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CONSTRUCTION OF UNITONS VIA PURELY ALGEBRAIC ALGORITHM**

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Abstract

A purely algebraic algorithm for constructing unitons is presented. It is shown that all unitons can be constructed by using this algorithm.

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§1. Introduction

Harmonic maps from R^2 (or its simply connected region Ω) to $U(N)$ are considered extensively. The conception of unitons are introduced by K. Uhlenbeck[7]. There are a series of papers[2,8,9] devoted to the construction of unitons. In our paper[4] the Darboux transformation has been used to construct harmonic maps and unitons via purely algebraic algorithm. Here the purely algebraic algorithm contains operations in linear algebra only, and no other operations such as differentiation and integral transformation are involved.

However, the formula for unitons in the previous paper[4] is too complicated, hence it is not convenient for applications. Further, it has not been proved that the construction exhausts all unitons. In the present paper we use a kind of renormalization procedure to get a purely algebraic formula for unitons which is much easier to be worked out and it is proved that all the unitons can be obtained by using this formula, starting with single unitons which have been constructed explicitly in [7]. It is noted that we keep our notations in previous papers[4−6]. There are differences in notations in comparison with [7] and related papers, in particular, the order of multiplication of matrices in the Lax equations etc. is reversed. For simplicity, we state the results for harmonic map from R^2 instead of simply-connected region $\Omega \subseteq R^2$, although they hold true for Ω as well as for R^2.

In §2 a brief review of fundamental facts is given. In §3 we use Darboux transformation to establish a purely algebraic formula for unitons. In §4 we prove that all unitons can be obtained by using the formula successively.

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§2. Harmonic Maps to $U(N)$ and Darboux Transformations

Let $\zeta, \bar{\zeta}$ be complex coordinates of R^2, i.e.

$$\zeta = \frac{1}{2}(x+iy), \quad \bar{\zeta} = \frac{1}{2}(x-iy) \tag{2.1}$$

with

$$\frac{\partial}{\partial \zeta} = \frac{\partial}{\partial x} - i\frac{\partial}{\partial y}, \quad \frac{\partial}{\partial \bar{\zeta}} = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}. \tag{2.2}$$

Consider a C^∞-map $\phi : R^2 \to U(N)$. Define

$$A = \frac{\partial \phi}{\partial \bar{\zeta}}\phi^{-1} = \phi_{\bar{\zeta}}\phi^{-1}, \quad B = \phi_\zeta \phi^{-1}. \tag{2.3}$$

Then A and B satisfy

$$A_\zeta - B_{\bar{\zeta}} + [A, B] = 0 \tag{2.4}$$

$$A^* + B = 0. \tag{2.5}$$

Moreover, if A and B satisfy

$$A_\zeta + B_{\bar{\zeta}} = 0, \tag{2.6}$$

then ϕ is called harmonic maps from R^2 to $U(N)$. Note that (2.4) is the integrability condition of (2.3) which is considered as a linear system of equations for ϕ. (2.5) implies $\phi \in U(N)$ if it holds at a fixed point, say $\zeta = \bar{\zeta} = 0$.

The Lax pair of harmonic maps from R^2 to $U(N)$ is

$$\Phi_{\bar{\zeta}} = \frac{1-\mu}{2}A\Phi, \quad \Phi_\zeta = \frac{1-\mu^{-1}}{2}B\Phi. \tag{2.7}$$

The integrability conditions of (2.7) are (2.4) and (2.5). A solution $\Phi(\mu)\,(\mu \neq 0)$ with $\det \Phi \neq 0$ is called extended harmonic map. In particular $\Phi(-1) = \phi\cdot$constant matrix, and ϕ is a harmonic map to $U(N)$.

Darboux transformation is a constructive method for obtaining new extended harmonic maps (and hence new harmonic maps) from a known extended harmonic map via purely algebraic algorithm. We sketch it as follows.

