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Regularity and Compactness of Stationary Map-Varifold Pairs*

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Abstract The authors introduce the conception of stationary map-varifold pairs and prove a compactness result. As applications, they analyse the asymptotic structure of the pseudo tangent map of stationary harmonic maps. For stationary pair, they also get a strong convergence criterion about the map part and introduce the stratification of the singular set.

Keywords Stationary map-varifold pairs, Regularity, Compactness
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1 Introduction

The regularity for area minimizing rectifiable currents (see [1–2]) and the regularity for energy minimizing maps (see [6, 10, 13–14, 16–17]) have been studied intensively and successfully. The results depend on the compactness theorem and the existence of tangent cones and the existence of tangent maps.

However, the set of stationary harmonic maps is not compact (see [4, 8–9]), the energy may concentrate on a blow-up set. If the target does not support any harmonic S^2, Lin [9] proved the compactness theorem for stationary harmonic maps, and consequently, he showed the regularity theorem in this case. Li-Tian [8] showed that the "sum" of the blow-up set and the weak limit is stationary. But, in general, every one may not be stationary (see [4]). Using the blow-up formula derived in [8], they proved that tangent maps and tangent cones of a stationary harmonic map at a singular point exist. A similar argument shows the existence of pseudo tangent maps (see Section 2 for the definition). In general, the pseudo tangent map and its corresponding detect measure are not conical. To investigate the asymptotic behavior of the pseudo tangent map, motivated by Li-Tian's blow-up formula, we introduce the conception of stationary map-varifold pair (see Definition 2.2) and establish the compactness for a subclass of such objects. A fundamental tool is Moser's (see [11]) theory on stationary measures and

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the rectifiability of the detect measure (see [9, 11]). The advantage of stationary harmonic maps is that they share the same small energy regularity with energy minimizing maps (see [3, 5]), which yields that the $(m-2)$-dimensional Hausdorff measure of the singular set of a stationary harmonic map vanishes, where $m = \dim M$. The small energy regularity is involved in the choice of subclass $\mathcal{E}(\varepsilon_0)$ (see Definition 2.2) of stationary map-varifold pairs, such that the compactness holds with rectifiability.

Theorem 1.1 *For any sequence $\{\mu_i = (u_i, V_i)\}_{i=1}^{\infty} \subset \mathcal{E}(\varepsilon_0)$ such that $\sup_i \overline{\mu}_i(K) \leq C_K < +\infty$ for any compact subset $K \subset M$, there exists a subsequence converges as Radon measure to a stationary map-varifold pair $(u, V) \in \mathcal{E}(\varepsilon_0)$.*

In this notes, we also adopt Simon's idea (see [16–17]) to get the stratification $\mathcal{S}_j$ ($\mathcal{S}_j \subset \mathcal{S}_{j+1}$) for singular set of stationary map-varifold pairs for $0 \leq j \leq m-2$, with $\dim \mathcal{S}_j \leq j$. In proving $\dim \mathcal{S}_j \leq j$, we use the compactness described above instead of the compactness theorem for energy minimizing maps. It is interesting to know whether the Hausdorff codimension of the singular set is at least 3, which was proved for energy minimizing maps by Schoen-Uhlenbeck [13–14]. In other words, we hope to know whether $\mathcal{S}_{m-2} = \mathcal{S}_{m-3}$. We show that

$$\mathcal{S}_{m-2} = \mathcal{S}_{m-3} \cup \mathcal{S}_{-1},$$

where

$$\mathcal{S}_{-1} = \{x \in \mathcal{S}_{m-2} \mid \text{all tangent maps are constant}\}.$$

Applying the compactness of stationary map-varifold pairs and a strong convergence criterion for the map part(see Corollary 2.4), we show that $\mathcal{S}_{-1} = \emptyset$ for strongly stable stationary harmonic map, which revisits Hong-Wang's theorem (see [7]) in the following.

Corollary 1.1 *Let M be a compact m-dimensional Riemannian manifold. Let N be a compact Riemannian manifold. Let u be a strongly stable stationary harmonic map from M to N. Then the Hausdorff codimension of the singular set of u is at least 3.*

In 3-dimensional case, we can show the following corollary.

Corollary 1.2 *Let M be a compact 3-dimensional Riemannian manifold. Let N be a compact analytic manifold with an analytic metric. Let u be a strongly stable stationary harmonic map from M to N. Then the singular set of u consists of at most finitely many points.*

2 Stationary Pairs, Compactness and Stratification

Let M and N be two compact Reimannian manifolds. Let $u(x)$ be a stationary harmonic map from M to N. The regular set $\text{reg}(u)$ of u is defined as the set of points $x \in M$ such that u is smooth in some neighborhood of x. It is clear that $\text{reg}(u)$ is open. The singular set $\text{sing}(u)$ of u is defined to be the complement of $\text{reg}(u)$.

We define the density function Θ_u of u by

$$\Theta_u(x) = \lim_{r\to 0} r^{2-m} \int_{B_r(x)} |\nabla u|^2 \mathrm{d}vol.$$

The monotonicity inequality (see [12]) for u yields that the limit exists. And it is proved (see [3, Theorem I.4]) that $x \in \mathrm{reg}(u)$ if and only if $\Theta_u(x) = 0$.

The first author and Tian [8] proved the existence of tangent maps and tangent cones at a singular point.

Let u be a stationary harmonic map from M to N. Assume that $x \in \mathrm{sing}(u)$. We set $u_{x,r}(y) = u(x + ry)$. Then for any sequence $r_{i'}$, there is a subsequence $r_i \to 0$ such that $u_{x,r_i} \to \phi$ weakly in $H^{1,2}(R^m, N)$ and $|\nabla u_{x,r_i}|^2 \mathrm{d}y \to |\nabla \phi|^2 \mathrm{d}y + \theta(y)\mathcal{H}^{m-2} \llcorner \Sigma_x$ in the sense of measure. It is proved in [8] that $\phi(\lambda y) = \phi(y)$ for all $\lambda > 0$, $y \in R^m$, $\theta(\lambda y) = \theta(y)$ for all $\lambda > 0$, $y \in R^m$, and Σ_x is a tangent cone, that is $\lambda\Sigma_x = \Sigma_x$ for all $\lambda > 0$.

We call ϕ a tangent map at x, (Σ_x, θ) a tangent cone at x.

