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PARALLEL STABILIZATION

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Abstract

A new algorithm for the stabilization of (possibly turbulent, chaotic) distributed systems, governed by linear or non linear systems of equations is presented.

The SPA (Stabilization Parallel Algorithm) is based on a systematic parallel decomposition of the problem (related to arbitrarily overlapping decomposition of domains) and on a penalty argument.

SPA is presented here for the case of linear parabolic equations, with distributed or boundary control. It extends to practically all linear and non linear evolution equations, as it will be presented in several other publications.

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§1. Introduction

Let us consider an evolution system whose state is given by the solution of a Partial Differential Equation (PDE) which is written (formally for the time being) as

$$\mathcal{C}\frac{\partial y}{\partial t} + \mathcal{A}(y) = \mathcal{B}v, \tag{1.1}$$

$$y\big|_{t=0} = 0. \tag{1.2}$$

In (1.1), (1.2), which may be a linear or a non linear PDE, y denotes the state, and v denotes the control.

The operator $\mathcal{A}$, which is linear or non linear, is a P.D. Operator, and the operator $\mathcal{B}$ maps the "space of controls" into the "space of the states". The operator $\mathcal{C}$ is linear, symmetric positive definite.

Of course one should add to (1.1), (1.2) the boundary conditions. They are here implicit. Of course this will be made precise in the examples.

Let y_d be a given state ($d =$"desired"). We want to choose v such that

(i) $y(t;v) =$ solution of (1.1), (1.2) remains as close as possible of y_d and

(ii) the goal (i) is achieved at not too large a cost.

In more precise terms, we introduce

$$J(t;v) = \frac{1}{2}\|y(t;v) - y_d\|^2 + \frac{c}{2}\|v\|^2, \quad c > 0 \text{ given}, \tag{1.3}$$

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(where the norms are taken in the appropriate spaces; again this will be made precise in the examples) and we look at any time t, for

$$\inf_v J(t; v). \tag{1.4}$$

Remark 1.1. In (1.1), (1.2) the state y can be a vector function. For instance, (1.1), (1.2) can be the Navier-Stokes equations. In such a case, it is not known in $3D$ if there is a unique (weak) solution of (1.1), (1.2). Then in (1.4) the inf is taken for all possible solutions of the state equation.

Remark 1.2. In (1.1) the control v can be distributed or a boundary control.

Remark 1.3. The formulation (1.4) is a little fuzzy. This will be made precise in Section 5 below.

Keeping with the (admitelly fuzzy) problem (1.4) our goal is to (try to) find a completely general parallel method to solve (1.4) (hopefully in real time).

We introduce to this effect a decomposition method in Section 2 below. It is presented in a completely general axiomatic way and illustrated by the example of multi-domain decomposition.

Using this decomposition and a penalty argument, an approximation result is given in Section 3. It becomes then possible to present a general parallel method for the solution of (1.1), (1.2) (to find v). This is the object of Section 4.

Using the algorithm of Section 4, one can finally present the parallel algorithm SPA (Stabilization Parallel Algorithm). SPA is presented here for the first time. It can be applied in many other situations, briefly treated in Section 5 and that will be the object of other publications.

Remark 1.4. Many other developments, including Numerical Computations, will be presented in joint work with R. Glowinski, J. Périaux and O. Jironneau and others.

Other possible applications of the ideas presented here have been found by J. Périaux.

Remark 1.5. The Decomposition Method introduced here in Section 4 could also be applied to non overlapping domains.

Remark 1.6. Decomposition methods are classically used in numerical analysis. General and systematic extensions of the Schwarz alternating method have been given in P. L. Lions[10,11,12]. Overlapping domain decomposition has also been studied in [5].

As far as we know, the "connections" between the various subdomains are made through boundary conditions on the (many) interfaces[4].

This is not the case here, making life a lot simpler but at the price of introducing a "penalty parameter", which may be painful to handle numerically. We will return to that in other publications.

Remark 1.7. If there is a huge amount of results and methods in Decomposition methods (one can consult also the bibliography of the works indicated in Remark 1.6), much less is known in the application of these methods to control problems. In these directions one can quote the very interesting work of J. E. Lagnese and G. Leugering[6], following the contributions of J. D. Benamou and B. Despres[2], J. D. Benamou[1].

The questions which are studied here are (sort of) Stabilization problems, and we obtain parallel algorithms.

