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APPLICATIONS OF THE THEORY OF CAMINA GROUPS**

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Abstract

In this paper, the author first gives a conjugacy-class version of Camina hypotheses and then applys the Camina group theory to discussing two classes of finite groups.

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This paper is divided into four sections. In Section 1, we present a conjugacy-class version of Camina Hypotheses . In Section 2, we give some basic facts about a Camina group G with the kernel G', which will be used in Sections 3 and 4. In Sections 3 and 4, we apply the Camina group theory to discussing two classes of finite groups.

All groups considered are finite and all group characters are ordinary characters. The letter G always denotes a group. The letters p and q denote two different prime numbers. For a normal subset N of G, we denote by $m(N)$ the number of the G-conjugacy classes which are contained in N. A pair (H, K) denotes a Frobenius group with the Frobenius kernel K and a Frobenius complement H. For $x \in G$, we denote by $\mathrm{Cl}_G(x)$ the conjugacy class of x in G. $\mathrm{Irr}^{\#}(G)$ denotes the set of non-principle irreducible characters of G. For a character χ of G, $\mathrm{Irr}(\chi)$ denotes the set of irreducible constituents of χ. Some additional notation will be introduced as we go along. The rest of our notation is standard and adapted from [1].

§1. A Conjugacy-Class Version of Camina Hypotheses†

A. R. Camina presented two hypotheses and proved that the two hypotheses are equivalent to each other[2,p.153−154,p.158−160]. In this section, we present a conjugacy-class version of Camina's hypotheses. Our version is suitable to character tables.

Throughout this section, let G be a non-Abelian group, and let $C_1 = \{1\}$, $C_2, \cdots, C_m$, $C_{m+1}, \cdots$, C_n be all the conjugacy classes of G, where $2 \le m \le n-1$.

Hypothesis (H1). G has a character X such that

(a) $X(C_2) = \cdots = X(C_m)$, (b) for every $\varphi \in \mathrm{Irr}(X), \varphi(C_{m+1}) = \cdots = \varphi(C_n) = 0$.

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Hypothesis (H2). $C_iC_j = C_j$ for $i = 1, \cdots, m$ and $j = m+1, \cdots, n$.

If G satisfies (H1) or (H2), somtimes we say G satisfies (H1) or (H2) with respect to $C_1, \cdots, C_m$.

Theorem 1.1. *G satisfies* (H1) *if and only if G satisfies* (H2).

Proof. Necessity. This part of the proof is provided in a series of steps. Suppose that G satisfies (H1).

Step 1 Let $h = X(C_2) = \cdots = X(C_m)$, where X is the character satisfying the conditions (a) and (b) in (H1). Then $h = -X(1)/(|C_2| + \cdots + |C_m|)$.

Since $1_G \notin \mathrm{Irr}(X)$, by the condition (b) in (H1) we have

$$0 = [1_G, X] = [X(1) + h(|C_2| + \cdots + |C_m|)]/|G|.$$

So, Step 1 is established.

Step 2 $\bigcup\limits_{i=1}^{m} C_i = \bigcap\{\ker\vartheta : \vartheta \in \mathrm{Irr}(G) - \mathrm{Irr}(X)\}$ (In particular, $\bigcup\limits_{i=1}^{m} C_i \lhd G$).

Since $1_G \notin \mathrm{Irr}(X), \mathrm{Irr}(G) - \mathrm{Irr}(X) \neq \emptyset$. For each $\vartheta \in \mathrm{Irr}(G) - \mathrm{Irr}(X)$, by (H1) we have

$$0 = [X, \vartheta] = X(1)\vartheta(1) + h\sum_{i=2}^{m} |C_i|\overline{\vartheta(C_i)},$$

where $h = X(C_2) = \cdots = X(C_m)$. So, by Step 1 we get

$$\sum_{i=2}^{m} |C_i|\overline{\vartheta(C_i)} = -X(1)\vartheta(1)/h = \sum_{i=2}^{m} |C_i|\vartheta(1).$$

Since $|\vartheta(C_i)| \leq \vartheta(1)$, from the above equality we obtain $\overline{|\vartheta(C_i)|} = |\vartheta(C_i)| = \vartheta(1)$ for $i = 2, \cdots, m$. Hence, by [1, (2.27)] we have $\vartheta(C_i) = \epsilon_i\vartheta(1)$, where ϵ_i is a complex number and $|\epsilon_i| = 1$ for $i = 2, \cdots, m$. It follows that $\sum\limits_{i=2}^{m} |C_i|\epsilon_i = \sum\limits_{i=2}^{m} |C_i|$ and thus $\epsilon_2 = \cdots = \epsilon_m = 1$. We therefore have $\vartheta(C_i) = \epsilon_i\vartheta(1) = \vartheta(1)$ for $i = 2, \cdots, m$. So, we get

$$\bigcup_{i=1}^{m} C_i \subseteq \bigcap\{\ker\vartheta : \vartheta \in \mathrm{Irr}(G) - \mathrm{Irr}(X)\}.$$

Take C_j with $j \in \{m+1, \cdots, n\}$ and suppose that $C_j \subseteq \bigcap\{\ker\vartheta : \vartheta \in \mathrm{Irr}(G) - \mathrm{Irr}(X)\}$. Then, by the condition (b) in (H1) and a formula on the regular character ρ_G, we have

$$0 = \rho_G(C_j) = \sum\{\vartheta(1)^2 : \vartheta \in \mathrm{Irr}(G) - \mathrm{Irr}(X)\},$$

contradicting the fact that $\mathrm{Irr}(G) - \mathrm{Irr}(X) \neq \emptyset$. Hence, for C_j with $j = m+1, \cdots, n$, we have $C_j \not\subseteq \bigcap\{\ker\vartheta : \vartheta \in \mathrm{Irr}(G) - \mathrm{Irr}(X)\}$ and thus

$$\Big(\bigcup_{j=m+1}^{n} C_j\Big) \bigcap \big(\bigcap\{\ker\vartheta : \vartheta \in \mathrm{Irr}(G) - \mathrm{Irr}(X)\}\big) = \emptyset.$$

This completes Step 2.