Let $\Phi(\mu)$ be an extended harmonic map and ϵ be a complex number ($\epsilon \neq 0, |\epsilon| \neq 1$), L_1 and L_2 be $N \times k$ and $N \times (N-k)$ constant matrices respectively such that

$$H = [\Phi(\epsilon)L_1,\ \Phi(\bar{\epsilon}^{-1})L_2] \tag{2.8}$$

is nondegenerated and

$$(\Phi(\bar{\epsilon}^{-1})L_2)^*\Phi(\epsilon)L_1 = 0. \tag{2.9}$$

It is shown in [4] that if (2.8) and (2.9) hold at a point of R^2, then they hold everywhere.

Let

$$\omega_1 = \frac{1-\epsilon}{2}, \quad \omega_2 = \frac{1-\bar{\epsilon}^{-1}}{2}, \tag{2.10}$$

$$\Lambda = \mathrm{diag}(\omega_1, \cdots, \omega_1; \omega_2, \cdots, \omega_2) \quad \text{(with } k\ \omega_1\text{'s and } N-k\ \omega_2\text{'s)}, \tag{2.11}$$

$$S = H\Lambda^{-1}H^{-1}. \tag{2.12}$$

We proved that

$$\Phi_1(\mu) = \left(I - \frac{1-\mu}{2}S\right)\Phi(\mu) \tag{2.13}$$

is a new extended harmonic map. The transformation $\Phi(\mu) \to \Phi_1(\mu)$ is called Darboux transformation and $D = I - \frac{1-\mu}{2}S$ Darboux matrix. In [4] we defined

$$\pi = H\begin{bmatrix} 0 & 0 \\ 0 & I_{N-k} \end{bmatrix}H^{-1}, \quad \pi^{\perp} = I - \pi = H\begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}H^{-1} \tag{2.14}$$

and (2.13) can be written as

$$\Phi_1(\mu) = (\pi + r\pi^{\perp})\Phi(\mu)\Big(1 - \frac{1-\mu}{1-\bar{\epsilon}^{-1}}\Big) \quad \Big(r = \frac{(\bar{\epsilon}-1)(\epsilon-\mu)}{(1-\epsilon)(1-\bar{\epsilon}\mu)}\Big). \tag{2.15}$$

We proved that

$$\pi^2 = \pi, \quad \pi^* = \pi, \tag{2.16}$$

$$\pi_{\bar{\zeta}} = -\omega_1\pi A + \omega_2 A\pi + (\omega_1 - \omega_2)\pi A\pi, \quad \pi_{\zeta} = \bar{\omega}_1 B\pi - \bar{\omega}_2\pi B + (\bar{\omega}_2 - \bar{\omega}_1)\pi B\pi. \tag{2.17}$$

(2.17) appeared firstly in [7] and it is the differential equation for the Bäcklund transformation for harmonic maps from R^2 to $U(N)$. In [4] it is proved that the Hermitian projection π defined by (2.14) is an explicit solution to the equations (2.17).

Remark 2.1. It is easily seen that (2.17) is completely integrable and π is determined by its value at a given point. Thus (2.14) exhausts all solutions of (2.17).

Remark 2.2. Let $S' = (\omega_2^{-1}\pi^{\perp} + \omega_1^{-1}\pi)$. we have

$$(I - \lambda S')(I - \lambda S) = (1 - \lambda(\omega_1^{-1} + \omega_2^{-1}) + \lambda^2\omega_1^{-1}\omega_2^{-1})I, \quad \lambda = \frac{1-\mu}{2}.$$

Hence $I - \lambda S'$ provides the inverse of the Darboux transformation (2.13) and $\pi^{\perp}, \pi$ can be constructed from $\Phi_1(\mu)$ via the formulas similar to (2.14), Thus we conclude that the inverse of a Darboux transformation (2.13) for harmonic maps from R^2 to $U(N)$ is a Darboux transformation too.

An extended harmonic map $\Phi(\mu)$ satisfying

$$\begin{aligned} &\text{(a)} \quad \Phi(1) = I; && \text{(b)} \quad \Phi(-1) = \phi \in U(N); \\ &\text{(c)} \quad \Phi(\mu) = \sum_{a=0}^{r} T_a\mu^a; && \text{(d)} \quad \Phi(\mu)^*\Phi(\bar{\mu}^{-1}) = I \end{aligned} \tag{2.18}$$

is called extended uniton and ϕ is called uniton (see [7]). It is known that a single extended uniton has expression

$$\Phi(\mu) = \pi + \mu\pi^{\perp}. \tag{2.19}$$

Here π is Hermitian projection to k planes of C^N which constitute an antiholomophic section of the trivial boundle $G_k(N) \times R^2$.

