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ON THE CONVERGENCE OF THE PARABOLIC APPROXIMATION OF A CONSERVATION LAW IN SEVERAL SPACE DIMENSIONS

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Abstract

The authors give a proof of the convergence of the solution of the parabolic approximation $u_t^\varepsilon + \mathrm{div} f(x,t,u^\varepsilon) = \varepsilon\Delta u^\varepsilon$ towards the entropic solution of the scalar conservation law $u_t + \mathrm{div} f(x,t,u) = 0$ in several space dimensions. For any initial condition $u_0 \in L^\infty(\mathbb{R}^N)$ and for a large class of flux f, they also prove the strong converge in any L^p_{loc} space, using the notion of entropy process solution, which is a generalization of the measure-valued solutions of DiPerna.

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§1. Introduction

We give a proof of the strong convergence in $L^p_{\mathrm{loc}}, p<\infty$ of the solution of the parabolic approximation

$$u_t^\varepsilon + \mathrm{div} f(x,t,u^\varepsilon) = \varepsilon\Delta u^\varepsilon,\ x\in\mathbb{R}^N,\ t>0, \tag{1.1}$$

$$u^\varepsilon(x,0) = u_0^\varepsilon(x),\ x\in\mathbb{R}^N \tag{1.2}$$

towards the entropic solution to the scalar conservation law

$$u_t + \mathrm{div} f(x,t,u) = 0,\ x\in\mathbb{R}^N,\ t>0, \tag{1.3}$$

$$u(x,0) = u_0(x),\ x\in\mathbb{R}^N, \tag{1.4}$$

where $u_0\in L^\infty(\mathbb{R}^N)$, u_0^ε denotes some approximation of u_0 such that

$$\begin{cases} \|u_0^\varepsilon\|_{L^\infty} \le M, \\ u_0^\varepsilon \to u_0 \text{ strongly in } L^1_{\mathrm{loc}}(\mathbb{R}^N); \end{cases} \tag{1.5}$$

and the flux f satisfies

$$\begin{cases} f\in\mathcal{C}^1(\mathbb{R}^N\times\mathbb{R}^+\times\mathbb{R}), \\ \mathrm{div}_x f(x,t,s) = 0, \quad \forall(x,t,s)\in\mathbb{R}^N\times\mathbb{R}^+\times\mathbb{R}, \\ \frac{\partial f}{\partial s}\in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^N\times\mathbb{R}^+\times\mathbb{R}). \end{cases} \tag{1.6}$$

The convergence of the approximate solutions (u^ε) of (1.1),(1.2) towards the entropic solution u of the hyperbolic equation with initial condition (1.3),(1.4) has been studied in several way. In 1970, Kruzkov[6] gave a proof of the convergence using the L^1_{loc}-compactness of the approximate solutions (see also [9]). Our proof follows ideas of Tartar[11] and DiPerna[2].

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For any uniformly bounded initial data $u_0^\varepsilon \in L^\infty(\mathbb{R}^n)$ ($\|u_0^\varepsilon\|_{L^\infty(\mathbb{R}^N)} \le M$), it is classical to see that the parabolic approximation (1.1), (1.2) admits a smooth solution, at least $\mathcal{C}^2(\mathbb{R}^N \times (0,+\infty))$, which is uniformly bounded : $\|u^\varepsilon\|_{L^\infty(\mathbb{R}^N\times\mathbb{R}^+)} \le M$ (see [8, 10]).