A Parallel algorithm for control problems (CPA) will be presented in other publications.

§2. Decomposition Method

Let us consider two real Hilbert spaces V and H

$$V \subset H, \quad V \text{ dense in } H. \tag{2.1}$$

We identify H to its dual. Then if V' is the dual of V, we have

$$V \subset H \subset V'. \tag{2.2}$$

We consider the bilinear forms

$$\begin{aligned} &y, \hat{y} \to c(y, \hat{y}) \text{ which is symmetric, continuous in } H \times H, \text{and such that} \\ &c(y, y) \geq \gamma \|y\|_H^2, \quad \gamma > 0, \quad \forall y \in H; \end{aligned} \tag{2.3}$$

$$\begin{aligned} &y, \hat{y} \to a(y, \hat{y}) \text{ continuous on } V \times V, \text{not necessarily symmetric and such that} \\ &a(y, y) \geq \alpha \|y\|_V^2, \quad \alpha > 0, \quad \forall y \in V. \end{aligned} \tag{2.4}$$

We then consider the evolution equation

$$c\Big(\frac{\partial y}{\partial t}, \hat{y}\Big) + a(y, \hat{y}) = (v, \hat{y}), \ \forall \hat{y} \in V, \quad y \in L^2(0, T; V) \quad (T > 0 \text{ arbitrary}), \tag{2.5}$$

$$y(0) = 0, \tag{2.6}$$

where the control v is given, such that

$$v \in L^2(0, T; V'). \tag{2.7}$$

Example 2.1. Let Ω be an open set of $\mathbb{R}^d$ ($d = 1, 2, 3$ in the applications). We take

$$H = L^2(\Omega), \quad V = H^1(\Omega) \ \text{ (Sobolev space of order 1 on } L^2(\Omega)),$$

$$c(y, \hat{y}) = \int_\Omega y\hat{y}\, dx, \tag{2.8}$$

$$a(y, \hat{y}) = \int_\Omega (\nabla y \nabla \hat{y} + y\hat{y})\, dx, \quad \nabla y = \Big\{\frac{\partial y}{\partial x_i}\Big\}, \tag{2.9}$$

and let us consider v defined by

$$(v, \hat{y}) = \int_{\Gamma_0} v\hat{y}\, d\Gamma_0, \tag{2.10}$$

where v is given in $L^2(\Gamma_0 \times (0, T))$, $\Gamma_0 \subset \partial\Omega$.

(One could take in (2.10) $v \in L^2(0, T; H^{-\frac{1}{2}}(\Gamma_0))$-but taking great care of the closure of functions of compact support in Γ_0 into $H^{\frac{1}{2}}(\Gamma_0)$-cf. for all that J. L. Lions and E. Magenes[13]).

Then (2.5) is equivalent to

$$\begin{aligned} \frac{\partial y}{\partial t} - \Delta y + y &= 0 \ \text{ in } \Omega \times (0, T), \\ y(x, 0) &= 0 \ \text{ in } \Omega, \\ \frac{\partial y}{\partial n} &= v \ \text{ on } \Gamma_0 \times (0, T) \\ &= 0 \ \text{ on } \Gamma/\Gamma_0 \times (0, T). \end{aligned} \tag{2.11}$$

Remark 2.1. Of course Example 2.1 is one of the simplest example one can think of the (2.5)!

Remark 2.2. Everything presented here can be extended to non linear problems-such as the Navier-Stokes systems (see [2]).

We now introduce the decomposition method in an axiomatic way (followed by an example!). We are given a family of N triple of Hilbert spaces

$$V_i \subset H_i \subset V_i', \quad i = 1, \cdots, N, \tag{2.12}$$

and we are also given Hilbert spaces $H_{ij},\ i, j = 1, \cdots, N$, such that

$$H_{ij} = H_{ji}, \quad \forall i, j. \tag{2.13}$$

We are given linear operators r_i and r_{ij} such that

$$r_i \in \mathcal{L}(H; H_i), \quad r_i \in \mathcal{L}(V; V_i), \quad r_{ij} \in \mathcal{L}(H_j; H_{ij}). \tag{2.14}$$

The two essential hypothesis are

$$r_{ji} r_i \varphi = r_{ij} r_j \varphi, \quad \forall \varphi \ \in \ H, \ \ \forall i, j, \tag{2.15}$$

and

$$\begin{aligned}
&\text{if } y_1, \cdots, y_N \text{ are given in } V_1 \times V_2 \times \cdots \times V_N \ \Big(\text{resp.} \prod_{i=1}^{N} H_i\Big) \text{ such that} \\
&r_{ij} y_j = r_{ji} y_i, \quad \forall i, j, \\
&\text{then there exists } y \text{ in } V \ (\text{resp.} H) \text{ such that } y_i = r_i y, \ \ \forall i, \\
&y \ \text{ is unique and depends continuously on } \ \{y_i\} \text{ in } \Pi V_i \ (\text{resp. } \ \Pi H_i).
\end{aligned} \tag{2.16}$$

No doubt an example is needed!