It is also known[7] that

$$\Phi_1(\mu) = (\pi + \mu\pi^{\perp})\Phi(\mu) \tag{2.20}$$

is an extended uniton iff π satisfies

$$(2\pi_{\bar{\zeta}} + \pi A)\pi^{\perp} = 0, \quad \pi^{\perp}A\pi = 0 \tag{2.21}$$

provided that $\Phi(\mu)$ is an extended uniton. (2.21) is the limit of (2.17) as $\epsilon \to 0$. The transformation (2.20) $\Phi(\mu) \to \Phi_1(\mu)$ is also called flag transformation[9].

§3. Algebraic Construction of Flag Transformations

Let $\Phi(\mu)$ be an extended uniton. We construct its Darboux transformation (2.13). Instead of (2.15) and (2.8) we use the notation

$$\Phi_{\epsilon}(\mu) = (\pi_{\epsilon} + r_{\epsilon}\pi_{\epsilon}^{\perp})\Phi(\mu)\Big(1 - \frac{1-\mu}{1-\bar{\epsilon}^{-1}}\Big) \tag{3.1}$$

$$H_{\epsilon} = [\Phi(\epsilon)L_1, \Phi(\bar{\epsilon}^{-1})L_2]. \tag{3.2}$$

Here π_ϵ is defined by (2.14) with $H = H_\epsilon$. Without loss of generality we assume that $[L_1, L_2] \in U(N)$ and $\Psi(\epsilon)L_1$ is of rank k everywhere. We also have

$$L_2^*\Phi(\bar{\epsilon}^{-1})^*\Phi(\epsilon)L_1 = 0. \tag{3.3}$$

We can calculate H_ϵ^{-1} explicitly

$$H_\epsilon^{-1} = \begin{bmatrix} C_1(\epsilon)L_1^*\Phi(\epsilon)^* \\ C_2(\epsilon)L_2\Phi(\bar{\epsilon}^{-1})^* \end{bmatrix}, \tag{3.4}$$

where $C_1(\epsilon)$ and $C_2(\epsilon)$ are $k \times k$ and $(N-k)\times(N-k)$ matrices defined by

$$C_1(\epsilon)L_1^*\Phi(\epsilon)^*\Phi(\epsilon)L_1 = I_k, \quad C_2(\epsilon)L_2^*\Phi(\bar{\epsilon}^{-1})^*\Phi(\bar{\epsilon}^{-1})L_2 = I_{N-k} \tag{3.5}$$

respectively. Then

$$\pi_\epsilon = H_\epsilon \begin{bmatrix} 0 & 0 \\ 0 & I_{N-k} \end{bmatrix} H_\epsilon^{-1} = \Phi(\bar{\epsilon}^{-1})L_2C_2(\epsilon)L_2^*\Phi(\bar{\epsilon}^{-1})^*,$$

$$\pi_\epsilon^\perp = H_\epsilon \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} H_\epsilon^{-1} = \Phi(\epsilon)L_1C_1(\epsilon)L_1^*\Phi(\epsilon)^*. \tag{3.6}$$

The Darboux matrix is

$$D_\epsilon = (\pi_\epsilon + r_\epsilon\pi_\epsilon^\perp)\left(\frac{\mu - \bar{\epsilon}^{-1}}{1 - \bar{\epsilon}^{-1}}\right) \tag{3.7}$$

with

$$r_\epsilon = \frac{(\bar{\epsilon} - 1)(\mu - \epsilon)}{(1-\epsilon)(\bar{\epsilon}\mu - 1)} \quad (\to \mu \quad \text{as} \quad \epsilon \to 0). \tag{3.8}$$

Consider the limit of $\pi_\epsilon^\perp$ (resp. π_ϵ) as $\epsilon \to 0$. At first we see that the k columns of $\Phi(\epsilon)L_1$ are invariant vectors of $\pi_\epsilon^\perp$. In fact

$$\pi_\epsilon^\perp\Phi(\epsilon)L_1 = \Phi(\epsilon)L_1C_1(\epsilon)L_1^*\Phi(\epsilon)^*\Phi(\epsilon)L_1 = \Phi(\epsilon)L_1. \tag{3.9}$$

Denote the image of $\pi_\epsilon^\perp$ which are spanned by the k columns of $\Phi(\epsilon)L_1$ by $P_k(\epsilon)$.

