 - \widetilde{u}\|_{H^2(\Omega)} \le \varepsilon.$$

Then

$$\|\widetilde{u}\|_{H^2(\Omega)} \le CM, \tag{4.4}$$

and by the trace theorem (see [1]),

$$\|\widetilde{f}_0 - f_0\|_{H^1(\Gamma)} + \|\widetilde{g}_0 - g_0\|_{L^2(\Gamma)} \le \varepsilon, \tag{4.5}$$

where $\widetilde{f}_0 = \widetilde{u}|_\Gamma$ and $\widetilde{g}_0 = \partial_\nu \widetilde{u}|_\Gamma$.

The definition of the minimizer yields

$$\begin{aligned} &\|f^* - f^\varepsilon\|_{H^1(\Gamma)}^2 + \|g^* - g^\varepsilon\|_{L^2(\Gamma)}^2 + \alpha\|u^*\|_{H^2(\partial\Omega)}^2 \\ &\le \|\widetilde{f}_{0,n}^h - f^\varepsilon\|_{H^1(\Gamma)}^2 + \|\widetilde{g}_{0,n}^h - g^\varepsilon\|_{L^2(\Gamma)}^2 + \alpha\|\widetilde{u}_{0,n}^h\|_{H^2(\partial\Omega)}^2, \end{aligned} \tag{4.6}$$

where $\widetilde{f}_{0,n}^h = \widetilde{u}_{0,n}^h|_\Gamma$ and $\widetilde{g}_{0,n}^h = \partial_\nu \widetilde{u}_{0,n}^h|_\Gamma$. Therefore,

$$\alpha\|u^*\|_{H^2(\partial\Omega)}^2 \le \|\widetilde{f}_{0,n}^h - f^\varepsilon\|_{H^1(\Gamma)}^2 + \|\widetilde{g}_{0,n}^h - g^\varepsilon\|_{L^2(\Gamma)}^2 + \alpha\|\widetilde{u}_{0,n}^h\|_{H^2(\partial\Omega)}^2.$$

We can estimate the first term on the right-hand side as

$$\|\widetilde{f}_{0,n}^h - f^\varepsilon\|_{H^1(\Gamma)} \le \|\widetilde{f}_{0,n}^h - \widetilde{f}_0\|_{H^1(\Gamma)} + \|\widetilde{f}_0 - f_0\|_{H^1(\Gamma)} + \|f_0 - f^\varepsilon\|_{H^1(\Gamma)}.$$

The second term $\|\widetilde{g}_{0,n}^h - g^\varepsilon\|_{L^2(\Gamma)}^2$ can be dealt with similarly. Based on Lemma 4.1, we obtain

$$\|\widetilde{f}_{0,n}^h - \widetilde{f}_0\|_{H^1(\Gamma)} + \|\widetilde{g}_{0,n}^h - \widetilde{g}_0\|_{L^2(\Gamma)} \le C_1\delta(n) + C_2 h,$$

where C_1, C_2 depend on $\Gamma, \widetilde{\Omega}, M$. By (4.5) and the assumption that $\|f^\varepsilon - f_0\|_{H^1(\Gamma)} + \|g^\varepsilon - g_0\|_{L^2(\Gamma)} \le \varepsilon$, we see

$$\|\widetilde{f}_{0,n}^h - f^\varepsilon\|_{H^1(\Gamma)} + \|\widetilde{g}_{0,n}^h - g^\varepsilon\|_{L^2(\Gamma)} \le C_3\delta(n) + C_4 h + C_5\varepsilon,$$

where the constants C_3, C_4, C_5 depend on $M, \Gamma, \widetilde{\Omega}$. Meanwhile,

$$\|\widetilde{u}_{0,n}^h\|_{H^2(\partial\Omega)} \le \|\widetilde{u}_{0,h}^h - \widetilde{u}_0\|_{H^2(\partial\Omega)} + \|\widetilde{u}_0 - u_0\|_{H^2(\partial\Omega)} + \|u_0\|_{H^2(\partial\Omega)} \le C_6 M.$$

Consequently,

$$\|u^*\|_{H^2(\partial\Omega)}^2 \le C_7(\Gamma, \widetilde{\Omega}, M)\frac{\delta^2(n) + h^2 + \varepsilon^2}{\alpha} + C_6(\Gamma, \widetilde{\Omega})M^2.$$