Step 3 G satisfies (H2).

Take C_i and C_j such that $1 \leq i \leq m$ and $m+1 \leq j \leq n$. Let $y \in C_j$ and $x \in C_i$. It follows from Step 2 that $xy \in \bigcup\limits_{r=m+1}^{n} C_r$. So, by Step 2 and the condition (b) in (H1) we have $\varphi(xy) = \varphi(y)$ for every $\varphi \in \mathrm{Irr}(G)$. This implies that xy is conjugate to y in G, and so G satisfies (H2).

Sufficiency. Suppose that G satisfies (H2). Let $X = \rho_G - \sum\{\vartheta(1)\vartheta : \vartheta \in \mathrm{Irr}(G)$ and $\bigcup_{i=1}^{m} C_i \subseteq \ker\vartheta\}$, where ρ_G is the regular character of G. Since $m \geq 2$, by [1, (2.11) and (2.21)] $X \neq 0$ and thus X is a character of G. For $i = 2, \cdots, m$, we obviously have

$$X(C_i) = -\sum\{\vartheta(1)^2 : \vartheta \in \mathrm{Irr}(G) \text{ and } \bigcup_{i=1}^{m} C_i \subseteq \ker\vartheta\}.$$

So, X satisfies the condition (a) in (H1).

For any conjugacy class C_r in G, put $\tilde{C_r} = \sum\{x : x \in C_r\}$. Since G satisfies (H2), by [1,(2.4)] we have $\tilde{C_i}\tilde{C_j} = |C_i|\tilde{C_j}$, for $i = 1, \cdots, m$ and $j = m+1, \cdots, n$. Let $\varphi \in \mathrm{Irr}(X)$. By [1, p.36] we have

$$\omega_\varphi(\tilde{C_i})\omega_\varphi(\tilde{C_j}) = |C_i|\omega_\varphi(\tilde{C_j}), \quad \text{for } i = 1, \cdots, m, \quad j = m+1, \cdots, n,$$
$$\omega_\varphi(\tilde{C_r}) = \varphi(C_r)|C_r|/\varphi(1), \quad \text{for } r = 1, \cdots, n.$$

So, we have

$$0 = [\varphi(C_i)/\varphi(1) - 1]\varphi(C_j)/\varphi(1), \quad \text{for } i = 1, \cdots, m, \quad j = m+1, \cdots, n. \qquad (*)$$

Since $\varphi \in \mathrm{Irr}(X)$, by the choice of X we know that $\varphi(C_i) \neq \varphi(1)$ for some i with $1 \leq i \leq m$. Hence, from the equality $(*)$ above, it follows that $\varphi(C_j) = 0$ for $j = m+1, \cdots, n$. This implies that X satisfies the condition (b) in (H1). So, G satisfies (H1). This completes the proof.

Corollary 1.1. *Suppose that G satisfies* (H1) *or* (H2) *with respect to $C_1 = \{1\}, C_2, \cdots, C_m$. Let $N = \bigcup_{i=1}^{m} C_i$. Then*

(1) *N is a normal subgroup of G and $xN \subseteq \mathrm{Cl}_G(x)$ for every $x \in G - N$.*

(2) *All the conjugacy classes of the factor group $\overline{G} := G/N$ are $\overline{C}_1, \overline{C}_{m+1}, \cdots, \overline{C}_n$, where $\overline{C}_r$ denotes the image of C_r in $\overline{G} = G/N$ for $r = 1, m+1, m+2, \cdots, n$.*

(3) *When G satisfies* (H1)*, let X be the character satisfying the conditions* (a) *and* (b) *in* (H1)*, then*

(a) $N = \bigcap\{\ker\vartheta : \vartheta \in \mathrm{Irr}(G) - \mathrm{Irr}(X)\}$ *and* $\mathrm{Irr}(G/N) = \{\vartheta \in \mathrm{Irr}(G) - \mathrm{Irr}(X)\}$. *In particular, χ vanishes on $G - N$ for every $\chi \in \mathrm{Irr}(G)$ such that $N \not\leq \ker\chi$.*

(b) (*Up to a rational number*)

$$X = \rho_G - \sum\{\vartheta(1)\vartheta : \vartheta \in \mathrm{Irr}(G), N \leq \ker\vartheta\} = \rho_G - \sum\{\vartheta(1)\vartheta : \vartheta \in \mathrm{Irr}(G/N)\}.$$

(c) $|\mathrm{Irr}(X)| = m - 1 = m(N) - 1$.

(4) *Let M be any normal subgroup of G. Then either $N \leq M$ or $M < N$.*

(5) $Z(G) \leq N \leq G'$.

Proof. From Theorem 1.1 and its proof we know that (1), (2) and (3)(a) hold.

Now, let us show that (3)(b) and (3)(c)are true. Assume that G satisfies (H1). By (1) we have $N \lhd G$. Let $Y = \rho_G - \sum\{\vartheta(1)\vartheta : \vartheta \in \mathrm{Irr}(G/N)\}$. Clearly, Y has the constant integer value $-|G/N|$ on $N - \{1\} = \bigcup_{i=2}^{m} C_i$, and Y vanishes on $G - N = \bigcup_{i=m+1}^{n} C_i$ because $\mathrm{Irr}(Y) = \mathrm{Irr}(X)$ by 3(a) and X satisfies the condition (b) in (H1). By Step 1 in the proof of Theorem 1.1 we know that X has a constant rantional value on $N - \{1\}$. On the other hand, the values of X are algebraic integers. So, X has a constant integer value on $N - \{1\}$. In addition, X vanishes on $G - N$. Hence, there exist integers a and b such that $aX - bY$

vanishes on $G-\{1\}$. It follows that there exists an integer c such that $aX-bY=c\rho_G$. Note that $\mathrm{Irr}(aX-bY)\bigcap\mathrm{Irr}(G/N)=\emptyset$ by (3)(a), while $\mathrm{Irr}(G/N)\subseteq\mathrm{Irr}(\rho_G)=\mathrm{Irr}(G)$. Hence, $c=0$ and thus $aX=bY$. This establishes (3)(b). For a group K, we denote by $\mathrm{Con}(K)$ the set of conjugacy classes of K. Then by (2) and (3)(a) we have $|\mathrm{Irr}(X)|=|\mathrm{Irr}(G)|-|\mathrm{Irr}(G/N)|=|\mathrm{Con}(G)|-|\mathrm{Con}(G/N)|=n-(n-m+1)=m-1=m(N)-1$. So, (3)(c) is true.