The expansion of $\Phi(\epsilon)L_1$ as a polynomial of ϵ is denoted by

$$\Phi(\epsilon)L_1 = X^0 = X_0^0 + X_1^0\epsilon + \cdots + X_n^0\epsilon^n, \tag{3.10}$$

where $X_0^0, \cdots, X_n^0$ are $N \times k$ matrices. Choose some columns of X^0 to constitute a matrix

$$\tilde{X}^1 = \tilde{X}_0^1 + \tilde{X}_1^1\epsilon + \cdots + \tilde{X}_n^1\epsilon^n \tag{3.11}$$

such that $\tilde{X}_0^1$ consists of linearly independent columns of X_0^0 and all other columns of X_0^0 are linear combinations of them.

The other column of X^0, after subtracting suitable linear combinations of the columns of $\tilde{X}^1$ from them, constitute a matrix of the form

$$\epsilon\tilde{\tilde{X}}^1 = \epsilon(\tilde{\tilde{X}}_0^1 + \tilde{\tilde{X}}_1^1\epsilon + \cdots). \tag{3.12}$$

Let

$$X^1 = [\tilde{X}^1, \tilde{\tilde{X}}^1]. \tag{3.13}$$

X^1 is of rank k and its columns span the plane $P_k(\epsilon)$ too. Using the same transformations as X^0 to X^1 successively, we obtain a series of matrices

$$X^2 = X_0^2 + X_1^2\epsilon + \cdots + X_n^2\epsilon^n, \quad X^3 = X_0^3 + X_1^3\epsilon + \cdots + X_n^3\epsilon^n, \quad \cdots.$$

Finally, we obtain a matrix

$$X = X_0 + X_1\epsilon + \cdots + X_n\epsilon^n \tag{3.14}$$

such that X_0 is of rank k and the columns of X span $P_k(\epsilon)$. Evidently, the columns of X_0 span a plane P_k which is the limit of $P_k(\epsilon)$ as $\epsilon \to 0$. Consequently,

$$\pi^\perp = X_0(X_0^*X_0)^{-1}X_0^* \tag{3.15}$$

is the limit of $\pi_\epsilon^\perp$, hence $\pi = I - \pi^\perp$ is the limit of π_ϵ and (2.21) is satisfied by π. We have

Theorem 3.1 *The algebraic formula*

$$\pi^\perp = X_0(X_0^* X_0)^{-1} X_0^*, \quad \pi = I - \pi^\perp \tag{3.16}$$

gives a solution to (2.21) *and* $\Phi_1(\mu) = (\pi + \mu\pi^\perp)\Phi(\mu)$ *is an extended uniton.*

Remark 3.1. The procedure to obtain $\pi^\perp$ and π from $\Phi(\epsilon)L_1$ (or $\Phi(\bar{\epsilon}^{-1})L_2$) is purely algebraic. In order to obtain $\Phi_1(\mu)$ we use $\Phi(\mu)$ only, i.e., in the algorithm, we do not need A, B or the differential of $\Phi(\mu)$.

Remark 3.2. As $\epsilon \to 0$, $C_1(\epsilon)$(or $C_2(\epsilon)$) may not have a regular limit, while the limit of $\pi_\epsilon^\perp$ (or π_ϵ) exists. The procedure to obtain X_0 from $\Phi(\epsilon)L_1$ is a kind of renormalization.

Remark 3.3. By using (3.14) we have $\pi_\epsilon^\perp = X(X^*X)^{-1}X^*$. As $\epsilon \to 0$, X approaches to X_0 and X_0 is of rank k. Hence $\pi_{\epsilon\bar{\zeta}}^\perp$ approaches to $\pi_{\bar{\zeta}}^\perp$.