Example 2.2. Let $\Omega_1, \cdots, \Omega_N$ (N is "large") be a family of open sets such that $\Omega_i \subset \Omega \subset U\overline{\Omega}_i$.

We say that "j" (in the set $1, 2, \cdots, N$) is a neighhour of "i" iff

$$\Omega_j \cap \Omega_i \neq \emptyset. \tag{2.17}$$

It is assumed that every "i" has at least one neighbour (and it can have several of them).

Then (actually weighted spaces are needed)

$$\begin{aligned}
&H_i = L^2(\Omega_i), \ \ V_i = H^1(\Omega_i), \ \ H_{ij} = L^2(\Omega_i \cap \Omega_j), \\
&r_i = \text{restriction to } \Omega_i, \\
&r_{ij} = \text{restriction to } \Omega_i \cap \Omega_j \ \ (r_{ij} = r_{ji}).
\end{aligned}$$

Then (2.15), (2.16) hold true.

Remark 2.3. One has

$$r_{ji} = 0 \ \text{ if "} j'' \text{ is not a neighbour of "} i''. \tag{2.18}$$

Then the matrix $\|r_{ij}\|$ will be in general a spare matrix.

Remark 2.4. Many other examples will be presented elsewhere.

We now introduce bilinear forms c_i, a_i such that

$$\begin{aligned}
&c_i(y_i, \hat{y}_i) \text{ is continuous and symmetric on } H_i \times H_i, \\
&\text{such that } c_i(y_i, y_i) \geq \gamma_i \|y_i\|_{H_i}^2, \ \ \gamma_i > 0,
\end{aligned} \tag{2.19}$$

$$\begin{aligned}
&a_i(y_i, \hat{y}_i) \text{ is continuous, not necessarily symmetric on } V_i \times V_i, \\
&\text{such that } a_i(y_i, y_i) \geq \alpha_i \|y_i\|_{H_i}^2, \ \ \alpha_i > 0,
\end{aligned} \tag{2.20}$$

and such that

$$c(y,\hat{y}) = \sum_{i=1}^{N} c_i(r_i y, r_i \hat{y}), \quad \forall y, \hat{y} \in H \times H,$$
$$a(y,\hat{y}) = \sum_{i=1}^{N} a_i(r_i y, r_i \hat{y}), \quad \forall y, \hat{y} \in V \times V. \tag{2.21}$$

Example 2.3. Let $\rho_i,\ i = 1,2,\cdots,N$ be a family of functions such that

$$\rho_i \in L^\infty(\Omega_i), \quad \rho_i = 0 \ \text{ outside } \Gamma_i, \quad \rho_i \geq \bar{\rho} \geq 0 \text{ a.e.}, \quad \text{in } \Gamma_i, \tag{2.22}$$
$$\sum_{i=1}^{N} \rho_i(x) = 1 \text{ in } \Omega.$$

(There are infinitely many such that sets of functions). Then, in the framework of Example 2.1, if one takes

$$c_i(y_i,\hat{y}_i) = \int_{\Omega_i} \rho_i y_i \hat{y}_i \, dx, \quad a_i(y_i,\hat{y}_i) = \int_{\Omega_i} \rho_i y_i (\nabla y_i \nabla \hat{y}_i + y_i \hat{y}_i) dx, \tag{2.23}$$

the conditions (2.21) are satisfied.

We have now all the tools needed to present the Penalty Approximation.

Remark 2.5. The systems considered here are "stable". But the methods presented here apply to unstable systems, or "turbulent" ones (see [8]).