With the choice of α, one reaches

$$\|u^*\|_{H^2(\partial\Omega)} \le C'(\Gamma, \widetilde{\Omega}, M). \tag{4.7}$$

For the residual part, in view of (4.6) and the above estimate we have

$$\begin{aligned}
&\|f^* - f^\varepsilon\|_{H^1(\Gamma)}^2 + \|g^* - g^\varepsilon\|_{L^2(\Gamma)}^2 \\
&\le \|\widetilde{f}_{0,n}^h - f^\varepsilon\|_{H^1(\Gamma)}^2 + \|\widetilde{g}_{0,n}^h - g^\varepsilon\|_{L^2(\Gamma)}^2 + \alpha\|\widetilde{u}_{0,n}^h\|_{H^2(\Omega)}^2 \\
&\le C_7(\delta^2(n) + h^2 + \varepsilon^2) + \alpha C_6 M^2.
\end{aligned}$$

Therefore, with the choice of α, we have

$$\begin{aligned}
&\|f^* - f^\varepsilon\|_{H^1(\Gamma)} + \|g^* - g^\varepsilon\|_{L^2(\Gamma)} \\
&\le C''(\Gamma, \widetilde{\Omega}, M)(\varepsilon + \delta(n) + h).
\end{aligned}$$

Finally, by combining the boundness of u^* and the estimate on the space of test functions, we have

$$\begin{aligned}
\|f^* - f_0\|_{H^1(\Gamma)} + \|g^* - g_0\|_{L^2(\Gamma)} &\le \|f^* - f^\varepsilon\|_{H^1(\Gamma)} + \|f^\varepsilon - f_0\|_{H^1(\Gamma)} \\
&\quad + \|g^* - g^\varepsilon\|_{L^2(\Gamma)} + \|g^\varepsilon - g_0\|_{L^2(\Gamma)} \\
&\le C(\Gamma, \widetilde{\Omega}, M)(\varepsilon + \delta(n) + h),
\end{aligned}$$

which is the conclusion of the lemma.

Lemma 4.3 *Denote $\boldsymbol{b}^* = (b_1^*, \cdots, b_n^*)$ as the vector corresponds to $u^* = \sum\limits_{i=1}^n b_i^* w_i^h(x)$ where u^* is the minimizer in Lemma 4.2, and $u_n^* = \sum\limits_{i=1}^n b_i^* w_i(x)$, where, for $i = 1, \cdots, n$, $w_i(x)$ are the base functions defined in Section 3 and $w_i^h(x)$ the numerical approximations. Under the assumption of Lemma 4.2, we have*

$$\|u_n^* - u_0\|_{C^1(\overline{\Omega})} \le C(\Gamma, \widetilde{\Omega}, M),$$

provided that n is sufficiently large.

Proof First,

$$\|u_n^*\|_{H^2(\partial\Omega)} \le \|u_n^* - u^*\|_{H^2(\partial\Omega)} + \|u^*\|_{H^2(\partial\Omega)}.$$

By means of the boundness (4.7) and the convergence of the FDM, we see

$$\|u_n^*\|_{H^2(\partial\Omega)} \le CM.$$

The assumption of u_0 and the Sobolev embedding (see [1]) yield

$$\|u_n^* - u_0\|_{C^1(\overline{\Omega})} \le \|u_n^* - u_0\|_{H^2(\partial\Omega)} \le \|u_n^*\|_{H^2(\partial\Omega)} + \|u_0\|_{H^2(\partial\Omega)} \le C(\Gamma, \widetilde{\Omega}, M).$$

After having got the estimate on Γ and the boundness on Ω, we can further have the reconstruction error on Ω.

