Chin. Ann. of Math.
20B: 1(1999),51-56.

A CHARACTERISTIC OF HYPERBOLIC SPACE FORM**

CHEN Zhihua* ZHOU Chaohui*

Abstract

In this paper, the authors obtain a characteristic that a complete Riemannian manifold turns into a hyperbolic space form.

Keywords Hyperbolic space form, Jacobi field, Hessian
1991 MR Subject Classification 53C20
Chinese Library Classification O186.12

§1. Introduction

M. Obata[1] proved the following characteristic of a sphere: In order for a complete Riemannian manifold M of dimension n to be a sphere of radius $a^{-1} > 0$ it is necessary and sufficient that M admits a C^2 real-valued function Φ with $D^2\Phi = -a^2\Phi g$, where g is the Riemannian metric on M and $D^2\Phi$ the Hessian of Φ. The similar result on the hyperbolic space form is still not obtained by now[2]. In this paper, we obtain a theorem on the hyperbolic space form, which is similar to the Obata's characteristic on the sphere.

§2. The Theorem and the Proof of its Necessity

Theorem 2.1. *Let M be a complete manifold of dimension n which is isomorphic to hyperbolic space form $\mathbf{H}^n(-c)$ iff M admits a C^2 real-valued function Φ satisfying*

$$D^2\Phi = c^2\Phi g \qquad (c \neq 0), \tag{2.1}$$

and such that Φ takes at least extreme value on some point of M, in (2.1) g is also the Riemannian metric of M.

Proof. The both sides of (2.1) are the bilinear symmetric form on $T_x(M), \forall x \in M$. So we can only consider the case $c = 1$. The proof of the case $c \neq 1$ is completely similar to the case $c = 1$. Here we take that the model of $\mathbf{H}^n(-1)$ is a spacelike hypersurface in the Lorentzian space $\mathbf{L}^{n,1}$

$$\mathbf{H}^n(-1) = \Big\{(x^0, x^1, \cdots\cdots, x^n) \in \mathbf{L}^{n,1}\Big|(x^0)^2 - \sum_{i=1}^{n}(x^i)^2 = 1, x^0 > 0\Big\}. \tag{2.2}$$

The Riemannian metric g on $\mathbf{H}^n(-1)$ is the restriction of the Lorentzian metric $ds_L^2 = \sum\limits_{i=1}^{n}(dx^i)^2 - (dx^0)^2$ of $\mathbf{L}^{n,1}$. Now we use $\dfrac{\partial}{\partial x^0}, \dfrac{\partial}{\partial x^1}, \cdots, \dfrac{\partial}{\partial x^n}$ to denote the natural basis of

Manuscript received September 24, 1997.
*Department of Applied Mathematics, Tongji University, Shanghai 200092, China.
E-mail: zzzhhc@online.sh.cn
**Project supported by the National Natural Science Foundation of China.

$\mathbf{L}^{n,1}$, and set $X = x^0 \dfrac{\partial}{\partial x^0} + \sum\limits_{i=1}^{n} x^i \dfrac{\partial}{\partial x^i}$, the position vector of $\mathbf{H}^n(-1)$, then $\langle X, X\rangle = -1$ with respect to the Lorentzian metric of $\mathbf{L}^{n,1}$. On the other hand, from the defining equation (2.2) of $\mathbf{H}^n(-1)$, we obtain

$$x^0 dx^0 - \sum_{i=1}^{n} x^i dx^i = 0, \tag{2.3}$$

hence the position vector X of $\mathbf{H}^n(-1)$ is orthogonal to $\mathbf{H}^n(-1)$ with respect to the Lorentzian metric. We choose a local normal basis $\{e_\alpha\}; \alpha = 1, \cdots, n$ on M, and

$$e_\alpha = \sum_{A=0}^{n} a_\alpha^A \frac{\partial}{\partial x^i}, \quad A = 0, 1, \cdots, n,$$

then we have

$$\langle e_\alpha, e_\beta\rangle = \sum_{i=1}^{n} a_\alpha^i a_\beta^i - a_\alpha^0 a_\beta^0 = \delta_{\alpha\beta}, \tag{2.4}$$

$$\langle e_\alpha, X\rangle = \sum_{i=1}^{n} a_\alpha^i x^i - a_\alpha^0 x^0 = 0. \tag{2.5}$$

By covariant differentiation of $\langle X, X\rangle = \sum\limits_{i=1}^{n} \left(x^i\right)^2 - (x^0)^2 = -1$ with respect to $\{e_\alpha\}$ we obtain

$$\sum_{i=1}^{n} x^i x_\alpha^i - x^0 x_\alpha^0 = 0. \tag{2.6}$$

By covariant differentiation of (2.6) we obtain

$$\sum_{i=1}^{n} x_\beta^i x_\alpha^i - x_\beta^0 x_\alpha^0 + \sum_{i=1}^{n} x^i x_{\alpha\beta}^i - x^0 x_{\alpha\beta}^0 = 0. \tag{2.7}$$