§4. Construction of all Unitons

In this section we prove the following theorem

Theorem 4.1. *All unitons can be constructed from single unitons via purely algebraic algorithm.*

We need following lemmas

Lemma 4.1. *Let* $\Phi(\mu)$ *be a polynomial of* μ *of degree* n*, valued in* $GL(N, C)$*. If* $\Phi(\mu)$ *satisfies* (i) $\Phi(1) = I$, (ii) $\Phi(\mu)^*\Phi(\bar{\mu}^{-1}) = I$ $(\mu \neq 0)$, *then*

$$\Phi(\mu) = (\pi_1 + \mu\pi_1^\perp)\cdots(\pi_n + \mu\pi_n^\perp), \tag{4.1}$$

where $\pi_1, \cdots, \pi_n$ *are Hermitian projections and* $\pi_1^\perp, \cdots, \pi_n^\perp$ *their complementary.*

Proof. For $n = 1$, it is proved in [7]. For general n, it can be proved by induction as follows. Suppose that Lemma 4.1 holds for n. Let $\Phi(\mu) = \sum_{a=0}^{a=n+1} T_a\mu^a$ satisfy (i), (ii). We assume $T_0 \neq 0, T_{n+1} \neq 0$, otherwise, the conclusion follows immediately. We note that $\det\Phi(\mu) \neq 0$ for $\mu \neq 0$. In fact, if $\det\Phi(\mu_0) = 0$ $(\mu_0 \neq 0)$, then $\Phi(\mu_0)^*\Phi(\bar{\mu}_0^{-1}) = I$ is impossible. Thus $\det\Phi(\mu) = 0$ iff $\mu = 0$. Hence $\det T_0 = 0$. Let P be the kernel of T_0 and $\pi_{N+1}^\perp$ be the Hermitian projection on P. Then $T_0\pi_{n+1}^\perp = 0$ and $\tilde{\Phi}(\mu) = \Phi(\mu)(\pi_{n+1} + \mu^{-1}\pi_{n+1}^\perp)$ is a polynomial of degree n. Evidently, $\tilde{\Phi}(\mu)$ satisfies (i), (ii) and $\Phi(\mu) = \tilde{\Phi}(\mu)(\pi_{n+1} + \mu\pi_{n+1}^\perp)$. Hence $\Phi(\mu)$ admits a factorization with $(n+1)$ factors, since the factorization of $\tilde{\Phi}(\mu)$ holds from the hypothesis of induction.

Remark 4.1. Lemma 4.1 is an algebraic proposition, since we do not assume that $\Phi(\mu)$ to be an extended uniton. For extended unitons, the factorization is established in [7].

Lemma 4.2. *If* $\Phi(\mu)$ *is an extended uniton of degree* n*, then there exists an Hermitian projection* π*, such that*

$$\Phi_1(\mu) = (\pi + \mu\pi^\perp)\Phi(\mu) \tag{4.2}$$

is an extended uniton whose degree is at most n*.*

Proof. By Lemma 4.1, we have

$$\Phi(\mu) = T_0 + T_1\mu + \cdots + T_n\mu^n = (\pi_1 + \mu\pi_1^\perp)\cdots(\pi_n + \mu\pi_n^\perp). \tag{4.3}$$

Hence

$$T_0 = \pi_1\cdots\pi_n, \quad T_n = \pi_1^\perp\cdots\pi_n^\perp. \tag{4.4}$$

Let rank $T_0 = k$ and take $[L_0, L_1] \in U(N)$ such that $\mathrm{rank}(T_0L_1) = k$. Hence $\mathrm{rank}\Phi(0)L_1 = k$. From the definition of $C(\epsilon)$, we see that $\lim_{\epsilon\to 0} C(\epsilon) = C(0)$ is a regular matrix

$$C(0) = (L_1^*T_0^*T_0L_1)^{-1} \tag{4.5}$$

and hence

$$\pi^{\perp} = T_0 L_1 C_1(0) L_1^* T_0^*. \tag{4.6}$$

On the other hand,

$$\pi^{\perp} T_n = T_0 L_1 C(0) L_1^* T_0^* T_n = T_0 L_1 C(0) L_1^* \pi_n^* \cdots \pi_1^* \pi_1^{\perp} \cdots \pi_n^{\perp} = 0. \tag{4.7}$$

The degree of $\Phi_1(\mu)$ cannot greater than n.