§3. Penalty Approximation

We consider the set of PDE's

$$c_i\Big(\frac{\partial y_i}{\partial t}, \hat{y}_i\Big) + a_i(y_i,\hat{y}_i) + \frac{1}{\varepsilon}\sum_j (r_{ji}y_i - r_{ij}y_j, r_{ji}\hat{y}_i)_{H_{ij}} = (v_i,\hat{y}_i), \quad \hat{y}_i \in V_i, \quad i = 1,\cdots,N, \tag{3.1}$$
$$y_i(0) = 0, \quad \forall i, \tag{3.2}$$

where $\varepsilon > 0$ is "small" and where the v_i's satisfy

$$v_i \in L^2(0,T;V_i'), \quad \sum_{i=1}^{N}(v_i, r_i\hat{y}) = (v,\hat{y}), \quad \forall \hat{y} \in V. \tag{3.3}$$

Example 3.1. In the framework of Example 2.1, one introduces the subset E of $\{1,2,\cdots,N\}$ defined by

$$i \in E \ \text{ iff } \ \Gamma_0 \cap \partial\Omega_i \neq \emptyset. \tag{3.4}$$

Then we introduce functions $\sigma_i,\ i \in E$, such that

$$\sigma_i \text{ has support in } \Gamma_0 \cap \partial\Omega_i,\ \sigma_i \in L^\infty(\Gamma_0 \cap \partial\Omega_i),$$
$$\text{and } \sum_{i\in E} \sigma_i = 1 \ \text{ on } \Gamma_0, \text{ and } \sigma_i = 0 \ \text{ if } \ i \notin E. \tag{3.5}$$

Then

$$(v,\hat{y}) = \int_{\Gamma_0} v\hat{y} d\Gamma_0 = \sum_i \int_{\Gamma_0 \cap \partial\Omega_i} (\sigma_i v)\hat{y}\, d\Gamma_0, \tag{3.6}$$

which gives one (of the infinitely many) decomposition satisfying (3.3).

We now prove

Theorem 3.1. *We assume that* (2.15), (2.16), (2.19)–(2.21) *hold true. Problem* (3.1), (3.2) *admits a unique solution*

$$y_i^\varepsilon \in L^2(0,T;V_i)\cap L^\infty(0,T;H_i), \quad i=1,\cdots,N. \tag{3.7}$$

As $\varepsilon \to 0$*, one has*

$$y_i^\varepsilon \to y_i \ \ in \ \ L^2(0,T;V_i) \ \ weakly, \ in \ L^\infty(0,T;H_i) \ weak \ star, \tag{3.8}$$

where

$$y_i = r_i y, \quad y = solution \ of \ (2.5),(2.6). \tag{3.9}$$

Proof. Step 1. A Priori Estimates

We write for the moment y_i instead of y_i^ε. We choose $\hat{y}_i = y_i$ in (3.1). This procedure can be justified, for instance by using a Galerkin method.

We obtain

$$\frac{1}{2}\frac{d}{dt}c_i(y_i) + a_i(y_i) + \frac{1}{\varepsilon}X_i = (v_i, y_i), \tag{3.10}$$

where $c_i(y_i) = c_i(y_i, y_i), a_i(y_i) = a_i(y_i, y_i)$ and where

$$\begin{aligned} X_i &= \sum_j (r_{ji}y_i - r_{ij}y_j, r_{ji}y_i)_{H_{ij}} \\ &= \frac{1}{2}\sum_j \|r_{ji}y_i - r_{ij}y_j\|^2_{H_{ij}} + \frac{1}{2}\sum \|r_{ji}y_i\|^2_{H_{ij}} - \frac{1}{2}\sum_j \|r_{ij}y_j\|^2_{H_{ij}}. \end{aligned} \tag{3.11}$$

But $\sum\limits_i\sum\limits_j \|r_{ji}y_i\|^2_{H_{ij}} = \sum\limits_{i,j}\|r_{ij}y_j\|^2_{H_{ji}}$ (by exchanging i and j) so that

$$\sum_i X_i = \frac{1}{2}\sum_{i,j}\|r_{ji}y_i - r_{ij}y_j\|^2_{H_{ij}}. \tag{3.12}$$