Lemma 2.1 *Suppose that $x_j \to x \in M$, and that u_j is a sequence of stationary harmonic maps from M to N which satisfies that $u_j \to u$ weakly in $H^{1,2}(M, N)$, $|\nabla u_j|^2 \mathrm{d}V \to |\nabla u|^2 \mathrm{d}V + \theta\mathcal{H}^{m-2} \llcorner \Sigma$ in the sense of measure with blow-up set Σ. Then*

$$\begin{aligned}\Theta_{u,\theta}(x) &= \lim_{r\to 0} r^{2-m} \Big(\int_{B_r(x)} |\nabla u|^2 \mathrm{d}vol + \int_{B_r(x)} \theta\mathcal{H}^{m-2} \llcorner \Sigma \Big) \\ &\geq \limsup_{j\to\infty} \Theta_{u_j}(x_j).\end{aligned}$$

Proof For any $\rho > 0$ and any $\varepsilon > 0$, by monotonicity inequality for stationary harmonic maps, we have

$$\begin{aligned}\Theta_{u_j}(x_j) &\leq \rho^{2-m} \int_{B_\rho(x_j)} |\nabla u_j|^2 \mathrm{d}vol \\ &\leq \rho^{2-m} \int_{B_{\rho+\varepsilon}(x)} |\nabla u_j|^2 \mathrm{d}vol\end{aligned}$$

for j sufficiently large. Letting $j \to \infty$, letting $\varepsilon \to 0$, and then letting $\rho \to 0$, we obtain the lemma.

Generally, we can consider pseudo tangent map similar to tangent map.

Definition 2.1 *Suppose that $x \in \mathrm{sing}(u)$, and $x_j \to x$ as $j \to \infty$. Let $u_{x_j,r}(y) = u(x_j + ry)$. Then for any sequence $r_j \to 0$ there is a subsequence which we also denote by r_j such that $u_{x_j,r_j} \to \phi$ weakly in $H^{1,2}(R^m, N)$ and $|\nabla u_{x_j,r_j}|^2 \mathrm{d}y \to |\nabla \phi|^2 \mathrm{d}y + \theta(y)\mathcal{H}^{m-2} \llcorner \Sigma_x$ in the sense of measure. We call ϕ a pseudo tangent map at x and $\nu = \theta\mathcal{H}^{m-2} \llcorner \Sigma_x$ the detect measure.*

In general, pseudo tangent map may not be conical. But by Li-Tian blow-up formula, when adding the detect measure together, there is still a monotonicity formula.

Lemma 2.2 *Let $x \in \mathrm{sing}(u)$, ϕ a tangent map or a pseudo tangent map at x, $(\Sigma_x, \theta) =$ the corresponding detect measure. For any $y_0 \in R^m$, the function*

$$\rho^{2-m}\Big(\int_{\Sigma_x \cap B_\rho(y_0)} \theta \mathcal{H}^{m-2} \llcorner \Sigma_x + \int_{B_\rho(y_0)} |\nabla \phi|^2 \mathrm{d}y\Big)$$

is increasing in $\rho > 0$.

Proof By [8, Theorem 2.1] we have,

$$\int_{\Sigma_x} \mathrm{div}_{\Sigma_x}(X) \theta \mathcal{H}^{m-2} \llcorner \Sigma_x + \int_{R^m} \Big(|\nabla \phi|^2 \mathrm{div}(X) - 2\Big\langle \mathrm{d}\phi(\nabla_\alpha X), \mathrm{d}\phi\Big(\frac{\partial}{\partial x^\alpha}\Big)\Big\rangle\Big) \mathrm{d}V = 0 \tag{2.1}$$

for any smooth vector field X with compact support in R^m.

We choose $X(y) = \xi(r) r \frac{\partial}{\partial r}$ where $r = |y - y_0|$ and

$$\xi(r) = \begin{cases} 1 & \text{if } r \le t', \\ \dfrac{t-r}{t-t'} & \text{if } t' < r < t, \\ 0 & \text{if } r \ge t. \end{cases}$$

Then we have

$$\begin{aligned} &\int_{\Sigma_x} (\xi' r + (m-2)\xi) \theta \mathcal{H}^{m-2} \llcorner \Sigma_x - \int_{\Sigma_x} \xi' r |\nabla_{\Sigma_x^\perp} r|^2 \theta \mathcal{H}^{m-2} \llcorner \Sigma_x \\ &+ \int_{R^m} (\xi' r + (m-2)\xi) |\nabla \phi|^2 \mathrm{d}y - 2 \int_{R^m} \xi' r \Big|\frac{\partial \phi}{\partial r}\Big|^2 \mathrm{d}y = 0. \end{aligned}$$

Letting $t' \to t$ and integrating, we get

$$\begin{aligned} &\rho^{2-m}\Big(\int_{\Sigma_x \cap B_\rho(y_0)} \theta \mathcal{H}^{m-2} \llcorner \Sigma_x + \int_{B_\rho(y_0)} |\nabla \phi|^2 \mathrm{d}y\Big) \\ &= \sigma^{2-m}\Big(\int_{\Sigma_x \cap B_\sigma(y_0)} \theta \mathcal{H}^{m-2} \llcorner \Sigma_x + \int_{B_\sigma(y_0)} |\nabla \phi|^2 \mathrm{d}y\Big) \\ &\quad + \int_{\Sigma_x \cap (B_\rho(y_0) \setminus B_\sigma(y_0))} r^{2-m} |\nabla_{\Sigma_x^\perp} r|^2 \theta \mathcal{H}^{m-2} \llcorner \Sigma_x \\ &\quad + 2 \int_{B_\rho(y_0) \setminus B_\sigma(y_0)} r^{2-m} \Big|\frac{\partial \phi}{\partial r}\Big|^2 \mathrm{d}y. \end{aligned} \tag{2.2}$$

This proves the lemma.

Corollary 2.1 *Assume $\phi \in H^1(\mathbb{R}^m, N)$ and a $(m-2)$-rectifiable varifold $V = \underline{v}(\Sigma, \theta)$ satisfying* (2.1)*, then for any $y \in R^m$, the density*

$$\Theta_{\theta,\phi}(y) = \lim_{\rho \to 0} \rho^{2-m}\Big(\int_{\Sigma_x \cap B_\rho(y)} \theta \mathcal{H}^{m-2} \llcorner \Sigma_x + \int_{B_\rho(y)} |\nabla \phi|^2 \mathrm{d}y\Big)$$

exists and is upper semi-continuous on $\mathbb{R}^m$.