Thanks to this L^∞ estimate, as in Tartar's work, one can construct a Young measure as the limit of a sequence of approximate solutions (u^ε) : for all $g \in \mathcal{C}^0(\mathbb{R}^N \times \mathbb{R}^+ \times \mathbb{R})$ and for all $\varphi : \mathbb{R}^N \times [0,\infty) \to \mathbb{R}$ compactly supported, $\varphi \in L^1(\mathbb{R}^N \times [0,\infty))$,

$$\int_0^\infty \int_{\mathbb{R}^N} g(x,t,u^\varepsilon(x,t))\,\varphi(x,t)\,dx \to \int_0^\infty \int_{\mathbb{R}^N} \langle g(x,t,\lambda), \nu_{x,t}(\lambda)\rangle\,\varphi(x,t)\,dx. \qquad (1.7)$$

This result is well-known for continuous functions on $\mathbb{R}$ (g does not depend on $(x,t) \in \mathbb{R}^N \times \mathbb{R}^+$) and has been generalized to functions in $\mathcal{C}^0(\mathbb{R}^N \times \mathbb{R}^+ \times \mathbb{R})$ in the work of [1] by using "process functions", which were introduced by Eymard-Gallouët-Herbin in [3].

Theorem 1.1. *Let $(u_n)_{n\in\mathbb{N}}$ be bounded in $L^\infty(\mathbb{R}^N \times \mathbb{R}^+)$, there exists a function $\mu \in L^\infty(\mathbb{R}^N \times \mathbb{R}^+ \times (0,1))$ and a subsequence, still denoted by (u_n), such that u_n converges towards μ in a "nonlinear weak-$\star$ sense" as $n \to \infty$, that is for any $g \in \mathcal{C}^0(\mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^N)$, and for any $\varphi \in L^1(\mathbb{R}^N \times \mathbb{R}^+)$ compactly supported*

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}^N} g(x,t,u_n(x,t))\,\varphi(x,t)\,dx\,dt \to \int_{\mathbb{R}^+}\int_{\mathbb{R}^N}\int_0^1 g(x,t,\mu(x,t,\alpha))\,\varphi(x,t)\,d\alpha\,dx\,dt. \qquad (1.8)$$

We call the function μ "process function".

Remarks.

(i) To each "process function" μ can be associated a Young measure family $\nu_{x,t}$ which satisfies (1.7) and such that

$$\int_0^1 g(x,t,\mu(x,t,\alpha))\,d\alpha = \langle \nu_{x,t}(\lambda), g(x,t,\lambda)\rangle \ \text{ a.e.}(x,t). \qquad (1.9)$$

(ii) Conversely, to each Young measure family $\nu_{x,t}$, associated to a bounded sequence (u_n), we can associate a "process function" which verifies (1.9).

(iii) Let u be in $L^\infty(\mathbb{R}^N \times \mathbb{R}^+)$. We suppose that "$u_n \to \mu$ " and so that "$u_n \to \nu_{x,t}$", then u_n in $L^p_{\text{loc}} \Leftrightarrow \mu(x,t,\alpha) = u(x,t)$ a.e. $(x,t,\alpha) \Leftrightarrow \nu_{x,t} = \delta_{u(x,t)}$ a.e. (x,t).

Tartar[11] shows, using compensated compactness, that in one space dimension, the Young measure $\nu_{x,t}$ is supported in an interval on which f is affine, if u_0^ε converges weakly to u_0 in L^1_{loc}.

In the next section, we prove that if u_0^ε converge strongly towards u_0, then for all flux f satisfying (1.6), $\nu_{x,t}$ is a Dirac measure at the point $u(x,t)$ for a.e. (x,t) and by the way the familly u^ε converges towards u, the unique entropic solution of the hyperbolic problem (1.3), (1.4).

For that, we prove that a "process function" μ associated to the familly (u^ε) is an "entropy process solution" of the hyperbolic problem (1.3), (1.4) (see [1, 3])

Definition 1.1. *A function $\mu \in L^\infty(\mathbb{R}^N \times \mathbb{R}^+ \times (0,1), \mathbb{R})$ is called "entropy process solution" of the hyperbolic problem* (1.3),(1.4) *if, for all $\varphi \in \mathcal{C}^1_c(\mathbb{R}^N \times \mathbb{R}^+, \mathbb{R}^+)$, and for all $\kappa \in \mathbb{R}$,*