Proof of Theorem 4.1 We note that $u_n^* - \widetilde{u}$ is harmonic and is bounded due to Lemma 4.3. By Lemma 4.2 we can obtain

$$\|f_n^* - f_0\|_{H^1(\Gamma)} + \|g_n^* - g_0\|_{H^1(\Gamma)} \le C(\Gamma, \widetilde{\Omega}, M)(\varepsilon + h + \delta(n)),$$

where we have used that

$$\|f_n^* - f_0\|_{H^1(\Gamma)} \le \|f_n^* - f^*\|_{H^1(\Gamma)} + \|f^* - f_0\|_{H^1(\Gamma)} \le C_1(\Gamma, \widetilde{\Omega}, M)h + C_2(\Gamma, \widetilde{\Omega}, M)(\varepsilon + h + \delta(n)).$$

Then we apply the conditional stability result in Theorem 2.1 to reach

$$|u_n^*(x) - u_0(x)| \le C_3(\Gamma, \widetilde{\Omega}, M)(\varepsilon + h + \delta(n))^{\tau(x)}.$$

Finally, for the approximation error by the FDM for u_n^*, we have

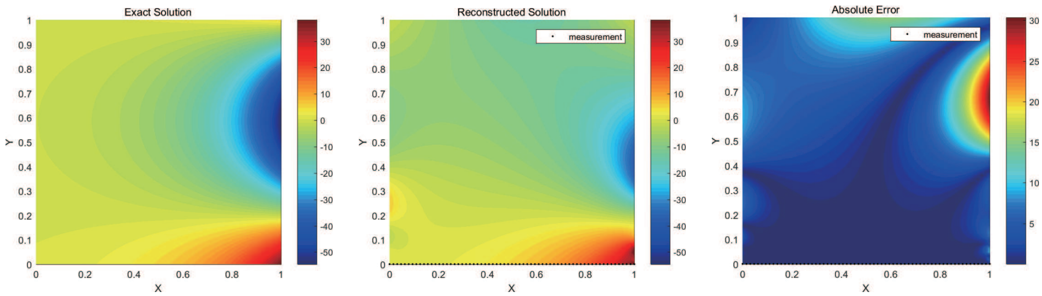
$$\begin{aligned} |u^*(x) - u_0(x)| &\le |u^*(x) - u_n^*(x)| + |u_n^*(x) - u_0(x)| \\ &\le C_4 h + C_3(\Gamma, \widetilde{\Omega}, M)(\varepsilon + h + \delta(n))^{\tau(x)} \le C(\Gamma, \widetilde{\Omega}, M)\varepsilon^{\tau(x)} \end{aligned}$$

for $x \in \Omega$, provided that $\delta(n)$ and h are sufficiently small.

5 Numerical Examples

Numerical example We have applied the above numerical methods to various cases. We will demonstrate the performances including the reconstruction evaluations.

We consider the domain $\Omega = (0,1)\times(0,1)$ and the measurement boundary $\Gamma = \{(x_1, 0); 0 < x_1 < 1\}$. The exact solution is selected as $u(x_1, x_2) = \mathrm{e}^{4x_1}\cos 4(x_2 + 0.2)$. The result with noise level 1% is displayed in Figure 1, where we adopt the method in Section 3. We choose $\varphi_i(x_1, x_2), i = 1, \cdots, n$ as linear interpolation bases along the boundary of the FEM grids. We discussed the case $n = 264$ and $h = \frac{1}{64}$ in the FDM calculation. The measurement points coincide with the FDM grids.

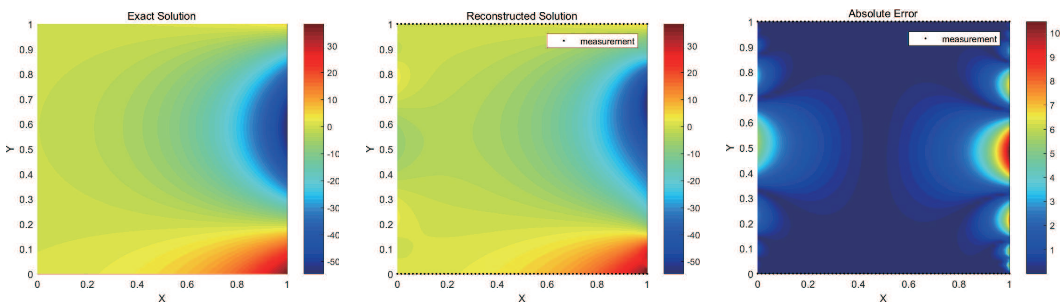


(a) Exact solution (b) Reconstruction (c) Absolute error

Figure 1 Reconstruction with one measurement boundary.