Proof Denote $\mu = |\nabla\phi|^2 \mathrm{d}y + \theta\mathcal{H}^{n-2}\llcorner\Sigma$, by Lemma 2.2, for any $y \in R^m$, $\Theta_{\theta,\phi}(y) = \lim\limits_{\rho\to 0} \frac{\mu(B_\rho(y))}{\rho^{m-2}}$ exists. Moreover, for any $y_i \to y$ and $\rho > 0$,

$$\frac{\mu(B_\rho(y))}{\rho^{m-2}} \geq \frac{\mu(B_{\rho-|y_i-y|}(y_i))}{(\rho-|y_i-y|)^{m-2}}\Big(\frac{\rho-|y_i-y|}{\rho}\Big)^{m-2} \geq \Theta_{\theta,\phi}(y_i)\Big(1-\frac{|y_i-y|}{\rho}\Big)^{m-2}.$$

Letting $i\to\infty$ and then $\rho \to 0$, we get

$$\Theta_{\theta,\phi}(y) \geq \limsup_{y_i\to y} \Theta_{\theta,\phi}(y_i).$$

In the following, we analyse the asymptotic behavior of the pseudo tangent map. Let ϕ be a pseudo tangent map and (Σ_x, θ) the corresponding detect measure at x. It is clear that

$$\begin{aligned}
&\rho^{2-m}\Big(\int_{\Sigma_x\cap B_\rho(0)} \theta\mathcal{H}^{m-2}\llcorner\Sigma_x + \int_{B_\rho(0)} |\nabla\phi|^2\mathrm{d}y\Big)\\
&-\sigma^{2-m}\Big(\int_{\Sigma_x\cap B_\sigma(0)} \theta\mathcal{H}^{m-2}\llcorner\Sigma_x + \int_{B_\sigma(0)} |\nabla\phi|^2\mathrm{d}y\Big)\\
={}&\lim_{i\to\infty}\Big(\rho^{2-m}\int_{B_\rho(0)} |\nabla u_{x_i,r_i}|^2\mathrm{d}y - \sigma^{2-m}\int_{B_\sigma(0)} |\nabla u_{x_i,r_i}|^2\mathrm{d}y\Big)\\
={}&\lim_{i\to\infty}\int_{B_{\rho r_i}(x_i)\setminus B_{\sigma r_i}(x_i)} \frac{1}{r^{m-2}}\Big|\frac{\partial u}{\partial r}\Big|^2.
\end{aligned}$$

In the case $x_i = x$, i.e., ϕ is a tangent map, we know the last term is zero by the absolutely continuity of integral. But in general, we can not hope the last term to be zero, since $r = |\cdot - x_i|$ depends on i. It can be expected that this pair of pseudo tangent map and the varifold is asymptotic to a cone at infinity. For this goal, we need to blow down this pair again. Motivated by this and Li-Tian's blow-up formula and the conception of stationary measure introduced by Moser [11], we define the following stationary conception of map-varifold pair and proves a compactness result.

Definition 2.2 *Assume $u \in W^{1,2}_{\mathrm{loc}}(M^m, N^n)$ is a weak harmonic map and $V = \underline{v}(\Sigma,\theta)$ is a rectifiable $(m-2)$-varifold. If the $TM\otimes T^*M$-valued measure*

$$\mu = \nabla u \otimes \mathrm{d}u \otimes \mathrm{d}vol + \frac{1}{2}\theta\mathcal{H}^{m-2}\llcorner\Sigma\cdot p^{\perp}_{T_x\Sigma}$$

is stationary in the sense of Moser [11], i.e., satisfying the Li-Tian blow-up formula

$$\int_M (|\nabla u|^2 \operatorname{div} X - 2\langle\nabla u\otimes \mathrm{d}u, \nabla X\rangle)\mathrm{d}vol(x) + \int_\Sigma \operatorname{div}^\Sigma X\theta \mathrm{d}\mathcal{H}^{m-2}(x) = 0$$

for any Lipschitz vector field X on M, then we call $\mu =: (u,V)$ a stationary map-varifold pair. The corresponding energy density measure is defined by

$$\overline{\mu} = \operatorname{tr}\mu = |\nabla u|^2\mathrm{d}vol(x) + \theta(x)\mathcal{H}^{m-2}\llcorner\Sigma.$$

Moreover, we denote the class

$$\mathcal{E}(\varepsilon_0) := \{\textit{stationary pair } \mu = (u, V) \textit{ such that } \theta \geq \varepsilon_0, \Sigma \textit{ closed, and } \operatorname{sing}(u) \subset \Sigma\},$$

where ε_0 is the constant in the ε-regularity theorem [B] *of stationary harmonic maps and* $\operatorname{sing}(u)$ *is the singular set of u.*

Example 2.1 By the blow-up formula (see [8, 11]), for any stationary harmonic map sequence $\{u_i\}$ with bounded energy, there exists a subsequence such that $\nabla u_i \otimes \mathrm{d}u_i$ converges as $TM \otimes T^*M$-valued measure to a stationary map-varifold pair $(u, V) \in \mathcal{E}(\varepsilon_0)$.

Proof More precisely, assume $\nabla u_i \otimes \mathrm{d}u_i \to \mu = (u, V)$, then the blow-up formula means μ is stationary and hence the monotonicity formula holds, which further implies the density $\Theta(\mu, x) = \lim\limits_{r\to 0} \frac{\overline{\mu}(B_r(x))}{\omega_{m-2} r^{m-2}}$ exists. If $\Theta(\mu, x) < \varepsilon_0$, then $\Theta(u_i, x, r) \leq \varepsilon_1 < \varepsilon_0$ for some fixed $r > 0$ and $i \gg 1$. Thus the ε-regularity (see [3]) implies u_i are smooth in $B(x, r)$ with uniform estimates, and hence u_i converges to u smoothly near x and $\Theta(\mu, x) = 0$. So, we know

$$\Sigma := \{x \in M \mid \Theta(\mu, x) > 0\} = \{x \in M \mid \Theta(\mu, x) \geq \varepsilon_0\} \text{ is a closed subset.}$$

By [8] or [11], Σ is $(m-2)$-rectifiable and $\mu \llcorner \Sigma = \frac{1}{2}\theta \mathcal{H}^{m-2} \llcorner \Sigma \cdot p^{\perp}_{T_x\Sigma}$. This implies

$$\mu = \mu \llcorner \Sigma^c + \mu \llcorner \Sigma = \nabla u \otimes \mathrm{d}u + \frac{1}{2}\Theta(\mu, x)\mathcal{H}^{m-2} \llcorner \Sigma \cdot p^{\perp}_{T_x\Sigma},$$

where Σ is a closed set, $\theta = \Theta(\mu, x) \geq \varepsilon_0$ for $x \in \Sigma$ and u is smooth in $M \backslash \Sigma$. This implies $(u, V) \in \mathcal{E}(\varepsilon_0)$.