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}^N}\int_0^1 [\eta_\kappa(\mu(x,t,\alpha))\varphi_t(x,t) + q_\kappa(x,t,\mu(x,t,\alpha))\nabla\varphi(x,t)]\,d\alpha\,dx\,dt$$
$$+ \int_{\mathbb{R}^N} \eta(u_0(x))\varphi(x,0)\,dx \ge 0,$$

where

$$\eta_\kappa(\mu) = \mid \mu - \kappa \mid \quad \text{and} \quad q_\kappa(x,t,\mu) = (f(x,t,\mu\top\kappa) - f(x,t,\mu\bot\kappa)), \tag{1.10}$$

with $a\top b = \max(a,b)$ *and* $a\bot b = \min(a,b)$.

We can check that the Young measure associated to an "entropy process solution" is a "measure valued solution" of the hyperbolic equation (1.3) in the sense of DiPerna[2] which satisfies the initial condition in the sense:

$$\lim_{\tau\to 0}\int_0^\tau \int_K \langle \nu_{x,t}(\lambda), |\lambda - u_0(x)|\rangle dxdt = 0, \quad \text{for all compact } K \subset \mathbb{R}^N. \tag{1.11}$$

To conclude, we use a generalization of the theorem of uniqueness of measure valued solutions of DiPerna.

Theorem 1.2. *If* μ *is an "entropy process solution" of the hyperbolic equation with initial condition* (1.3), (1.4), *then* $\mu(x,t,\alpha) = u(x,t)$ *for almost every* $(x,t,\alpha) \in \mathbb{R}^N \times \mathbb{R}^+ \times (0,1)$, *where* u *denotes the unique entropic solution of* (1.3), (1.4).

This result, first proved in [3] and then extended in [1,4,5], enables us to deduce that the parabolic approximation (1.1), (1.2) converges towards the entropic solution of (1.3), (1.4). This convergence is, moreover, strong in each L^p_{loc}-space for $p < \infty$.

§2. Convergence of the Parabolic Approximation towards the Entropic Solution of the Conservation Law

2.1. Convergence towards the "Entropy Process Solution"

Theorem 2.1. *Under Assumptions* (1.5), (1.6), *the solution* u^ε *of the parabolic approximation* (1.1), (1.2) *converges in the "nonlinear weak-$\star$ sense" towards an "entropy process solution" of the hyperbolic problem* (1.3), (1.4).

Sketch of Proof. Under Assumptions (1.5), (1.6), the problem (1.1), (1.2) has a smooth bounded solution $u^\varepsilon \in L^\infty(\mathbb{R}^N \times (0,+\infty)) \cap \mathcal{C}^2(\mathbb{R}^N \times (0,+\infty))$ such that $\|u^\varepsilon\|_\infty \leq M$. Moreover, we can derive an energy estimate : for any compact set K of $\mathbb{R}^N \times [0,+\infty)$,

$$\varepsilon \int_K |\nabla u^\varepsilon|^2 dx\, dt \leq C, \tag{2.1}$$

where C only depends on K and on M.

We can then easily check that for any function $\varphi \in \mathcal{C}^1_c(\mathbb{R}^N \times [0,+\infty), \mathbb{R}^+)$ and for any constant $\kappa \in \mathbb{R}$,

$$\int_0^\infty \int_{\mathbb{R}^N} \eta_\kappa(u^\varepsilon)\,\varphi_t + q_\kappa(x,t,u^\varepsilon)\,\nabla\varphi\, dx\, dt + \int_{\mathbb{R}^N} \eta_\kappa(u_0^\varepsilon(x))\,\varphi(x,0)\,dx \geq \xi(\varepsilon), \tag{2.2}$$

where η_κ and q_κ are defined by (1.10) and

$$\xi(\varepsilon) = \varepsilon \int_0^\infty \int_{\mathbb{R}^N} \text{sgn }(u^\varepsilon - \kappa)\nabla u \cdot dt \to 0 \quad \text{as} \ \ \varepsilon \to 0.$$

Therefore

$$|\xi(\varepsilon)| \leq \sqrt{\varepsilon} \quad \|\nabla\varphi\|_\infty \left[\varepsilon \int_K |\nabla u^\varepsilon|^2 dx\, dt\right]^{\frac{1}{2}} \text{mes}(K)^{\frac{1}{2}},$$

where K denotes the support of φ.