Then from $e_\alpha = \sum\limits_{i=1}^{n} a_\alpha^i \frac{\partial}{\partial x^i} + a_\alpha^0 \frac{\partial}{\partial x^0}$ we obtain $e_\alpha(x^A) = a_\alpha^A,\ A = 0, \cdots, n$. Since $x^0, x^1, \cdots, x^n$ are the coordinate functions, so $a_\alpha^A = x_\alpha^A,\ 1 \le \alpha \le n;\ 0 \le A \le n$. Then from (2.7) we obtain

$$\sum_{i=1}^{n} x_{\alpha\beta}^i x^i - x_{\alpha\beta}^0 x^0 = -\delta_{\alpha\beta}. \tag{2.8}$$

On the other hand,

$$x_{\alpha\beta}^A = e_\beta x_\alpha^A - \sum_{r}^{n} x_r^A \Gamma_{\alpha\beta}^r, \quad A = 0, \cdots, n, \tag{2.9}$$

where $\Gamma_{\alpha\beta}^r$ are connection coefficients with respect to $\{e_\alpha\}$. By the definition,

$$\Gamma_{\alpha\beta}^r = \langle e_r, \triangledown_{e_\beta} e_\alpha\rangle = \langle e_r, e_\beta e_\alpha\rangle = \sum_{j=1}^{n} x_r^i e_\beta x_\alpha^j - x_r^0 e_\beta x_\alpha^0. \tag{2.10}$$

Substituting (2.10) into (2.9) we have

$$x_{\alpha\beta}^A = e_\beta x_\beta^A - \sum_{r} \Big(\sum_{j=1}^{n} x_r^A x_r^j e_\beta x_\alpha^j - x_r^0 e_\beta x_\alpha^0 x_r^A \Big). \tag{2.11}$$

Now we calculate

$$\begin{aligned}\sum_{i=1}^{n} x_r^i x_{\beta\beta}^i - x_r^0 x_{\beta\beta}^0 &= \sum_{i=1}^{n} x_r^i (e_\beta x_\beta^i - \Gamma_{\beta\beta}^{\alpha} x_\alpha^i) - x_r^0 (e_\beta x_\beta^0 - \Gamma_{\beta\beta}^{\alpha} x_\alpha^0) \\ &= \sum_{i=1}^{n} x_r^i e_\beta x_\beta^i - x_r^0 e_\beta x_\beta^0 - \sum_{i=1}^{n} \Gamma_{\beta\beta}^{\alpha} x_r^i x_\alpha^i + \Gamma_{\beta\beta}^{\alpha} x_\alpha^0 x_r^0 \\ &= \sum_{i=1}^{n} x_r^i e_\beta x_\beta^i - x_r^0 e_\beta x_\beta^0 - \Gamma_{\beta\beta}^{r} = 0, \end{aligned} \tag{2.12}$$

where the last equality follows from (2.11).(2.12) indicates that the vectors $Y_\alpha = \sum_{A=0}^{n} x_{\alpha\alpha}^A \frac{\partial}{\partial x^A}$ are orthogonal to $e_\beta; 1 \le \beta \le n$ with respect to Lorentzian metric ds_L^2, i.e. $\langle Y_\alpha, e_\beta \rangle = 0$ for all $1 \le \alpha, \beta \le n$. Thus Y_α are all orthogonal to $\mathbf{H}^n(-1)$, then Y_α are all parallel to the position vector X. So $Y_\alpha = \lambda_\alpha X$. By (2.8) we have $\langle Y_\alpha, X \rangle = \lambda_\alpha \langle X, X \rangle = -\lambda_\alpha = -1$, and so $\lambda_\alpha = 1 (1 \le \alpha \le n)$. From $Y_\alpha = X$, we obtain

$$D^2 x^A (e_\alpha, e_\alpha) = x^A g(e_\alpha, e_\alpha),\ A = 0, \cdots, n \tag{2.13}$$

for all coordinate functions x^A. Since the equation (2.1) is the symmetric bilinear quadratic form on every $T_p(\mathbf{H}^n(-1)), \forall p \in M$, for all coordinate functions x^A, $0 \le A \le n$, the equality (2.1) is satisfied by $(e_\alpha, e_\alpha), 1 \le \alpha \le n$, i.e. by all (e_α, e_β), furthermore by every $X, Y \in T_p(\mathbf{H}^n(-1))$ and $\forall p \in \mathbf{H}^n(-1)$, especially x^0 is the coordinate function which takes the minimum value at $(1, 0, \cdots, 0) \in \mathbf{H}^n(-1)$. Hence we complete the proof of the necessity of Theorem 2.1.