Therefore, it follows from (3.10) that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\sum_i c_i(y_i) + \sum_i a_i(y_i) + \frac{1}{2\varepsilon}\sum_{i,j}\|r_{ji}y_i - r_{ij}y_j\|^2_{H_{ij}} \\ &= \sum_i (v_i, y_i) \le \sum_i \|v_i\|_{V_i'}\|y_i\|_{V_i} \le \sum \frac{1}{2}a_i(y_i) + \frac{c_i}{2}\|v_i\|^2_{V_i'}, \end{aligned} \tag{3.13}$$

where the c_i's are suitable constants. It follows that (by using (2.19) and (2.20))

$$y_i = y_i^\varepsilon \text{ is bounded (as } \to 0) \text{ in } L^2(0,T;V_i)\cap L^\infty(0,T;H_i), \tag{3.14}$$

$$\frac{1}{\varepsilon}(r_{ji}y_i^\varepsilon - r_{ij}y_j^\varepsilon) \text{ is bounded (as } \varepsilon \to 0) \text{ in } L^2(0,T;H_{ij}). \tag{3.15}$$

Step 2. It follows from (3.14) and (3.15) that one can extract a subsequence, still denoted by y_i^ε, such that (3.8) holds true and such that

$$r_{ji}y_i - r_{ij}y_j = 0, \quad \forall i,j. \tag{3.16}$$

Therefore, by using (2.16), there exists y such that

$$y_i = r_i y, \quad i = 1,\cdots,N. \tag{3.17}$$

It remains to show that y is the solution of (2.5), (2.6).

Step 3. Let φ be a smooth function from $[0,T] \to V$, such that $\varphi(T) = 0$.

We choose in (3.1), for every t, $\hat{y}_i = r_i\varphi(t)$. We obtain after integration on $(0,T)$ (and integration by parts)

$$-\int_0^T c_i\Big(y_i^\varepsilon, \frac{\partial r_i\varphi}{\partial t}\Big)dt + \int_0^T a_i(y_i^\varepsilon, r_i\varphi)dt + \frac{1}{\varepsilon}Y_{i\varepsilon} = \int_0^T (v_i, r_i\varphi)dt, \tag{3.18}$$

where

$$Y_{i\varepsilon} = \int_0^T \sum_j (r_{ji}y_i^\varepsilon - r_{ij}y_j^\varepsilon, r_{ji}r_i\varphi)_{H_{ij}}dt. \tag{3.19}$$

Then

$$\sum_i Y_{i\varepsilon} = \sum_{i,j}\int_0^T (r_{ji}y_i^\varepsilon, r_{ji}r_i\varphi)_{H_{ij}}dt - \sum_{i,j}\int_0^T (r_{ij}y_j^\varepsilon, r_{ji}r_i\varphi)_{H_{ij}}dt. \tag{3.20}$$

After exchanging i and j in the second term in (3.20), it becomes

$$\sum_{i,j}\int_0^T (r_{ji}y_i^\varepsilon, r_{ij}r_j\varphi)_{H_{ij}}dt$$

and since by hypothesis, one has (2.15), it follows that

$$\sum_i Y_{i\varepsilon} = 0. \tag{3.21}$$

Therefore (3.18) gives

$$-\sum_i\int_0^T c_i\Big(y_i^\varepsilon, \frac{\partial r_i\varphi}{\partial t}\Big)dt + \sum_i\int_0^T a_i(y_i^\varepsilon, r_i\varphi)dt = \sum_i\int_0^T (v_i, r_i\varphi)dt = \int_0^T (v,\varphi)dt. \tag{3.22}$$

Using (3.8) one can pass to the limit in (3.22). One obtains

$$-\sum_i\int_0^T c_i\Big(y_i, \frac{\partial r_i\varphi}{\partial t}\Big)dt + \sum_i\int_0^T a_i(y_i, r_i\varphi)dt = \int_0^T (v,\varphi)dt. \tag{3.23}$$

But by using (3.17) and (2.21), (3.22) is identical with

$$-\int_0^T c_i\Big(y, \frac{\partial\varphi}{\partial t}\Big)dt + \int_0^T a(y,\varphi)dt = \int_0^T (v,\varphi)dt. \tag{3.24}$$

which is the standard weak formulation of problem (2.5), (2.6). The proof is completed.

Remark 3.1. The previous type of proof applies in very many other situations for parabolic equations, linear or non linear. Cf. in particular the case of the Navier-Stokes equations in [8].

§4. Parallel Algorithm

We introduce now a time-discretization of the penalty approximation (3.1).