It is enough to show (4) under the assumption that G satisfies (H1) because of Theorem 1.1. Since a normal subgroup of G is the joint of the kernels of some irreducible characters of G, it follows from (3)(a) that (4) is true.

By (H1) we have $Z(G)\leq N$, and by (H2) we have $N\leq G'$. So, by Theorem 1.1 it follows that (5) holds. This completes the proof.

It is well-known that an irreducible character χ of G has at least two non-zero values, that is, χ does not vanish on at least two conjugacy classes of G. In [3], the extreme case where G has an irreducible character vanishing on all but two conjugacy classes was investigated. Note that if G has an irreducible character χ such that χ does not vanish on exactly two conjugacy classes $C_1=\{1\}$ and C_2 of G, then G satisfies (H1) with $X=\chi$ with respect to C_1, C_2.

Corollary 1.2. *G has an irreducible character χ which does not vanish on exactly two conjugacy classes $\{1\}$ and D of G if and only if $DC=C$ for every conjugacy class C of G such that $\{1\}\neq C\neq D$.*

Proof. Necessity. By hypothesis G satisfies (H1) with $X=\chi$ with respect to $C_1=\{1\},C_2=D$. So, by Theorem 1.1 we have $DC=C$ for every conjugacy class of G such that $\{1\}\neq C\neq D$.

Sufficiency. By hypothesis G satisfies (H2) with respect to $C_1=\{1\},C_2=D$. So, by Theorem 1.1 G satisfies (H1) with respect to $C_1=\{1\},C_2=D$, and hence G has an irreducible character χ not vanishing on exactly two conjugacy classes $C_1=\{1\}$ and $C_2=D$ of G. This completes the proof.

Corollary 1.3. *Suppose that G has an irreducible character χ such that χ has the same value on $C_2,\cdots,C_m$ and vanishes on $C_{m+1},\cdots,C_n$. Then the following statements hold*:

(1) *$m=2$, that is, χ does not vanish on exactly two conjugacy classes $C_1=\{1\}$ and C_2 of G.*

(2) *$N:=C_1\bigcup C_2=\{1\}\bigcup C_2$ is a unique minimal normal subgroup of G and is an elementary Abelian p-group.*

(3) *χ is a unique faithful irreducible character of G.*

(4) $\chi(C_2)=-\chi(1)/|C_2|=-\chi(1)/(|N|-1)$. (5) $\chi(1)^2=|G|(|N|-1)/|N|$.

Proof. By hypothesis G satisfies (H1) with $X=\chi$ with respect to $C_1=\{1\},\cdots,C_m$. So, in view of (3)(c) of Corollary 1.1 we have that $m-1=|\mathrm{Irr}(X)|=|\mathrm{Irr}(\chi)|=1$ and thus $m=2$, establishing (1). From (1), (4) and 3(a) of Corollary 1.1, it follows that (2) and (3) are true. By (1) and Step 1 in the proof of Theorem 1.1 we get (4). From (2) and (3) it follows that $\mathrm{Irr}(G/N)=\mathrm{Irr}(G)-\{\chi\}$. So, we have

$$|G|=\sum\{\varphi(1)^2:\varphi\in\mathrm{Irr}(G)\}=\sum\{\varphi(1)^2:\varphi\in\mathrm{Irr}(G/N)\}+\chi(1)^2=|G/N|+\chi(1)^2,$$

and thus $\chi(1)^2=|G|(|N|-1)/|N|$. This establishes (5), completing the proof.

Theorem 1.2. *Let $\chi \in \mathrm{Irr}(G)$. Then the following statements hold*:

(1) *If χ does not vanish on exactly two conjugacy classes $C_1 = \{1\}$ and C_2 of G and $\chi(C_2) = -\chi(1)$, then G is a 2-group.*

(2) *If $Z(G) \neq 1$ and χ does not vanish on exactly two conjugacy classes $C_1 = \{1\}$ and C_2 of G, then G is a 2-group and $\chi(C_2) = -\chi(1)$.*

(3) *Suppose that G is nilpotent and χ does not vanish on exactly two conjugacy classes $C_1 = \{1\}$ and C_2 of G. Then G is a 2-group and $\chi(C_2) = -\chi(1)$.*

(4) *Suppose that G is a 2-group. Then χ does not vanish on exactly two conjugacy classes of G if and only if $|G| = 2\chi(1)^2$.*

Proof. Assume that χ does not vanish on exactly two conjugacy classes $C_1 = \{1\}$ and C_2 of G, and that $\chi(C_2) = -\chi(1)$. Then, G satisfies (H1) with $X = \chi$ with respect to $C_1 = \{1\}$, C_2. Put $N = \{1\} \bigcup C_2$. By Corollary 1.3, N is a unique minimal normal subgroup of G and $|C_2| = |N| - 1 = 1$ because $\chi(C_2) = -\chi(1)$. Hence, N is central with order 2. On the other hand, by Corollary 1.1(5) we have $Z(G) \leq N$. We therefore have $Z(G) = N$ and $|Z(G)| = 2$. From this and [3, Theorem 2.5(b)] it follows that G has a normal Sylow 2-subgroup P. So, $G = PK$, where K is a 2-complement in G. Suppose $K \neq 1$ and take a $2'$-element $g \in K - \{1\}$. Then we have $gN = gZ(G) \subseteq \mathrm{Cl}_G(g)$ (Corollary 1.1(1)). It follows that the elements in gN have the same order. Clearly, this is impossible. Therefore, $K = 1$ and $G = P$ is a 2-group. This completes the proof of (1).