§3. Proof of the Sufficiency

Let M be a complete Riemannian manifold of dimension n and admit C^2 real-valued function Φ satisfying the equation (2.1), and Φ takes the extreme value at some point $p \in M$. Now assume that γ_X is a geodesic which issues from the point p and in the direction of unit vector $X \in T_pM, \dot{\gamma}(0) = X$. We denote its arc length parameter by ρ. When the equation (2.1) restricts to γ_X, it reduces to an ordinary differential equation

$$\frac{d^2\Phi}{d\rho^2} = \Phi(\rho). \tag{3.1}$$

Thus Φ is given by

$$\Phi = A \cosh \rho + B \sinh \rho. \tag{3.2}$$

p is an extreme point of Φ, so that $d\Phi(p) = 0 = B$. On the other hand, the solutions of (2.1) admit up to a constant multiplied factor, without loss of generality, we can assume $A = \Phi(p) = 1$, thus $\Phi|_{\gamma_X(\rho)} = \cosh \rho$. Now we define $\Phi|_{\gamma_X(\rho)} = \cosh \rho, 0 \le \rho \le +\infty$ for every unit vector $X \in T_pM$. Since M is complete for $\forall q \in M$, there exists a geodesic γ_X which issues from p and $\rho > 0$ such that $q = \gamma_X(\rho)$ so that $\Phi(q) = \cosh \rho$.

Now we prove that for every point $q \in M - \{p\}$ there exists only one geodesic which issues from p pass through the point q, e.g. for the point p, there is no conjugate point of p for every geodesic γ_X which issues from p. Otherwise, we can assume on some geodesic γ_X from p, there exists $q = \gamma_X(b)$ which is the nearest conjugate point of $p = \gamma_X(0)$ on γ_X, now there exists a nontrivial proper Jacobi field $Y(\rho)$ along $\gamma_X([0, b])$ such that $Y \perp \dot{\gamma}$, and the index form I vanishes on (Y,Y)[3,4] , where the index form is

$$I(Y,Y)=\int_0^b [\langle \dot{Y},\dot{Y}\rangle - R(\dot{\gamma}\wedge Y)|Y|^2]d\rho \tag{3.3}$$

and $\dot{Y}=\dfrac{d}{d\rho}Y=D_{\dot{\gamma}}Y$, $R(\dot{\gamma}\wedge Y)$ is the radical curvature decided by $\dot{\gamma}\wedge Y$.

Now we choose sufficiently small $\delta,\delta_1>0$ so that there is no conjugate point of $\gamma_X(b-\delta_1)$ on $\gamma_X((b-\delta_1,b+\delta])$. Since $\gamma_X(b)$ is the conjugate point of $p=\gamma_X(0)$, so we can choose a nontrivial proper vector field $Z(\rho)$ perpendicular to γ_X on $\gamma_X([0,b+\delta])$ and

$$I(Z,Z)=\int_0^{b+\delta} [\langle \dot{Z},\dot{Z}\rangle - R(\dot{\gamma}\wedge Z)|Z|^2]d\rho<0, \tag{3.4}$$

and the choice of $Z(\rho)$ can equal to Y in (3.3) on $\gamma_X([0,b-\delta_1])$. Since $Z(b+\delta)=0$ and $D_Z Z(b+\delta)=0$, we choose $0<\delta_2<\delta$ satisfying

(1) $Z(b+\delta_2)\neq 0$, (2) $\dfrac{d}{d\rho}\left|Z\right|^2(b+\delta_2)<0$,

$$\text{(3)}\qquad \int_0^{b+\delta_2} [\langle \dot{Z},\dot{Z}\rangle - R(\dot{\gamma}\wedge Z)|Z|^2]d\rho+\left.\langle D_Z Z,\dot{\gamma}\rangle\right|_{b+\delta_2}<0. \tag{3.5}$$

It is obvious that there exists δ_2 satisfying these conditions. At first, Z is not identically vanishing on $[b,b+\delta]$, (2) is proved by $Z(b+\delta)=0$, and (3) is obtained by the continuity of integral in (3.4).

Now let $W(\rho)$ be a Jacobi vector field along $\gamma_X([b-\delta_1,b+\delta_2])$, which is determined by $W(b-\delta_1)=Z(b-\delta_1)=Y(b-\delta_1)$ and $W(b+\delta_2)=Z(b+\delta_2)$. Set

$$\overline{Y}(\rho)=\begin{cases} Y(\rho), & 0\le\rho\le b-\delta_1,\\ W(\rho), & b-\delta_1\le\rho\le b+\delta_2.\end{cases} \tag{3.6}$$

$\overline{Y}(\rho)$ is a Jacobi vector field along $\gamma_X([0,b+\delta_2])$. So $\overline{Y}$ is a geodesic variation vector field with the base curve γ_X, and we denote this variation by $F:[0,\varepsilon]\times[0,b+\delta_2]\longrightarrow M$, $F(0,\rho)=\gamma_X(\rho)$, $F(s,\rho)=C_s(\rho)$.