The advantage of introducing the class $\mathcal{E}(\varepsilon_0)$ is that it is a compact class and describes some regularities.

Theorem 2.1 *For any sequence $\{\mu_i = (u_i, V_i)\}_{i=1}^{\infty} \subset \mathcal{E}(\varepsilon_0)$ such that $\sup\limits_i \overline{\mu}_i(K) \leq C_K < +\infty$ for any compact subset $K \subset M$, there exists a subsequence such that it converges as Radon measure to a stationary map-varifold pair $(u, V) \in \mathcal{E}(\varepsilon_0)$.*

Proof Since $\mu_i = (u_i, V_i)$ are stationary with locally uniformly bounded mass, by passing to subsequence, we know μ_i converges to a $TM \otimes T^*M$ valued measure μ such that μ is stationary in the following sense defined by Moser [11]:

$$\int_M \operatorname{div} X \mathrm{d}\overline{\mu} - 2\langle \nabla X, \mathrm{d}\mu \rangle = 0, \quad \forall X = \text{Lipschitz vector field.}$$

Moreover, by the monotonicity formula, we know

$$\Theta(\mu, x) = \lim_{r\to 0} \frac{\overline{\mu}(B_r(x))}{\omega_{m-2} r^{m-2}}$$

exists for any $x \in M$. If $\Theta(\mu, x) < \varepsilon_0$, then there exists $r > 0$ such that

$$\Theta(\mu_i, x, r) = \frac{\mu_i(B_r(x))}{\omega_{m-2} r^{m-2}} \le \varepsilon_1 < \varepsilon_0$$

for some fixed $r \ll 1$ and $\forall i \gg 1$. By the monotonicity formula, we know there exists $r_1 > 0$ and $\varepsilon_2 \in (\varepsilon_1, \varepsilon_0)$, such that $\Theta(\mu_i, y) \le \varepsilon_2 < \varepsilon_0, \forall y \in B(x, r_1)$. By the definition of $\mathcal{E}(\varepsilon_0)$, we know $\Sigma_i \cap B(x, r_1) = \emptyset$ and hence $\mu_i = \nabla u_i \otimes \mathrm{d}u_i$ in $B(x, r_1)$ for some regular harmonic maps $u_i : B(x, r_1) \to N$ with uniformly bounded energy and $\Theta(u_i, y, r_1) \le \varepsilon_2 < \varepsilon_0$. So, by Bethuel's uniform estimate (see [3]), we know u_i converges strongly to a smooth harmonic map $u : B(x, r_1) \to N$ and hence $\mu = \nabla u \otimes \mathrm{d}u$ in $B(x, r_1)$. As a result,

$$\Sigma = \{x \in M \mid \Theta(\mu, x) > 0\} = \{\Theta(\mu, x) \ge \varepsilon_0\} = \text{ a closed subset.}$$

Again by [11, Theorem 1.1], we know Σ is $(m-2)$-rectifiable and $\mu \llcorner \Sigma = \frac{1}{2} \theta \mathcal{H}^{m-2} \llcorner \Sigma \cdot p^{\perp}_{T_x \Sigma}$, where $\theta \ge \varepsilon_0$. So, we know $\mu = (u, V) \in \mathcal{E}(\varepsilon_0)$.

With this compactness theorem. We can analyse the asymptotic behavior of a pseudo-tangent object. Recall that the definition of blow down of a $\mathbb{R}^n \otimes \mathbb{R}^{n*}$ valued measure μ is defined by $\frac{1}{R_i}\mu \to \mu_\infty$ for a sequence $R_i \to \infty$, where

$$\Big(\frac{1}{R}\mu\Big)(E) := \frac{1}{R^{m-2}} \mu(RE), \quad \forall \text{ Borel set } E \subset \mathbb{R}^m.$$

Corollary 2.2 *Assume $u \in H^{1,2}_{\mathrm{loc}}(M, N)$ is a stationary harmonic map, $x \in \mathrm{sing}(u)$, $x_i \to x$ and $r_i \to 0$ and u_{x_i, r_i} converges weakly to a stationary map-varifold pair $\mu = (u_\infty, V) \in \mathcal{E}(\varepsilon_0)$. Then, for any sequence $R_i \to \infty$, there exists a subsequence (still denoted as R_i) such that $\frac{1}{R_i}\mu$ converges to $\mu_\infty = (\phi, \underline{v}(\Sigma, \theta)) \in \mathcal{E}(\varepsilon_0)$ such that μ_∞ is a cone in the following sense*:

(1) $\phi(\lambda y) = \phi(y)$ *for all* $\lambda > 0$, $y \in R^m$,

(2) $\theta(\lambda y) = \theta(y)$ *for all* $\lambda > 0$, $y \in \Sigma_x$,

(3) Σ_x *is conical with respect to* $0 \in \mathbb{R}^m$, *that is* $\lambda \Sigma_x = \Sigma_x$ *for all* $\lambda > 0$.

Proof Since u_{x_i, r_i} converges to (u_∞, V), we know that

$$|\nabla u_{x_i, r_i}|^2 \mathrm{d}x \to \overline{\mu} = |\nabla u_\infty|^2 \mathrm{d}x + \theta_\mu \mathcal{H}^{m-2} \llcorner \Sigma,$$

which implies

$$\begin{aligned} \frac{\overline{\mu}(B_r(0))}{\omega_{m-2} r^{m-2}} &= \lim_{i \to \infty} \frac{\int_{B_r(0)} |\nabla u_{x_i, r_i}|^2(y) \mathrm{d}y}{\omega_{m-2} r^{m-2}} \\ &\le \lim_{i \to \infty} \frac{\int_{B_\delta(x_i)} |\nabla u|^2(y) \mathrm{d}y}{\omega_{m-2} \delta^{m-2}} \end{aligned}$$

$$= \Theta(u, x, \delta).$$

Letting $\delta \to 0$ and then $r \to \infty$, we get

$$\Theta(\mu, 0, r) \le \Theta(\mu, \infty) \le \Theta(u, x) < \infty.$$

So, for any compact set $K \subset \mathbb{R}^m$, there exists $r = r_K$ such that $K \subset B_r(0)$ and hence for $\mu_i = \frac{1}{R_i}\mu$, there holds

$$\overline{\mu}_i(K) \le \frac{\overline{\mu}(B_{rR_i})}{R_i^{m-2}} \le \Theta(\mu, \infty)\omega_{m-2} r^{m-2} < +\infty.$$