Next, let us show (2). Put $N = C_1 \bigcup C_2 = \{1\} \bigcup C_2$. Since $Z(G) \neq 1$ by the assumption of (2), by using the arguments in the above paragraph we get $N = Z(G)$ and $|N| = |Z(G)| = 2$. Hence, by Corollary 1.3(4) we have $\chi(C_2) = -\chi(1)$. From this and (1) it follows that G is a 2-group. This establishes (2).

Since a nilpotent group has a non-trivial center, from (2) we get (3).

Finally, let us prove (4). Assume that χ does not vanish on exactly two conjugacy classes $C_1 = \{1\}$ and C_2 of G. Then, since G is a 2-group by the assumption of (4), by (3) we have $\chi(C_2) = -\chi(1)$. So, from (4) and (5) of Corollary 1.3 it follows that $|G| = 2\chi(1)^2$. Now, we assume that $|G| = 2\chi(1)^2$. By [1, (2.27), (2.30)] we have $\chi(1)^2 \leq |G : Z(\chi)| \leq |G : Z(G)|$. So, we get $Z(\chi) = Z(G)$, $\chi(1)^2 = |G : Z(\chi)|$ and $|Z(G)| = 2$. Hence, by [1, (2.30)], we know that χ does not vanish on exactly two conjugacy classes. This establishes (4), completing the proof of the theorem.

From Corollary 1.3(1) and Theorem 1.2 we immediately get the following

Corollary 1.4. *Let $\chi \in \mathrm{Irr}(G)$. Then the following statements hold*:

(1) *Suppose that G is a nilpotent group and χ has exactly two non-zero values. Then $G/\ker\chi$ is a 2-group and the two non-zero values of χ are $\chi(1)$ and $-\chi(1)$.*

(2) *Suppose that G is a 2-group. Then χ has exactly two non-zero values if and only if $|G/\ker\chi| = 2\chi(1)^2$.*

(3) *If χ has exactly two non-zero values and the two non-zero values are $\chi(1)$ and $-\chi(1)$, then $G/\ker\chi$ is a 2-group.*

Definition 1.1. *G is called a Camina group if G satisfies* (H1) *or* (H2) *with respect to $C_1 = \{1\}, C_2, \cdots, C_m$; $N := \bigcup_{i=1}^{m} C_i$ is called the kernel of the Camina group G.*

By Corollary 1.1(1) the kernel of a Camina group G is a non-trivial normal subgroup of G.

Remark 1.1. If G has an irreducible character χ not vanishing on exactly two conjugacy classes $C_1 = \{1\}$ and C_2 of G, then G is a Camina group with the kernel $N := \{1\} \bigcup C_2$. This is because in this case G satisfies (H1) with $X = \chi$ with respect to $C_1 = \{1\}$, C_2. Conversely, if G is a Camina group with the kernel N and $m(N) = 2$, then by (H1) G has an irreducible χ such that χ does not vanish on exactly two conjugacy classes of G.

By Theorem 1.1 and the definition of Camina groups, the following Corollary 1.5 is obviously true.

Corollary 1.5. *Let $M < N$ be non-trivial normal subgroups of G. If G is a Camina group with the kernel N, then G/M is a Camina group with the kernel N/M.*

The following Corollary 1.6 is a combination of [4,Proposition 3.1] and [5, Lemma 6.1(iv)]. We give here a proof for it from our point of view.

Corollary 1.6. *Let N be a non-trivial normal subgroup of G. Put $N = \bigcup\limits_{i=1}^{m} C_i$. Then the following conditions on (G, N) are equivalent.*

(1) *G is a Camina group with the kernel N.* (2) *If $g \in G - N$ then $gN \subseteq Cl_G(g)$.*

(3) *The conjugacy classes of $\overline{G} := G/N$ are $\overline{C}_1 = \{\overline{1}\}, \overline{C}_{m+1}, \cdots, \overline{C}_n$. In other words, if xN and yN are conjugate in G/N and are nontrivial, then x and y are conjugate in G.*

(4) *If $x \in G - N$, then $|C_G(x)| = |C_{G/N}(xN)|$.*

(5) *If χ is an irreducible character of G with $N \not\leq \ker\chi$, then χ vanishes on $G - N$.*

(6) *If χ and φ are irreducible characters of G such that $\varphi \in \mathrm{Irr}(G/N)$ and $N \not\leq \ker\chi$, then $\varphi\chi = \varphi(1)\chi$.*

Proof. Note that (2) and (3) are equivalent to (H2) with respect to $C_1 = \{1\}, C_2, \cdots, C_m$. So, (1), (2), and (3) are equivalent. Using the Second Orthogonality Relation twice, we get the equivalence of (4) and (5). By Corollary 1.1 we know that (1) implies (5). If G satisfies (5), then G satisfies (H1) with $X = \rho_G - \sum\{\vartheta(1)\vartheta : \vartheta \in \mathrm{Irr}(G/N)\}$ with respect to $C_1 = \{1\}, \cdots, C_m$, that is, G satisfies (1). So, (1) and (5) are eqiuvalent. Simple calculations for character values show that (5) and (6) are eqiuvalent. This completes the proof.

From Corollary 1.6(2) and Remark 1.1, we immediately get the following

Lemma 1.1. *Let N be a non-trivial normal subgroup of G. Assume that G is a Camina group with the kernel N. If $m(N) = 3$ and G has a normal subgroup M such that $1 < M < N$, then G is a Camina group with the kernel M. In particular, G has an irreducible character χ which does not vanish on exactly two conjugacy classes of G.*

Let G be a p-group, and N be a non-trivial normal subgroup of G. According to Remark 1.1 and the proof of Theorem 1.2, we know that if $m(N) = 2$ and G is a Camina group with the kernel N, then $p = 2$, $N = Z(G)$ and $|N| = 2$. Similarly, we have the following

Theorem 1.3. *Let G be a p-group, and N be a non-trivial normal subgroup of G. Assume that $m(N) = 3$ and G is a Camina group with the kernel N. Then $p = 3, N = Z(G)$ and $|N| = 3$.*

Proof. By hypothesis and Corollary 1.1(5) we see that either $1 < Z(G) < N$ or $Z(G) = N$. In order to complete the proof, it is enough to show that $Z(G) = N$.