With standard notations of numerical analysis, y_i^n denotes the (hopefully) approximation of y_i at time $n\Delta t$. We define y_i^n by

$$c_i\Big(\frac{y_i^n - y_i^{n-1}}{\Delta t}, \hat{y}_i\Big) + a_i(y_i^n, \hat{y}_i) + \frac{1}{\varepsilon}\sum_j (r_{ji}y_i^n - r_{ij}y_j^{n-1}, r_{ji}\hat{y}_i)_{H_{ij}}$$

$$= (v_i^n, \hat{y}_i), \quad \forall \hat{y}_i \in V_i,\ n = 1, 2, \cdots, \tag{4.1}$$

$$y_i^0 = 0. \tag{4.2}$$

Remark 4.1. The bilinear form $\frac{1}{\Delta t}c_i(y_i,\hat{y}_i)+a_i(y_i,\hat{y}_i)+\frac{1}{\varepsilon}\sum\limits_j(r_{ji}y_i,r_{ji}\hat{y}_i)_{H_{ij}}$ is continuous and coercive on $V_i\times V_i$. Therefore, given y_i^{n-1} and v_i^n = approximation of v_i at time $n\Delta t$ (for instance average around $n\Delta t$), (4.1) uniquely defines y_i^n.

Remark 4.2. The algorithm (4.1) is parallel.

Moreover, in (4.1), only are used the y_j^{n-1} such that "j" is a "neighbour" of "i".

We now prove

Theorem 4.1. *Under the hypothesis of Theorem* 3.1, *algorithm* (4.1) *is stable.*

Proof. We choose $\hat{y}_i=y_i^n$ in (4.1). We obtain

$$\frac{1}{2\Delta t}(c_i(y_i^n-y_i^{n-1})+c_i(y_i^n)-c_i(y_i^{n-1}))+a_i(y_i^n)+\frac{1}{\varepsilon}Z_i^n=(v_i^n,y_i^n), \tag{4.3}$$

where

$$Z_i^n=\sum_j(r_{ji}y_i^n-r_{ij}y_j^{n-1},r_{ji}y_i^n)_{H_{ij}}. \tag{4.4}$$

We write

$$Z_i^n=\frac{1}{2}\sum_j\|r_{ji}y_i^n-r_{ij}y_j^{n-1}\|^2_{H_{ij}}+\frac{1}{2}\sum_j\|r_{ji}y_i^n\|^2_{H_{ij}}-\frac{1}{2}\sum_j\|r_{ji}y_i^{n-1}\|^2_{H_{ij}}. \tag{4.5}$$

Let us define

$$\xi^n=\sum_i\sum_j\|r_{ji}y_i^n\|^2_{H_{ij}}. \tag{4.6}$$

We observe that

$$\sum_{i,j}\|r_{ij}y_i^{n-1}\|^2_{H_{ij}}=\ \text{(after exchange of } i \text{ and } j)\ \sum_{i,j}\|r_{ji}y_j^{n-1}\|^2_{H_{ij}}=\xi^{n-1},$$

so that

$$\sum_i Z_i^n=\frac{1}{2}\sum\|r_{ji}y_j^n-r_{ij}y_j^{n-1}\|^2_{H_{ij}}+\frac{1}{2}\xi^n-\frac{1}{2}\xi^{n-1}. \tag{4.7}$$

Using (4.3), (4.7), we obtain, after summation in n

$$\frac{1}{2\Delta t}\sum_i c_i(y_i^n)+\frac{1}{2\Delta t}\sum_i\sum_{k=1}^n c_i(y_i^k-y_i^{k-1})+\sum_i\sum_{k=1}^n a_i(y_i^k)$$
$$+\frac{1}{2\varepsilon}\sum_{i,j}\sum_{k=1}^n\|r_{ji}y_i^k-r_{ij}y_j^{k-1}\|^2_{H_{ij}}+\frac{1}{2\varepsilon}\xi^n=\sum_i\sum_{k=1}^n(v_i^k,y_i^k).$$

Stability immediately follows.

Remark 4.3. Of course other time-discretization schemes can be used in (4.1), with a similar result.

Remark 4.4. The next step is to prove convergence (in a suitable sense) of the step functions y_i^k in $((k-1)\Delta t,k\Delta t)$, towards y_i as $\Delta t\to 0$, where $y_i=y_i^\varepsilon$, ε fixed. This will be presented elsewhere.

Remark 4.5. Of course one can also introduce space approximations, which can differ in different spaces V_i. This will also be presented elsewhere.