So, by the compactness Theorem 2.1, we know $\mu_i \to \mu_\infty = (\phi, V = \underline{v}(\Sigma, \theta)) \in \mathcal{E}(\varepsilon_0)$ such that for any $r > 0$, there holds

$$\begin{aligned}\Theta(\mu_\infty, 0, r) &= \frac{\overline{\mu}_\infty(B_r(0))}{\omega_{m-2} r^{m-2}} \\ &= \lim_{i\to\infty} \frac{\overline{\mu}_i(B_r(0))}{\omega_{m-2} r^{m-2}} \\ &= \lim_{i\to\infty} \frac{\overline{\mu}(B_{rR_i}(0))}{\omega_{m-2}(rR_i)^{m-2}} \\ &= \Theta(\mu, \infty).\end{aligned}$$

Thus by (2.2), we know $\nabla_{\Sigma_x^\perp} r = 0$ and $\frac{\partial \phi}{\partial r} = 0$ for $r = |y|$, which implies $\phi(\lambda y) = \phi(y)$ for $\lambda > 0, y \in \mathbb{R}^m$. Similar to the proof of [15, Theorem 19.3], to show the varifold part is conical, i.e.,

$$\lambda \Sigma_x = \Sigma_x \quad \text{and} \quad \theta(\lambda y) = \theta(y), \quad \forall \lambda > 0,\ y \in \Sigma_x,$$

it is enough to prove that for any homogeneous function $h \in C^1(\mathbb{R}^m \backslash \{0\})$ of degree zero, i.e., $h(y) = h\big(\frac{y}{|y|}\big) =: h(\omega)$,

$$\rho^{-(m-2)} \int_{B_\rho(0)} h(y)\theta(y) \mathrm{d}\mathcal{H}^{m-2} \llcorner \Sigma_x = \text{constant (independent of } \rho).$$

For this, choose $X_h = h(\omega)\xi(r) r \frac{\partial}{\partial r} = hX$ in (2.1). Note $\nabla_{\Sigma_x^\perp} r = 0$, $\frac{\partial \phi}{\partial r} = 0$ and $\frac{\partial h}{\partial r} = 0$. By direct calculation, we get

$$\mathrm{div}_{\Sigma_x}(X_h) = (\xi' r + (m-2)\xi) h,$$

$$\mathrm{div}(X_h) = (\xi' r + m\xi) h$$

and

$$g^{\alpha\beta} \Big\langle \mathrm{d}\phi\Big(\nabla_\alpha X_h, \mathrm{d}\phi\Big(\frac{\partial}{\partial x^\beta}\Big)\Big)\Big\rangle = h(\xi' r + (m-2)\xi)|\nabla \phi|^2,$$

which implies

$$\int_{\mathbb{R}^m} (\xi' r + (m-2)\xi) h(\theta \mathcal{H}^{m-2} \llcorner \Sigma_x + |\nabla \phi|^2 \mathrm{d}y) = 0.$$

As before, letting $t' \to t$ and integrating, we know

$$\rho^{2-m} \int_{B_\rho(0)} h(\theta \mathcal{H}^{m-2} \llcorner \Sigma_x + |\nabla \phi|^2 \mathrm{d}y) \text{ is independent of } \rho.$$

Since $h(y) = h(\omega)$ and $\phi(y) = \phi(\omega)$, we know

$$\begin{aligned}
&\rho^{2-m} \int_{B_\rho(0)} h |\nabla \phi|^2 \mathrm{d}y \\
=&\rho^{2-m} \int_0^\rho \int_{S^{m-1}} h(\omega) \frac{|\nabla^{S^{m-1}} \phi|^2(\omega)}{r^2} r^{m-1} \mathrm{d}\omega \mathrm{d}r \\
=&\int_{S^{m-1}} h(\omega) |\nabla^{S^{m-1}} \phi|^2(\omega) \mathrm{d}\omega
\end{aligned}$$

is independent of ρ and hence so is $\rho^{2-m} \int_{B_\rho(0)} h\theta \mathcal{H}^{m-2} \llcorner \Sigma_x$. This completes the proof.

By the same argument, we can prove the tangent measure of a stationary pair $(u, V) \in \mathcal{E}(\varepsilon_0)$ is conical. For $x \in \Sigma$ and $r > 0$, we denote

$$\mu_{x,r}(E) := r^{2-m} \mu(rE + x), \quad \forall \, \text{Borel set } E \subset \mathbb{R}^m.$$

Corollary 2.3 *Assume $\mu = (u, V) \in \mathcal{E}(\varepsilon_0)$, $x \in \Sigma$, i.e., $\Theta(\mu, x) > 0$ and $r \to 0$, then $\mu_{x,r} \to \mu_x = (u_x, V_x) \in \mathcal{E}(\varepsilon_0)$ and*

$$\frac{\overline{\mu}_x(B_\rho(0))}{\omega_{m-2}\rho^{m-2}} \equiv \Theta(\mu, x), \quad \forall \rho > 0.$$

The measure $\mu_x = (u_x, V_x)$ is called the tangent pair of $\mu = (u, V)$.

Introducing the conception of map-varifold pair provides a simple strong convergence criterion.

Corollary 2.4 *Assume $\mu_i = (u_i, V_i) \in \mathcal{E}(\varepsilon_0)$ such that $\mu_i \to \mu \in (u, V) \in \mathcal{E}(\varepsilon_0)$. If $V = \underline{v}(\Sigma, \theta)$ such that $\mathcal{H}^{n-2}(\Sigma) = 0$, then u_i converges to u strongly in $W^{1,2}$.*

Proof $V = \underline{v}(\Sigma, \theta)$ is rectifiable $(m-2)$ varifold and $\mathcal{H}^{m-2}(\Sigma) = 0$ implies $V = 0$ and hence u is a stationary weak harmonic map with $\mathrm{sing}(u) \subset \Sigma$ and $\overline{\mu} = |\nabla u|^2 \mathrm{d}x$. Moreover, $\mu_i \to \mu$ implies $\overline{\mu}_i = |\nabla u_i|^2 \mathrm{d}x + \theta_i \mathcal{H}^{n-2} \llcorner \Sigma_i$ converges as a Radon measure to the limit $\overline{\mu} = |\nabla u|^2 \mathrm{d}x$. So, we know for any $r > 0$, there holds

$$\limsup_{i \to \infty} \int_{B_r(p)} |\nabla u_i|^2 \mathrm{d}x$$

$$\leq \lim_{i\to\infty} \overline{\mu}_i(B_r(p))$$
$$= \int_{B_r(p)} |\nabla u|^2 \mathrm{d}x.$$

On the other hand, the weak lower semi-continuity of Dirichlet energy implies u_i converges weakly to some limit v and

$$\int_{B_r(p)} |\nabla v|^2 \mathrm{d}x \leq \liminf_{i\to\infty} \int_{B_r(p)} |\nabla u_i|^2 \mathrm{d}x.$$

Noting that the compactness theorem implies u_i converges strongly to u out side the closed subset Σ with $\mathcal{H}^{n-2}(\Sigma)=0$. So, $u(x)=v(x)$ on Σ^c, which implies $v=u$ as a $W^{1,2}$ maps and hence

$$\int_{B_r(p)} |\nabla u|^2 \mathrm{d}x = \int_{B_r(p)} |\nabla v|^2 \mathrm{d}x = \lim_{i\to\infty} \int_{B_r(p)} |\nabla u_i|^2 \mathrm{d}(x).$$

So, we know $\|u_i - u\|_{W^{1,2}(B_r(p))} \to 0, \forall r>0$.