Suppose that $1 < Z(G) < N$. Then $|Z(G)| = 2$ and so G is a 2-group. Further, by Lemma

1.1 and Theorem 1.2(4) we know that G has an irreducible character χ such that $|G| = 2\chi(1)^2$. Since $G/Z(G)$ is a Camina group with the kernel $N/Z(G)$ and $m_{G/Z(G)}(N/Z(G)) = 2$, by Remark 1.1 and Theorem 1.2(4) we know that $G/Z(G)$ has an irreducible character φ such that $|G/Z(G)| = 2\varphi(1)^2$. So, letting $|G| = 2^r$, we get $\chi(1)^2 = 2^{r-1}$ and $\varphi(1)^2 = 2^{r-2}$. It follows that both $r-1$ and $r-2$ are even, a contradiction. So, we have $Z(G) = N$, completing the proof.

§2. Some Basic Facts about a Camina Groups G with the Kernel G'

For the reader's convenience, we mention some basic facts about a Camina group G with the kernel G', which will be used in Sections 3 and 4.

Lemma 2.1.[6] *Suppose that $1 < G' < G$ and G is a Camina group with the kernel G'. Then one of the following assertions is true*:

(1) *$G = (C, G')$, a Frobenius group with the Frobenius kernel G' and a cyclic Frobenius complement C.*

(2) *G is a p-group.*

(3) *$G = RP$, where P is a Sylow p-subgroup of G for some prime p and R is the normal p-complement in G with $R < G'$. In addition, if P has class 2, then $P = Q_8$, the quaternion group of order 8, and $G = (Q_8, R)$, a Frobenius group with the Frobenius kernel R and a Frobenius complement $P = Q_8$.*

We omit an easy proof of the following Lemma 2.2.

Lemma 2.2. *Let G be a p-group. If G has class 2 and G is a Camina group with the kernel G', then G is semi-extraspecial.* (*See* [7] *for the definition of semi-extraspecial p-groups.*)

Lemma 2.3. *Let G be a non-Abelian 2-group. If G is a Camina group with the kernel G', then G is semi-extraspecial.*

Proof. By hypothesis and [8, Theorem 3.1], G has class 2. So, by Lemma 2.2, G is semi-extraspecial. This completes the proof.

§3. Restricted V^3-Groups

Following [9], we introduce some additional notations. For a group G, $\mathrm{Irr}_1(G)$ denotes the set of all nonlinear irreducible characters of G. We write $E(p^n)$ and $C(n)$ to denote an elementary Abelian group of order p^n and a cyclic group of order n, respectively.

For $\chi \in \mathrm{Irr}_1(G)$, put $V(\chi) = |\{\chi(x) : x \in G\}|$. A non-Abelian group G is said to be a V^3-group if $V(\chi) = 3$ for all $\chi \in \mathrm{Irr}_1(G)$. In this section we discuss a subclass of the class of V^3-groups, called restricted V^3-groups. Applying the Camina group theory, we establish a structure theorem on restricted V^3-groups. By using this structure theorem, the structure theorem on V^3-groups[9,main Theorem] is easily obtained. It turns out that the subclass of restricted V^3-groups is almost the class of V^3-groups with some exceptions in 2-groups. Furthermore, it will be seen that for restricted V^3-groups we can get more information than for V^3-groups.

A group G is said to be a restricted V^3-group if G is a V^3-group and satisfies the following condition: $\ker\chi < G'$ for every $\chi \in \mathrm{Irr}_1(G)$. Let G be a V^3-group. If G is not a restricted

V^3-group, then we say that the V^3-group G is non-restricted.

Before establishing our results in this section, we make some remarks. Let G be a V^3-group. Then, according to [1, (3.15)], each $\chi \in \mathrm{Irr}_1(G)$ has exactly two non-zero values. This fact allows us to apply Corollary 1.3 and other results on Camina groups to the study of groups $G/\ker\chi$ (see Remark 1.1), where $\chi \in \mathrm{Irr}_1(G)$. In particular, if G has a faithful $\chi \in \mathrm{Irr}_1(G)$ or if G has only one minimal normal subgroup (in this case G has a faithful $\chi \in \mathrm{Irr}_1(G)$), we can apply Corollary 1.3 and other results on Camina groups to the study of G.

Lemma 3.1.[9,p.21] *The following two statements hold*:

(1) *Non-Abelian factor groups of V^3-groups are V^3-groups.* (2) *V^3-groups are solvable.*

Lemma 3.2.[9,Lemma 3] *The group $G = (Q_8, E(3^4))$ has a faithful irreducible character.*

Lemma 3.3. *Let G be a V^3-group. If there exists a $\chi \in \mathrm{Irr}_1(G)$ such that χ does not vanish on $G - G'$, then G is non-restricted and $G/(G' \cap \ker\chi)$ is a 2-group.*

Proof. Note that for an arbitrary $\varphi \in \mathrm{Irr}_1(G)$, if $\varphi(x) \neq \varphi(1)$ for $x \in G$, then $\varphi(x)$ is either zero or a negative rational integer (Corollary 1.3(4)).

Take $g \in G - G'$ such that $\chi(g) \neq 0$, and take $\lambda \in \mathrm{Irr}(G/G')$ such that $\lambda(g) \neq 1$. Considering χ and $\lambda\chi$, by virtue of the remark in the above paragraph it is easy to check that $\lambda(g) = -1$, $\chi(g) = \pm\chi(1)$. It follows that G is non-restricted and the only non-zero values of $\lambda\chi$ are $\chi(1)$ and $-\chi(1)$ for every $\lambda \in \mathrm{Irr}(G/G')$. By Corollary 1.4(3), $G/\ker\lambda\chi$ is a 2-group for every $\lambda \in \mathrm{Irr}(G/G')$. So, noticing that $G' \cap \ker\chi = \bigcap\{\ker\lambda\chi : \lambda \in \mathrm{Irr}(G/G')\}$, we see that $G/(G' \cap \ker\chi)$ is a 2-group. The proof is complete.