For the error in the reconstruction domain, it is expected that the error in Ω will be amplified significantly when getting far from the measurement boundary, due to the Hölder-type stability index indicated in Section 2. Since the estimate is sharp, even though the error is small somewhere far from the bottom side, the result is not reliable.

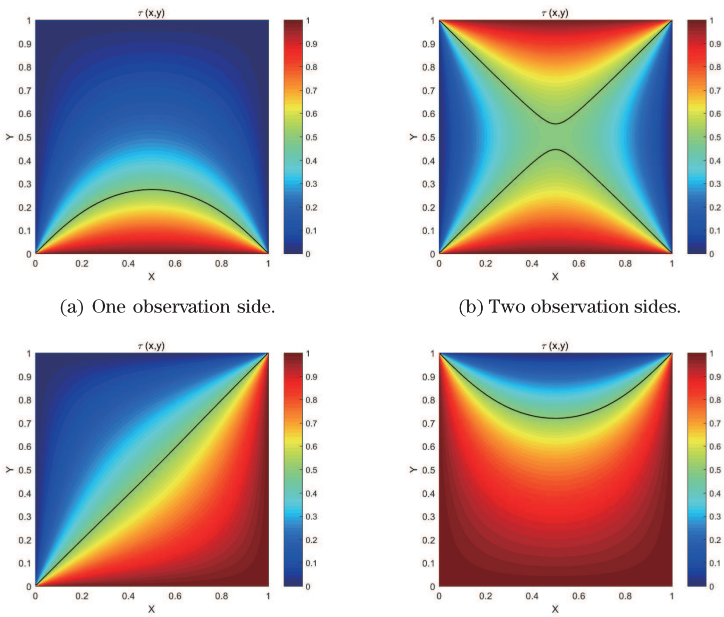
Figure 2 gives the case with an additional measurement boundary on the upper side. The result near the upper boundary is greatly improved comparing with the single measurement case. On the lateral sides, there is no measurement and the result there is not reliable, although the error level is not large in some places. This will be further illustrated in the following.



(a) Exact solution (b) Reconstruction (c) Absolute error

Figure 2 Reconstruction with two measurement boundaries.

Indicate function and reliable reconstruction domain The error estimate implies that the reconstruction error will no longer be improved by increasing the discrete accuracy once the observation error becomes dominant. Meanwhile, even if the observation error is small, the error far from the measurement area may be enlarged significantly, that is, the reconstruction there is not reliable. These phenomena are caused by the ill-posedness of the problem. Now that the reconstruction accuracy on the whole reconstruction domain are not ensured, one hopes to know where the reconstruction error has an acceptable convergence rate with observation noises and discrete errors in real applications. The reliable domain can be determined by the pointwise error estimate in Theorem 4.1. Since the error growth rate depends on $\tau(x_1, x_2)$, it further depends on the shape of the reconstruction domain. Figure 3 indicates profiles of the harmonic measure corresponding to the present case with a rectangle computational area. The area bounded by the black curve and the measurement boundary corresponds to $\tau(x_1, x_2) > 0.5$, which may be regarded as confidence area in practice. This is consistent with the error distributions in the examples (see Figures 1–2). The area with convergence rate higher than 0.5 is enlarged significantly by adding measurement boundaries.



(a) One observation side. (b) Two observation sides.

(c) Two observation sides. (d) Three observation sides.

Figure 3 Indicate function $\tau(x)$ with various characteristic boundaries. The black curve is the contour of $\tau(x) = 0.5$.

6 Concluding Remarks

The Cauchy problem of the Laplace equation often appears in real applications, and provides ways to infer global information from local measurement, which is also an ill-posed problem. The conditional stability estimates are proved, by which we can design stable numerical algorithms and estimate errors. The numerical treatment and the corresponding error estimate are presented. The estimate is featured by an indicate function constructed by the harmonic measure. This facilitates the evaluation of the numerical results, based on which how to improve the numerical results are proposed. Although we treat only the Laplace equation in this work, similar results can be proved for more general elliptic equations. The Cauchy problem is closely related to the unique continuation problem, and for the latter, conditional stability results and the numerical treatments are presented (e.g., [6, 15]).

Acknowledgement The authors thank the anonymous referee for helpful comments.

Declarations

Conflicts of interest The authors declare no conflicts of interest.