Remark 2.1 In the setting of the above corollary, we also know that

$$\lim_{i\to\infty} \mathcal{H}^{n-2}(\Sigma_i) = 0.$$

But we are not able to show that $\mathcal{H}^{n-2}(\Sigma_i)=0$. So, we do not know whether u_i is stationary.

The tangent map-varifold pair and the blow down of the pseudo tangent map-varifold pair are both conical, i.e., symmetric in the radical direction. In the following, Simon's idea (see [16–17]) is used to describe the full symmetry of map-varifold pair satisfying volume cone condition, see also [11]. The only difference we want to show is that both the map part and the varifold part admit symmetries although both of them may not be stationary.

Definition 2.3 *Assume (ϕ, V) is a stationary map-varifold pair from $\mathbb{R}^m$ to N, where $V=\underline{v}(\Sigma,\theta)$. If*

$$\rho^{2-m}\Big(\int_{\Sigma\cap B_\rho(0)} \theta\mathcal{H}^{m-2}\llcorner\Sigma + \int_{B_\rho(0)} |\nabla\phi|^2 \mathrm{d}y\Big) \equiv \Theta_{\theta,\phi}(0), \quad \forall\, \rho>0, \tag{2.3}$$

we say (ϕ, V) satisfies volume cone condition respect to 0*.*

Lemma 2.3 *Assume (ϕ, V) is a stationary map-varifold pair from $\mathbb{R}^m$ to N satisfying the volume cone condition. Then, for any $Y\in R^m$, we have*

$$\Theta_{\theta,\phi}(Y) \leq \Theta_{\theta,\phi}(0).$$

The equality holds if and only if

(i) $\phi(Y+\lambda X)=\phi(Y+X)$ *for all* $\lambda>0$, $X\in R^m$,

(ii) $\theta(Y+\lambda X)=\theta(Y+X)$ *for all* $\lambda>0$, $X\in R^m$,

(iii) Σ_x *is a also tangent cone at* Y.

Proof Let $r=|X-Y|$. By (2.2), we have

$$\rho^{2-m}\Big(\int_{\Sigma_x\cap B_\rho(Y)}\theta\mathcal{H}^{m-2}\llcorner\Sigma_x+\int_{B_\rho(Y)}|\nabla\phi|^2\mathrm{d}y\Big)-\Theta_{\theta,\phi}(Y)$$
$$=\int_{\Sigma_x\cap(B_\rho(Y))}r^{2-m}|\nabla_{\Sigma_x^\perp}r|^2\theta\mathcal{H}^{m-2}\llcorner\Sigma_x+2\int_{B_\rho(Y)}r^{2-m}\Big|\frac{\partial\phi}{\partial r}\Big|^2\mathrm{d}y.$$

Note that $B_\rho(Y)\subset B_{\rho+|Y|}(0)$, we have

$$\rho^{2-m}\Big(\int_{\Sigma_x\cap B_\rho(Y)}\theta\mathcal{H}^{m-2}\llcorner\Sigma_x+\int_{B_\rho(Y)}|\nabla\phi|^2\mathrm{d}y\Big)$$
$$\le\rho^{2-m}\Big(\int_{\Sigma_x\cap B_{\rho+|Y|}(0)}\theta\mathcal{H}^{m-2}\llcorner\Sigma_x+\int_{B_{\rho+|Y|}(0)}|\nabla\phi|^2\mathrm{d}y\Big)$$
$$=\Big(1+\frac{|Y|}{\rho}\Big)^{m-2}(\rho+|Y|)^{2-m}\Big(\int_{\Sigma_x\cap B_{\rho+|Y|}(0)}\theta\mathcal{H}^{m-2}\llcorner\Sigma_x+\int_{B_{\rho+|Y|}(0)}|\nabla\phi|^2\mathrm{d}y\Big)$$
$$\equiv\Big(1+\frac{|Y|}{\rho}\Big)^{m-2}\Theta_{\theta,\phi}(0)\quad\text{by (2.3).}$$

Letting $\rho\to\infty$ we obtain

$$\Theta_{\theta,\phi}(0)-\Theta_{\theta,\phi}(Y)\ge\int_{\Sigma_x}r^{2-m}|\nabla_{\Sigma_x^\perp}r|^2\theta\mathcal{H}^{m-2}\llcorner\Sigma_x+2\int_{R^2}r^{2-m}\Big|\frac{\partial\phi}{\partial r}\Big|^2\mathrm{d}y.$$

So we have

$$\Theta_{\theta,\phi}(0)\ge\Theta_{\theta,\phi}(Y).$$

The equality holds if and only if $\nabla_{\Sigma_x^\perp}r=0$ on Σ_x and $\frac{\partial\phi}{\partial r}=0$. So (i) holds. Similar to the proof of Corollary 2.2, choosing $X_{h,Y}=h(\omega_Y)X(r)$ for $\omega_Y=\frac{y-Y}{|y-Y|}$ and applying (2.1), we know

$$\rho^{2-m}\int_{B_\rho(Y)}h(\omega_Y)\theta\mathrm{d}\mathcal{H}^{m-2}\llcorner\Sigma_x=\text{ constant independent of }\rho,$$

which implies (ii) and (iii). This proves the lemma.