Theorem 3.1. *If G is a restricted V^3-group, then one of the following assertions holds*:

(1) *G is a semi-extraspecial 2-greoup.*

(2) *$G = (Q_8, E(3^2))$, a Frobenius group with the Frobenius kernel $E(3^2)$ and a Frobenius complement Q_8.*

(3) *$G = (C(p^n - 1), G')$, a Frobenius group with the Frobenius kernel G' and a Frobenius complement $C(p^n - 1)$, and $G' \in \mathrm{Syl}_p(G)$. In addition, if G' is Abelian or if $p \neq 2$, then G' is elementary Abelian and for each $\chi \in \mathrm{Irr}_1(G)$, $G/\ker\chi \cong (C(p^n - 1), E(p^n))$.*

(Note: It is easy to see that groups (1) and (2) are restricted V^3-groups.)

Proof. Since G is restricted, by Lemma 3.3 and Corollary 1.6(5) G is a Camina group with the kernel G' (we have $1 < G' < G$ by Lemma 3.1(2)). So, noticing that $|G|$ must be even, by Lemma 2.1 we need to distinguish the following three cases.

(a) G is a 2-group.

In this case, by Lemma 2.3 we see that G is a semi-extraspecial 2-group, the type (1) in the statement of the theorem.

(b) $G = RP$, where $P \in \mathrm{Syl}_p(G)$, R is the normal p-complement in G and $1 < R < G'$.

Note that $P \cong G/R$ is a Camina group with the kernel $P \bigcap G' \cong G'/R$ and thus $P' = P \bigcap G' \neq 1$ (Corollary 1.5 and Corollary 1.1(5)). Hence, $P \cong G/R$ is also a restricted V^3-group, and so P is a semi-extraspecial 2-group. Then by Lemma 2.1 we get $P = Q_8$ and $G = (Q_8, R)$. In particular, R is Abelian and every normal subgroup of G either contains R or is contained in R.

By induction on $|G|$ we shall verify that G is the type (2) in the statement of the theorem.

Suppose that G has two different minimal normal subgroup, say B and H. Then we have $BH \leq R$. By induction we see that both G/B and G/H are the type (2). It follows that $|B| = |H| = 3^2$ and $R = H \times B$. So, $G = (Q_8, E(3^4))$ and hence G has a faithful $\chi \in \mathrm{Irr}_1(G)$ by Lemma 3.1, contradicting Corollary 1.3. Hence, G has only one minimal normal subgroup, say H, and thus R is a q-group for some prime $q \neq 2$. Then, since R is Abelian, by virtue of Corollary 1.6(4) (or [3, Theorem 2.5(b)]) we get $R = H$ and hence R is the unique minimal normal subgroup of G. So, by Corollary 1.3(2) we conclude that $R = E(3^2)$ and G is the type (2).

(c) $G = (C, G')$, a Frobenius group with the Frobenius kernel G' and a cyclic Frobenius complement C.

By induction on $|G|$ we shall verify that G is the type (3) in the statement of the theorem.

Since G' is nilpotent, G' is a p-group by induction. Let us show that $|C| = p^n - 1$ for some positive integer n. For this, we may assume, without loss of generality, that G' is a minimal normal subgroup of G. So, G' is a unique minimal normal subgroup of G and thus by Corollary 1.3(2) we get $|C| = p^n - 1$.

Now we assume that G' is Abelian. If $\chi \in \mathrm{Irr}_1(G)$, then by virtue of Corollary 1.6(4) (or [3, Theorem 2.5(b)]) and Corollary 1.3(2) we have $G/\ker\chi \cong (C(p^n - 1)), E(p^n))$. Then, since the intersection of the kernels of all non-linear irreducible characters of G is equal to 1, we conclude that G' is elementary Abelian. Note that if $p \neq 2$, then G' is Abelian. So, we have proved that G is of the type (3). This completes the proof.

Lemma 3.4 *If G is a non-restricted V^3-group, then G is a (non-Abelian) 2-group.*

Proof. Since G is a non-restricted V^3-group, there exists a $\chi \in \mathrm{Irr}_1(G)$ such that $\ker\chi \not\leq G'$. So, by Lemma 3.3 $G/(G' \cap \ker\chi)$ is a non-Abelian 2-group. If $G' \cap \ker\chi = 1$, then we are done. So we assume that $G' \cap \ker\chi \neq 1$. Let $R \leq G' \cap \ker\chi$ be a minimal normal subgroup of G. Clearly, G/R is a non-restricted V^3-group and hence G/R is a (non-Abelian) 2-group by induction. Note that R is a p-group. Set $|R| = p^n$. If $p = 2$, we are done. So, we assume that $p \neq 2$. Suppose that R is a unique minimal normal subgroup of G. Then G is a doubly transitive Frobenius group with the Frobenius kernel R (Remark 1.1, Corollary 1.3 and [2, Proposition 1, p.156]). It follows that $p^n - 1$ is a power of 2, and thus $p^n = 3^2$ and $G \cong (Q_8, E(3^2))$. This implies that G is a restricted V^3-group, a contradiction. Hence, R can not be a unique minimal normal subgroup of G. Let $R_1 \neq R$ be a minimal normal subgroup of G. Then $|R_1| = 2$. Clearly, G/R_1 is not Abelian. If G/R_1 is non-restricted, then by induction we are done. So, we assume that G/R_1 is restricted. Further, we can assume that G/R_1 is not a 2-group. Then, since a Sylow 2-subgroup of G/R_1 is not cyclic, by Theorem 3.1 we have $G/R_1 \cong (Q_8, E(3^2))$. It follows that $R_1 = Z(G)$ and R and R_1 are the only minimal normal subgroups of G. Hence, the sum of squares of degrees of all faithful irreducible characters of G is greater than $|G| - |G : R| - |G : R_1| (> 0)$. This implies that G has a faithful irreducible character, and hence G has only one minimal normal subgroup (Corollary 1.3), a contradiction. This completes the proof.