Lemma 4.3. *If $\Phi(\mu)$ is an extended uniton of degree n $(n > 1)$, then there exists an Hermitian projection σ such that*

$$\Phi(\mu) = (\sigma + \mu\sigma^{\perp})\Phi_{-1}(\mu) \tag{4.8}$$

and $\Phi_{-1}(\mu)$ is an extended uniton of degree $\le n-1$.

Proof.

$$\Phi_{-1}(\mu) = \Big(\sigma + \frac{1}{\mu}\sigma^{\perp}\Big)\Phi(\mu). \tag{4.9}$$

By calculations it is seen that $\Phi_{-1}(\mu)$ is an extended uniton iff σ satisfies

$$\sigma^{\perp} A \sigma - 2\sigma^{\perp}\sigma_{\bar\zeta} = 0, \quad \sigma A \sigma^{\perp} = 0. \tag{4.10}$$

It is easily verified that if π is a solution to (2.21), then $\sigma = \pi^{\perp}$ $\quad(\sigma^{\perp} = \pi)$ is a solution to (4.10). Hence $\Phi_{-1}(\mu) = \big(\pi^{\perp} + \frac{1}{\mu}\pi\big)\Phi(\mu)$ is an extended uniton. We take the Hermitian projection π of Lemma 4.2, then

$$\Phi_1(\mu) = (\pi + \mu\pi^{\perp})\Phi = (\pi + \mu\pi^{\perp})(\pi^{\perp} + \mu\pi)\Phi_{-1}(\mu) = \mu\Phi_{-1}(\mu).$$

Hence the degree of $\Phi_{-1}(\mu)$ is less than n.

Thus, by using the procedure of Lemma 4.3, any given extended uniton can be reduced to an extended uniton of lower degree via purely algebraic algorithm. Continuing the algorithm successively, we will reach a single uniton. From Remark 2.1 it can be seen that $\pi^{\perp}$, π can be constructed from $\Phi_{-1}(\mu)$ by the algebraic algorithm in Section 5. Hence starting from single unitons we can obtain all unitons via purely algebraic algorithm. The proof of the theorem 4.1 is completed.

The study of $U(N)$ unitons has been generalized to compact symmetric spaces[1]. It is interesting to find an algebraic algorithm for obtaining these unitons. The method of this paper has been already extended to the study of unitons of the noncompact group $U(p, q)$ (see [3]).

References

[1] Burstall, F. E. & Guest, M. A., Harmonic two-spheres in compact symmetric space, revisited, *Math. Ann.,* **309**(1997), 541–572.

[2] Dong Yuxin & Shen Yibin, Factorization and uniton numbers for harmonic maps into the unitary group $U(N)$, *Science in China(Series A),* **39**(1996), 589–597.

[3] Gu Chaohao, Unitons of harmonic maps from R^2 to $U(p, q)$, to appear in *Letters in Math. Phys.*

[4] Gu Chaohao & Hu Hesheng, The soliton behavior of the principal chiral fields, *Int. J. Mod. Phys A,* (Proc. of Inter. Conf. or Diff. Geom. Method in Theor. Phys., Tianjin 1992), **3**(1997), 501–510.

[5] Gu Chaohao & Hu Hesheng, Explicit construction of harmonic maps from R^2 to $U(N)$, *Chin. Ann. of Math.,* **16B:**2(1995), 139–152.

[6] Gu Chaohao & Zhou, Z. X., Darboux transformations for principal chiral fields, Proc. of Inter. Conf. "Nonlinear Evolution Equation Como, 1988", Manchester Univ. Press 1990.

[7] Uhlenbeck, K., Harmonic maps onto Lie groups (classical solutions of the chiral model), *J. Diff. Geom.,* **30**(1989), 1–50.

[8] Valli, G., On the energy spectrum of harmonic 2-spheres in unitary groups, *Topology,* **27**(1988), 129–136.

[9] Wood, J. C., Explicit construction and parametrization of harmonic maps in the unitary group, *Proc. London Math. Soc.,* **58**(1989), 608–624.