If (u_ε) converges (as $\varepsilon \to 0$) in the "nonlinear weak-$\star$ sense" towards μ, we then deduce, passing to the limit in Equation (2.2) and using the fact that u_0^ε converges strongly in

$L^1_{\rm loc}(\mathbb{R}^N)$ towards u_0, that for any function $\varphi \in L^1(\mathbb{R}^N \times \mathbb{R}^+)$ and any $\kappa \in \mathbb{R}$,

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}^N}\int_0^1 [\eta_\kappa(\mu(x,t,\alpha))\,\varphi_t(x,t) + q_\kappa(x,t,\mu(x,t,\alpha))\,\nabla\varphi(x,t)]\,d\alpha\,dx\,dt$$
$$+\int_{\mathbb{R}^N}\eta_\kappa(u^0(x))\,\varphi(x,0)\,dx \geq 0. \tag{2.3}$$

Hence, u^ε converges in the "nonlinear weak-$\star$ sense" towards an "entropy process solution" of the hyperbolic problem (1.3), (1.4).

2.2. Convergence to the Entropic Solution

Theorem 2.2. *Let* $u_0 \in L^\infty(\mathbb{R}^N)$*, and assume* (1.5), (1.6)*. The solution* u^ε *of the parabolic approximation* (1.1), (1.2) *converges strongly in* $L^p_{\rm loc}(\mathbb{R}^N \times \mathbb{R}^+)$ *for all* $p < \infty$ *towards the entropic solution of the problem* (1.3), (1.4).

Proof of Theorem 2.2. The uniqueness of the "entropy process solution" proved in [1,3] ensures the convergence in the "nonlinear weak-$\star$ sense" of u^ε towards an "entropy process solution" μ of the hyperbolic problem (1.3),(1.4) and this entropy process solution is given by the entropic solution u ($\mu(x,t,\alpha) = u(x,t)$ a.e.) of the hyperbolic problem (1.3), (1.4). The strong convergence remains to be proved. So, if K denotes any compact set of $\mathbb{R}^N \times \mathbb{R}^+$, using the nonlinear weak-$\star$ convergence of u^ε towards μ for $g(s) = s^2$, we check that $u^\varepsilon \to u$ weakly in $L^2(K)$ and $\|u^\varepsilon\|_{L^2(K)} \to \|u\|_{L^2(K)}$.

We immediately conclude that u^ε strongly converges towards u in $L^2(K)$ for any compact set K of $\mathbb{R}^N \times \mathbb{R}^+$. As the family (u^ε) is bounded in $L^\infty(K)$ (for any K), we deduce that the convergence is strong in all $L^p_{\rm loc}$, $p < \infty$.

Remarks. (i) For an initial condition $u_0 \in BV(\mathbb{R}^N)$, and $u_0^\varepsilon = u_0$ *a.e.*, Kuznetsov[7] got some estimate of the error $u^\varepsilon - u$ in the space $L^\infty(0,T;L^1(\mathbb{R}^N))$,

$$\|u^\varepsilon - u(.,t)\|_{L^1(\mathbb{R}^N)} \leq C(\|u_0\|_{BV(\mathbb{R}^N)})\sqrt{t\varepsilon}.$$

(ii) For an initial data $u_0 \in L^\infty(\mathbb{R}^N)$, we can also obtain an estimate in the space $L^\infty(0,T;L^1_{\rm loc}(\mathbb{R}^N))$, see the proof of [4,5].

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