From Theorem 3.1, Lemma 3.4 and Corollary 1.4(2) we immediately get the following Corollary 3.1. It is [9, Main Theorem], but the mistake in the statement (c) of [9, Main Theorem] has been corrected.

Corollary 3.1.[9,Main Theorem] *If G is a V^3-group, then one of the following assertions holds*:

(1) *G is a 2-group such that $|G/\ker\chi| = 2\chi(1)^2$ for all $\chi \in \mathrm{Irr}_1(G)$.* (2) *$G = (Q_8, E(3^2))$, a Frobenius group with the kernel $E(3^2)$ and a Frobenius complement Q_8.* (3) *$G = (C(p^n - 1), G')$, a Frobenius group with the Frobenius kernel G' and a Frobenius complement $C(p^n - 1)$, and $G' \in \mathrm{Syl}_p(G)$. In addition, if G' is Abelian or if $p \neq 2$, then G' is elementary Abelian.*

§4. *D*-Groups

A nonAbelian group G is called a D-group if G satisfies the following condition D.

D: If $1 < N \le G'$, N is normal in G, and $\lambda \in \mathrm{Irr}^{\#}(N)$, then the degrees of the irreducible constituents of the induced character λ^G are distinct.

A non-Abelian group G is called a $\overline{D}$ group if G satisfies the following condition $\overline{D}$.

$\overline{D}$: If $N > 1$ is any normal subgroup of G and $\lambda \in \mathrm{Irr}^{\#}(N)$, then the degrees of the non-linear irreducible constituents of the induced character λ^G are distinct.

In [10, Main Theorem], the solvable D-groups were classified. In this section, applying the Camina group theory, we not only give a simple proof for [10, Main Theorem] but also weaken the condition in the theorem. More precisely, the condition that G is solvable in [10, Main Theorem] is replaced by the condition that $G' < G$. Also, we shall see that [10, Theorem 9] is just an immediate consequence of the following Theorem 4.1.

Lemma 4.1. *Let $G = (H, R)$, a Frobenius group with the Frobenius kernel R and a Frobenius complement H. Assume that R is Abelian and $R \le G'$. If G is a D-group, then R is a minimal normal subgroup of G, that is, R is an elementary Abelian p-group and is an irreducible H-module over the prime field F_p.*

Proof. Let $M \le R$ be a minimal normal subgroup of G. Take $\lambda \in \mathrm{Irr}^{\#}(M)$. Since R is Abelian, there exists a $\varphi \in \mathrm{Irr}(R)$ such that $\varphi_M = \lambda$. By [1, (6.17)] we have $\mathrm{Irr}(\lambda^R) = \{\beta\varphi : \beta \in \mathrm{Irr}(R/M)\}$. By [1, (6.34)], $(\beta\varphi)^G \in \mathrm{Irr}(G)$. Note that $\beta\varphi \in \mathrm{Irr}^{\#}(R)$ for every $\beta \in \mathrm{Irr}(R/M)$ and $R \le G'$ by hypothesis. Then, since $(\beta\varphi)^G(1) = |G : R|$ and G is a D-group, we see that $(\beta\varphi)^G = \varphi^G$ for every $\beta \in \mathrm{Irr}(R/M)$, and so $\lambda^G = |R/M|\chi$, where $\chi = \varphi^G \in \mathrm{Irr}(G)$. Hence, χ vanishes on $G - M$ and G is a Camina group with the kernel M (Corollary 1.6(5)). So, if $R \neq M$, then for $x \in R - M$ we have $|C_{G/M}(xM)| = |C_G(x)| = |C_R(x)| = |R|$ (Corollary 1.6(4)). This is impossible. Therefore, $R = M$ and R is a minimal normal subgroup of G. This completes the proof.

Theorem 4.1. *Let G be a non-Abelian group, and assume that $G' < G$. If G is a D-group, then one of the following assertions holds*:

(1) *G is an extra-special p-group.*

(2) *$G = (Q_8, E(p^n))$, a Frobenius group with the Frobenius kernel $E(p^n)$ and a Frobenius complement Q_8, and Q_8 acts on $E(p^n)$ irreducibly.*

(3) *$G = (C(s), E(p^n))$, a Frobenius group with the Frobenius kernel $E(p^n)$ and a cyclic Frobenius complement $C(s)$, and $C(s)$ acts on $E(p^n)$ irreducibly.*

Proof. At first we show that G is a Camina group with the kernel G'. Let $\lambda \in \mathrm{Irr}^{\#}(G')$, and $\vartheta \in \mathrm{Irr}(G/G')$. Since G is a D-group, degrees of all the irreducible constituents of λ^G

are distinct. On the other hand, we have $\vartheta\lambda^G = (\vartheta_{G'}\lambda)^G = \lambda^G$. So, we conclude that for every irreducible constituent χ of λ^G, $\vartheta\chi = \chi$. Note that for any $\chi \in \mathrm{Irr}_1(G)$, there exists a $\lambda \in \mathrm{Irr}^{\#}(G')$ such that $\chi \in \mathrm{Irr}(\lambda^G)$. So, for any $\chi \in \mathrm{Irr}_1(G)$ and any $\vartheta \in \mathrm{Irr}(G/G')$, we have $\vartheta\chi = \chi$, and thus G is a Camina group with the kernel G' (Corollary 1.6(6)). Hence, in view of Lemma 2.1 we have to consider the following possibilities for G.

(a) G is a p-group.

Let $x \in Z(G)$ be an element of order p, and $\lambda \in \mathrm{Irr}^{\#}(\langle x\rangle)$. Put $\mathrm{Irr}(\lambda^G) = \{\chi^1, \cdots, \chi^s\}$. Note that χ^i's are non-linear as $\langle x\rangle \le Z(G) \le G'$ (Corollary 1.1(5)) and $\lambda \ne 1_{\langle x\rangle}$. We have $\chi^i_{\langle x\rangle} = \chi^i(1)\lambda$, and so $\lambda^G = \chi^1(1)\chi^1 + \cdots + \chi^s(1)\chi^s$. Set $|G| = p^n$ and $\chi^i(1) = p^{r_i}$ for $i = 1, \cdots, s$. Then we have $p^{n-1} = \lambda^G(1) = p^{2r_1} + \cdots + p^{2r_s}$. By noticing that r_i's are distinct (because G is a D-group), from the above eqaulity it follows that $s = 1$ and thus $\lambda^G = \chi^1(1)\chi^1$.