References

[1] Adams, R. A. and Fournier, J. J. F., Sobolev Spaces, Elsevier, Amsterdam, 2003.

[2] Alessandrini, G., Rondi, L., Rosset, E. and Vessella, S., The stability for the Cauchy problem for elliptic equations, *Inverse Problems*, **25**, 2009, 123004.

[3] Burman, E., Hansbo, P. and Larson, M., Solving ill-posed control problems by stabilized finite element methods: An alternative to Tikhonov regularization, *Inverse Problems*, **34**(3), 2018, 035004.

[4] Chakib, A. and Nachaoui, A., Convergence analysis for finite element approximation to an inverse Cauchy problem, *Inverse Problems*, **22**, 2006, 1191–1206.

[5] Cheng, J., Hon, Y. C., Wei, T. and Yamamoto, M., Numerical computation of a Cauchy problem for Laplace's equation, *Z. Angew. Math. Mech.*, **81**, 2001, 665–674.

[6] Cheng, J. and Yamamoto, M., Unique continuation on a line for harmonic functions, *Inverse Probl.*, **14**(4), 1998, 869–882.

[7] Cheng, J. and Yamamoto, M., One new strategy for a priori choice of regularizing parameters in Tikhonov's regularization, *Inverse Problems*, **16**(4), 2000, L31–L38.

[8] Friedman, A. and Vogelius, M., Determining cracks by boundary measurements, *Indiana University Mathematics Journal*, **38**(3), 1989, 527–556.

[9] Gilbarg, D. and Trudinger, N. S., Elliptic Partial Differential Equations of Second Order, Springer-Verlag, Berlin, 1983.

[10] Hadamard, J., Sur les problèmes aux dérivées partielles et leur signification physique, *Princeton University Bulletin*, **13**, 1902, 49–52.

[11] Hadamard, J., Lectures on Cauchy's Problem in Linear Partial Differential Equations, Yale University Press, New Haven, 1923.

[12] Hrycak, T. and Isakov, V., Increased stability in the continuation of solutions to the Helmholtz equation, *Inverse Problems*, **20**(3), 2004, 697–712.

[13] Isakov, V., Inverse Problems for Partial Differential Equations, Springer-Verlag, Berlin, 2006.

[14] Johansson, T., An iterative procedure for solving a Cauchy problem for second order elliptic equations, *Mathematische Nachrichten*, **272**(1), 2004, 46–54.

[15] Ke, Y. and Chen, Y., Unique continuation on quadratic curves for harmonic functions, *Chinese Annals of Mathematics*, *Series B*, **43**, 2022, 17–32.

[16] Kellogg, O. D., Foundations of Potential Theory, Dover Publications, Inc., New York, 1953.

[17] Kozlov, V. A. and Maz'ya, V. G., On iterative procedures for solving ill-posed boundary value problems that preserve differential equations, *Algebra i Analiz*, **1**, 1989, 144–170. English transl.: *Leningrad Math. J.*, **1**, 1990, 1207–1228.

[18] Larsson, S. and Thomé, V., Partial Differential Equations with Numerical Methods, Springer-Verlag, Berlin, 2003.

[19] Lattès, R. and Lions, J.-L., The Method of Quasi-Reversibility, Applications to Partial Differential Equations, American Elsevier Publishing Co., New York, 1969.

[20] Lax, P. D., A stability theorem for solutions of abstract differential equations, and its application to the study of the local behavior of solutions of elliptic equations, *Comm. Pure Appl. Math.*, **9**(4), 1956, 747–766.

[21] Natterer, F., The finite element method for ill-posed problems, R.A.I.R.O., *Analyse Numérique*, **11**(1), 1977, 271–278.

[22] Payne, L. E., Bounds in the Cauchy problem for the Laplace equation, *Archive for Rational Mechanics and Analysis*, **5**(1), 1960, 35–45.

[23] Rüland, A. and Salo, M., Quantitative runge approximation and inverse problems, *International Mathematics Research Notices*, **20**, 2019, 6216–6234.

[24] Yang, X., Choulli, M. and Cheng, J., An iterative method for the inverse problem of detecting corrosion in a pipe, *Numerical Mathematics-A Journal of Chinese Universities* (*English Series*), **14**(3), 2005, 252–266.