Remark 4.6. Because of the necessary cancellations in the terms $\frac{1}{\varepsilon}(\xi^k-\xi^{k-1})$, the same ε has to be taken during the computation in the interval $(0,T)$. This (negative) aspect of the parallel algorithm (4.1) is not harmful in the method SPA which is presented now.

§5. Method SPA

We present now the Stabilization Parallel Algorithm (SPA). At step n, the approximation of the state $y(n\Delta t)$ is provided by the various $\{y_i^n\}$, given by (4.1). They depend on the v_i^n. We want to minimize

$$J(n\Delta t, v) = \frac{1}{2}\|y(n\Delta t); v) - y_d\|_H^2 + \frac{c}{2}\|v(n\Delta t)\|_H^2. \tag{5.1}$$

We replace (5.1) by

$$\mathcal{J}(n\Delta t, v^n) = \sum_{i=1}^{N} \mathcal{J}_i(n\Delta t, v_i^n), \quad v^n = \{v_i^n\}, \tag{5.2}$$

where

$$\mathcal{J}_i(n\Delta t, v_i^n) = \frac{1}{2}\|y_i^n - r_i y_d\|_{H_i}^2 + \frac{c}{2}\|v_i^n\|_{H_i}^2. \tag{5.3}$$

We are looking for $\{v_i^n\}$ such that it achieves

$$\inf_{\{v_i^n\}} \mathcal{J}(n\Delta t, v^n). \tag{5.4}$$

But this is equivalent to computing

$$\inf_{v_i^n} \mathcal{J}_i(n\Delta t, v_i^n), \tag{5.5}$$

since

$$\inf_{\{v_i^n\}} \mathcal{J}(n\Delta t, v^n) = \sum_{i=1}^{N} \inf_{v_i^n} \mathcal{J}_i(n\Delta t, v_i^n). \tag{5.6}$$

Then SPA is as follows:

(1) y_i^n is defined by (4.1). It depends on v_i^n. It uses the previous computation of $y_j^{n-1}, j \in$ neighborhood of i.

(2) Compute v_i^n by solving (5.5), and proceed.

Several remarks are in order.

Remark 5.1. In the decomposition, there will be in general a large number of "i", such that

$$(v_i, \hat{y}_i) = 0, \quad \forall \hat{y}_i \in V_i, \tag{5.7}$$

say when $i \in F \subset [1, 2, \cdots, N]$.

Then (4.1) reduces to the equation

$$c_i\Big(\frac{y_i^n - y_i^{n-1}}{\Delta t}, \hat{y}_i\Big) + a_i(y_i^n, \hat{y}_i^n) + \frac{1}{\varepsilon}\sum_j (r_{ji}y_i^n - r_{ij}y_i^{n-1}, r_{ji}\hat{y}_i)_{H_{ij}} = 0 \tag{5.8}$$

and there is no step 2 in SPA when $i \in F$.

The role of the control v, which is now $\{v_i^n\}$, is propagated by the "neighbours" (summation in j in (4.1))-starting from the computations of step 2) of SPA for those i's which do not belong to F, i.e. $(v_i, \hat{y}_i) \not\equiv 0, \ \forall \hat{y}_i \in V_i$.

Remark 5.2. All possible minimization algorithms can be used in step 2.

Remark 5.3. In (5.8) the same ε has to be chosen for each fixed time but ε may depend on n.

Remark 5.4. One can use (if convenient) a local feedback for the solution of (5.5).

Remark 5.5. The avove method is completely general. Cf. for instance J. L. Lions[8] for the control of Navier-Stokes equations.

Remark 5.6. Error estimates with respect to the solution of the continuous problem (1.4) are not yet obtained (of course one needs to make (1.4) more precise). We hope to return on this (non trivial) task.

§6. Further Remarks

Remark 6.1. As we have already said, SPA can be applied to non-linear problems.

Remark 6.2. All the techniques introduced here can be used, after suitable adaptation, for second-order in time evolution equations: hyperbolic equations, well set PDE in the sense of Petrowsky. This is presented in [9].

Remark 6.3. For each t we may have to deal with a multicriteria problem, and we may want to solve a min-max problem, or find a Nash or a Pareto equilibrium (see [3] for the formulation of such problems). The SPA can be adapted to such situations.

Remark 6.4. The techniques presented here can be used for control problems. The Control Parallel Algorithms method (CPA) will be presented elsewhere.

Remark 6.5. SPA also extends to the case of systems where the state equation is a Variational Inequality.

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