Assume (ϕ,V) is a stationary map-varifold pair from $\mathbb{R}^m$ to N satisfying the volume cone condition. We set

$$S(\phi,\theta)=\{Y\in R^m\mid\Theta_{\theta,\phi}(0)=\Theta_{\theta,\phi}(Y)\}.$$

Lemma 2.4 *Assume* (ϕ,V) *is a stationary map-varifold pair from* $\mathbb{R}^m$ *to* N *satisfying the volume cone condition. Then,* $S(\phi,\theta)$ *is a linear subspace of* R^m, $\phi(X+Y)=\phi(X)$, $\theta(X+Y)=\theta(X)$, *for all* $X\in R^m$, $Y\in S(\phi,\theta)$, *and* $\Sigma_x=S(\phi,\theta)\times\Sigma_x^\perp$ *for some cone* $\Sigma_x^\perp$ *in* $S^\perp(\phi,\theta)$.

Proof Since for any $X \in R^m$, any $\lambda > 0$, $\phi(Y+\lambda X) = \phi(Y+X)$, $\theta(Y+\lambda X) = \theta(Y+X)$, $\lambda(\Sigma_x - Y) = \Sigma_x - Y$ and $\phi(\lambda X) = \phi(X)$, $\theta(\lambda X) = \theta(X)$, $\lambda\Sigma_x = \Sigma_x$, we have

$$\begin{aligned}\phi(X) &= \phi(\lambda X) = \phi(Y + (\lambda X - Y)) = \phi(Y + \lambda^{-2}(\lambda X - Y))\\ &= \phi(\lambda(Y + \lambda^{-2}(\lambda X - Y))) = \phi(X + (\lambda - \lambda^{-1})Y),\\ \theta(X) &= \theta(\lambda X) = \theta(Y + (\lambda X - Y)) = \theta(Y + \lambda^{-2}(\lambda X - Y))\\ &= \theta(\lambda(Y + \lambda^{-2}(\lambda X - Y))) = \theta(X + (\lambda - \lambda^{-1})Y)\end{aligned}$$

and

$$\begin{aligned}\Sigma_x &= \lambda\Sigma_x = \lambda(\Sigma_x - Y) + \lambda Y = \lambda^{-1}(\Sigma_x - Y) + \lambda Y\\ &= \lambda^{-1}\Sigma_x - (\lambda^{-1} - \lambda)Y = \Sigma_x - (\lambda^{-1} - \lambda)Y.\end{aligned}$$

So, for any $\mu \in \mathbb{R}$, $X \in R^m$, $Y \in S(\phi,\theta)$, we have $\phi(X+\mu Y) = \phi(X)$, $\theta(X+\mu Y) = \theta(X)$ and $\Sigma_x = \Sigma_x - \mu Y$. It is clear that, if $Y, Z \in S(\phi,\theta)$, then $\phi(X+Y+Z) = \phi(X+Y) = \phi(X)$, $\theta(X+Y+Z) = \theta(X+Y) = \theta(X)$ for all $X \in R^m$ and $\Sigma_x - (Y+Z) = \Sigma - Y - Z = \Sigma - Z = \Sigma$. So, for $Y, Z \in S(\phi,\theta)$ and $\mu \in \mathbb{R}$, we have

$$\begin{aligned}&\Theta_{\theta,\phi}(\mu Y)\\ =&\lim_{\rho\to 0}\rho^{2-m}\int_{B_\rho(\mu Y)}(|\nabla\phi|^2(y)\mathrm{d}y + \theta(y)\mathrm{d}\mathcal{H}^{m-2}\llcorner\Sigma_x(y))\\ =&\lim_{\rho\to 0}\rho^{2-m}\int_{B_\rho(0)}(|\nabla\phi|^2(z)\mathrm{d}z + \theta(z)\mathrm{d}\mathcal{H}^{m-2}\llcorner\Sigma_x(z))\\ =&\Theta_{\theta,\phi}(0)\end{aligned}$$

and, for the same reason,

$$\Theta_{\theta,\phi}(Y+Z) = \Theta(\phi,\theta)(0),$$

which implies $S(\phi,\theta)$ is a linear space of R^m. Since $\Theta_{\theta,\phi}(Y) = \Theta_{\theta,\phi}(0) > 0$ for all $Y \in S(\phi,\theta)$, we have $S(\phi,\theta) \subset \Sigma_x$. For any $X \in \Sigma_x$, there is a unique orthogonal decomposition $X = X^\top + X^\perp \in S(\phi,\theta) \oplus S^\perp(\phi,\theta)$. Denote $\Sigma_x^\perp = \{X^\perp \mid X \in \Sigma\}$. Then $\Sigma_x \subset S(\phi,\theta) \times \Sigma_x^\perp$. On the other hand, for any $Y + X^\perp \in S(\theta,\phi) \times \Sigma_x^\perp$, by $X^\top - Y \in S(\phi,\theta)$, we know

$$Y + X^\perp = X - (X^\top - Y) \in \Sigma_x - (X^\top - Y) = \Sigma_x.$$

So we know $\Sigma_x = S(\phi,\theta) \times \Sigma_x^\perp$ and hence $\Sigma_x^\perp$ is a cone in $S^\perp(\phi,\theta)$. This proves the lemma.

With the compactness theorem for map-varifold pairs and the description of the symmetry, as the arguments for energy minimizing maps (see [17]), we can also introduce the stratification of $\mathrm{sing}(\mu) = \Sigma$ for $\mu = (u, V) \in \mathcal{E}(\varepsilon_0)$. For each $j = 0, 1, \cdots, m$, we define

$$\mathcal{S}_j = \{x \in \Sigma \mid \dim S(\phi, \theta) \le j \text{ for all tangent pair } (\phi, \underline{v}(\Sigma, \theta)) \text{ of } \mu \text{ at } x\}.$$

It is clear that

$$\mathcal{S}_0 \subset \mathcal{S}_1 \subset \cdots \subset \mathcal{S}_m.$$

By Lemma 2.4, we have

$$\mathcal{S}_{m-2} = \mathcal{S}_{m-1} = \mathcal{S}_m = \Sigma.$$

Proposition 2.1 *For each $j = 1, 2, \cdots, m-2$, the Hausdorff dimension of $\mathcal{S}_j$, $\dim \mathcal{S}_j \le j$. For each $\alpha > 0$, $\mathcal{S}_0 \cap \{x \mid \Theta(\mu, x) = \alpha\}$ is a discrete set.*

Proof The proof can be followed line by line from [17, pages 200–203], applying the compactness Theorem 2.1 for stationary map-varifold pairs instead of that for energy minimizing maps.

Set

$$\mathcal{S}_{-1} = \{x \in \mathcal{S}_{m-2} \setminus \mathcal{S}_{m-3} \mid \text{all tangent maps are constant}\}.$$

We have the following decomposition of the singular set.