So, $p^{n-1} = |G/\langle x\rangle| = \lambda^G(1) = (\chi^1(1))^2$ and thus n is odd. If $G/\langle x\rangle$ is not Abelian, then by the conclusion just obtained we see that $n-1$ is odd, a contradiction. Hence we have $G' \le \langle x\rangle \le Z(G)$, and so $G' = Z(G)$ and $|Z(G)| = p$. This means that G is of the type (1) in the statement of the theorem.

(b) $G = RP$, where $P \in \mathrm{Syl}_p(G)$, R is the normal p-complement in G and $1 < R < G'$.

Since $P \cong G/R$ is also a D-group, by (a) P is an extraspecial p-group. So, by Lemma 2.1 we get $P \cong Q_8$ and $G = (Q_8, R)$. In particular, R is Abelian, and so by Lemma 4.1 G is of the type (2) in the statement of the theorem.

(c) $G = (C, G')$, a Frobenius group with the Frobenius kernel G' and a cyclic Frobenius complement C. In particular, G' is nilpotent.

Applying induction on $|G|$ we shall verify that G is of the type (3) in the statement of the theorem.

Suppose that G' is not of prime-power order. Then by induction we conclude that G' is Abelian and so G' is of prime-power by Lemma 4.1, a contradiction. G' is therefore a p-group.

Suppose that G' is not Abelian, and let M be any normal subgroup of G with $1 < M < G'$. By induction, G'/M is a minimal normal subgroup of G/M. It follows from this that $G'' = \Phi(G') = Z(G')$ and G'' is a unique minimal normal subgroup of G. Hence, both G'' and G'/G'' are faithful irreducible C-modules over the prime field F_p , and thus by [11, Satz 3.10, p.165] we have $|G''| = |G'/G''| = p^m$, where m is the least positive integer such that $p^m \equiv 1 \pmod{|C|}$.

Take any $\lambda \in \mathrm{Irr}^{\#}(G'')$. Suppose that $\lambda^G = e_1\chi^1 + \cdots + e_s\chi^s$, $\quad \mathrm{Irr}(\lambda^G) = \{\chi^1, \cdots, \chi^s\}$. Note that $\chi^1(1), \cdots, \chi^s(1)$ are distinct because G is a D-group. Let $\chi^i_{G''} = e_i(\lambda_1 + \cdots + \lambda_n)$ be the Clifford's decomposition, $\lambda_1 = \lambda$. Then $\chi^i(1) = e_i n\lambda(1) = e_i n$. Since the semidirect product $G'' \rtimes C$ is a Frobenius group and $G'' = Z(G')$, we have $I_G(\lambda) = G'$. It follows that $\chi^i(1) = e_i n = e_i|G : I_G(\lambda)| = e_i|G : G'| = e_i|C|$ for $i = 1, \cdots, s$. Then, since all the $\chi^i(1)$'s are distinct, all the e_i's are distinct. On the other hand, we have

$$|C||G'/G''| = \lambda^G(1) = e_1\chi^1(1) + \cdots + e_s\chi^s(1) = |C|(e_1^2 + \cdots + e_s^2),$$
$$|G'/G''| = e_1^2 + \cdots + e_s^2. \qquad (*)$$

Note that $|G'/G''|$ is a power of p, and that e_i's are also powers of p because e_i's are degrees of irreducible projective representations of the p-group $I_G(\lambda)/G'' = G'/G''$. Then, since all the e_i's are distinct, from the above equality $(*)$ we get $s = 1$ and so $\lambda^G = e_1\chi^1$. From this and Corollary 1.6(5) it follows that G is a Camina group with the kernel G'', and thus for $x \in G' - G''$ by Corollary 1.6(4) we have

$$|C_G(x)| = |C_{G/G''}(xG'')| = |G'/G''| = p^m = |G''| = |Z(G')| < |C_G(x)|,$$

a contradiction. Hence, G' is Abelian. Then by Lemma 4.1 G' is a faithful irreducible C-module over the prime field F_p. So, by [11, Satz 3.10, p.165] we have $|G'| = p^m$, where m is the order of p (mod $|C|$). Therefore G is the type (3). This completes the proof.

Corollary 4.1. *If G is a non-solvable D-group, then $G' = G$.*

A $\overline{D}$-group is obviously a D-group. So, we immediately obtain the following

Corollary 4.2.[10,Theorem 9] *If G is a non-solvable $\overline{D}$-group, then $G' = G$.*

Corollary 4.3.[12,Theorem] *Let G be a non-Abelian group. Suppose that the degrees of the non-linear irreducible characters of G are distinct. Then one of the following assertions holds*:

(1) *G is an extraspecial 2-group.* (2) $G \cong (Q_8, E(3^2))$. (3) $G \cong (C(p^n - 1), E(p^n))$.

Proof. It is easy to show that $G' < G$ (see [12, Lemma 1(3)]). Also, G is clearly a D-group of even order. Hence, by Theorem 4.1 we have

(a) G is an extraspecial 2-group.

(b) $G \cong (Q_8, E(p^n))$, a Frobenius group with the Frobenius kernel $E(p^n)$.

In this case, by [1, (6.32) and (6.34)] we get $(p^n - 1)/8 = 1$, $p^n = 9$, $G \cong (Q_8, E(3^2))$.

(c) $G \cong (C(s), E(p^n))$, a Frobenius group with the Frobenius kernel $E(p^n)$.

In this case, by [1, (6.32) and (6.34)] we get $(p^n - 1)/s = 1$, $s = p^n - 1$, $G \cong (C(p^n - 1), E(p^n))$. The proof is complete.

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