Theorem 2.2 *Let M be a compact m-dimensional Riemannian manifold. Let N be a Riemannian manifold. Let $\mu = (u, V)$ be a stationary map-varifold pair from M to N. We have*

$$\mathrm{sing}(\mu) = \mathcal{S}_{m-3} \cup \mathcal{S}_{-1}.$$

Proof Since $\mathrm{sing}(\mu) = \mathcal{S}_{m-2}$ it suffices to show that $\mathcal{S}_{m-2} \setminus \mathcal{S}_{m-3} = \mathcal{S}_{-1}$. Suppose that $x \in \mathcal{S}_{m-2} \setminus \mathcal{S}_{m-3}$. Then there is a tangent map ϕ and a tangent cone (Σ_x, θ) at x such that $\dim S(\phi, \theta) = m - 2$. By Lemma 2.4, we know that $\theta \equiv$ constant on $S(\phi, \theta)$ and $\Sigma_x = S(\phi, \theta) \times \Sigma_x^{\perp}$ for some cone $\Sigma_x^{\perp}$ in the 2-dimensional subspace $S(\phi, \theta)^{\perp}$. But we know Σ_x has finite $\mathcal{H}^{m-2}$ measure in local. So, $\Sigma_x^{\perp}$ is discrete and must be $\{0\}$ since it is a cone. Thus $\underline{v}(\Sigma_x, \theta)$ is stationary, combining with the stationary property of μ, we can see that ϕ is stationary. It is clear that $\Theta_\phi(y) = \Theta_\phi(0)$ for any $y \in S(\phi, \theta)$ by Lemma 2.4. If $\phi \not\equiv$ constant, we have $0 \in \mathrm{sing}(\phi)$, hence $S(\phi, \theta) \subset \mathrm{sing}(\phi)$. However, by Bethuel's theorem [3] (also see [5]) we know that $H^{m-2}(\mathrm{sing}(\phi)) = 0$. So $\phi \equiv$ constant.

This proves the theorem.

Remark 2.2 Moser [11] has proved that the top-dimensional part of the singular set for stationary measures consists of points whose tangent measure are all constant times of a plane , i.e., the map part vanishes. Besides the such constant theorem for the top-dimensional stratum, the above theorem also tells that the singular set is of dimension $(m-3)$ out of $\mathcal{S}_{-1}$.

3 Stability

For Hong-Wang's stability condition (see [7]), we can similarly consider strongly stable map-varifold pair. However, we observe that such stability condition (3.1) on the map-varifold pair implies the vanishing of the varifold part, and hence under the stability condition the conception of stationary map-varifold pair reduced to the classical stationary harmonic map.

Definition 3.1 *Assume $\mu=(u,V)$ is a stationary map-varifold pair, we call μ A-strongly stable if the map part u of μ is strongly stable as defined in* [7]*, i.e., there exists $A>0$ such that for any smooth function ϕ on M with compact support, there holds*

$$\int_M |\nabla\phi|^2 \mathrm{d}x \geq A\int_M \phi^2 \mathrm{d}\overline{\mu}. \tag{3.1}$$

Remark 3.1 Assume $\mu_i \in \mathcal{E}(\varepsilon_0)$ is a sequence of A-strongly stable stationary map-varifold pair, and μ_i converges to $\mu\in\mathcal{E}(\varepsilon_0)$. Then μ is also A-strongly stable since $\overline{\mu}_i$ converges to $\overline{\mu}$ as Radon measures.

Proposition 3.1 *Assume $\mu=(u,V)\in\mathcal{E}(\varepsilon_0)$ is an A-strongly stable stationary map-varifold pair and $V=\underline{v}(\Sigma,\theta)$. Then $\mathcal{H}^{m-2}(\Sigma)=0$, u is stationary and $\Sigma=\mathrm{sing}(u)$.*

Proof Using the same argument as the case without varifold part in [7], we know $\mathrm{Cap}_2(\Sigma)=0$ and hence $\mathcal{H}^{m-2}(\Sigma)=0$. This implies V is stationary and hence u is stationary. Thus by Bethuel's small energy regularity theorem, we know for any $x\in\Sigma\setminus \mathrm{sing}\ (u)$, there holds $\Theta_u(x)=0$ and hence $\Theta(\mu,x)=0$, i.e., $x\notin\Sigma$. This implies $\Sigma\setminus\mathrm{sing}(u)=\emptyset$ and hence

$$\mathrm{sing}(u)=\Sigma.$$

Corollary 3.1 *Let M be a compact m-dimensional Riemannian manifold. Let N be a Riemannian manifold. If u is a strongly stable stationary harmonic map from M to N, then*

$$\mathrm{sing}(u)=\mathcal{S}_{m-3}.$$

Proof If $\mathcal{S}_{-1}\neq\emptyset$, then there exits $x_0\in\mathcal{S}_{-1}\subset\mathrm{sing}(u)$ and $r_i\to 0$ such that

$$u_{x_0,r_i}\to\mu=(\phi,V),$$

where $V = \underline{v}(\Sigma, \theta)$. Since $x_0 \in \mathcal{S}_{-1}$, we have ϕ =constant. The stability of u_{x_0,r_i} implies μ is strongly stable. By Proposition 3.1, we know $\mathcal{H}^{m-2}(\Sigma) = 0$. So, Corollary 2.4 implies u_{x_0,r_i} converges to ϕ = constant strongly in $W^{1,2}$, which further implies

$$\Theta(u, x_0) \leq \lim_{r \to 0} \lim_{i \to \infty} \Theta(u_{x_0,r_i}, 0, r) = 0.$$

This contradicts to $x_0 \in \mathrm{sing}(u)$. So, we conclude $\mathcal{S}_{-1} = \emptyset$ and $\mathrm{sing}(u) = \mathcal{S}_{m-3}$ follows from Theorem 2.2.

Corollary 3.2 *Let M be a compact 3-dimensional Riemannian manifold. Let N be a compact analytic manifold with a analytic metric of positive sectional curvature. Let u be a stable stationary harmonic map from M to N. Then the singular set of u consists of at most finitely many points.*

Proof By Corollary 3.1, we know that $\mathrm{sing}(u) = \mathcal{S}_0$, by Proposition 2.1, we know that $\mathcal{S}_0 \cap \{x \mid \Theta_u(x) = \alpha\}$ is a discrete set. Note that $\Theta_u(x) = E(\phi)$, where ϕ is a harmonic map from S^2 to N. Because N is an analytic manifold with analytic metric, $E(\phi)$ takes discrete values. So, we know that $\mathcal{S}_0$ is a discrete set. This proves the corollary.

Declarations

Conflicts of interest The authors declare no conflicts of interest.

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