Now by Sygne formula, we get

$$\begin{aligned} &D^2\rho\left(\overline{Y}(b+\delta_2),\overline{Y}(b+\delta_2)\right)\\ &=\int_0^{b+\delta_2}[\langle \overline{Y},\overline{Y}\rangle - R(\dot{\gamma}\wedge\overline{Y})|\overline{Y}|^2]d\rho\\ &\le \int_0^{b-\delta_1}[\langle \dot{Z},\dot{Z}\rangle - R(\dot{\gamma}\wedge Z)|Z|^2]d\rho+\int_{b-\delta_1}^{b+\delta_2}[\langle \dot{Z},\dot{Z}\rangle - R(\dot{\gamma}\wedge Z)|Z|^2]d\rho<0. \end{aligned} \tag{3.7}$$

(3.7) is valid, since $\left.(D_Z Z,\dot{\gamma})\right|_{b+\delta_2}=-\frac{1}{2}\frac{d}{d\rho}\left|Z\right|^2>0$ and there is no conjugate points of $\gamma_X(b-\delta_1)$ on $\gamma_X(b-\delta_1,b+\delta_2)$, therefore the index form I of $\gamma_X([b-\delta_1,b+\delta_2])$ takes the minimum value on Jacobi vector field among all vector fields which have the same values on the two ends.

Now substitute $\left(\overline{Y}(b+\delta_2),\overline{Y}(b+\delta_2)\right)$ into (2.1). The right-hand side of (2.1) is

$$\begin{aligned} &D^2\Phi(\overline{Y}(b+\delta_2),\overline{Y}(b+\delta_2))\\ &=\frac{d\Phi}{d\rho}D^2\rho(\overline{Y}(b+\delta_2),\overline{Y}(b+\delta_2))+\frac{d^2\Phi}{d\rho^2}d\rho\otimes d\rho(\overline{Y}(b+\delta_2),\overline{Y}(b+\delta_2))\\ &=\sinh(b+\delta_2)\int_0^{b+\delta_2}[\langle \dot{\overline{Y}},\dot{\overline{Y}}\rangle - R(\dot{\gamma}\wedge\overline{Y})|Y|^2]d\rho<0, \end{aligned}$$

but the left-hand side of (2.1) is $\cosh(b+\delta_2)|\overline{Y}(b+\delta_2)|^2 > 0$, thus it is a contradiction. So there is no conjugate point of $p=\gamma_X(0)$ on γ_X, i.e. we complete the proof that there is only one geodesic which issues from p passing through q for every $q \in M-\{p\}$. Thus $\exp_p : T_pM \longrightarrow M$ is a diffeomorphism.

Now we choose an orthonormal basis $e_1,\cdots,e_n$ on T_pM. If X is a unit vector on M, then $X=\sum\limits_{i=1}^{n} a^i e_i$, $\sum\limits_{i=1}^{n}\left(a^i\right)^2=1$. So every point on T_pM can be represented by ρ and $a=(a^1,\cdots,a^n)$, $\sum\limits_{i=1}^{n}(a^i)^2=1$, where ρ is the distance from this point to the origin, a is the direction of this point, (ρ,a) is called the polar coordinate.

Now $\exp_p : T_pM \longrightarrow M$ is a diffeomorphism, so the polar coordinate (ρ,a) on T_pM can be considered as the global coordinates on M, which is called geodesic polar coordinate of M.

Now let φ be a mapping

$$\varphi : M \longrightarrow \mathbf{H}^n(-1),\quad \exp_p\Big(\rho\sum_{i=1}^{n} a^i e_i\Big) \longmapsto (\cosh\rho, a^1\sinh\rho,\cdots,a^n\sinh\rho).$$

Since $(\cosh\rho)^2-\sum\limits_{i=1}^{n}\left(a^i\sinh\rho\right)^2=\cosh^2\rho-\sinh^2\rho\sum\limits_{i=1}^{n}(a^i)^2=1$, φ is a mapping from M to $\mathbf{H}^n(-1)$. Evidently φ is injective, and the surjection of φ requires that for any $x=(x^0,x^1,\cdots,x^n)\in\mathbf{H}^n(-1)$, there exist one $\rho\in\mathbf{R}$ satisfying $\cosh\rho=x^0$ and $(a^1,\cdots,a^n)\in\mathbf{S}^{n-1}(1)$ satisfying $(a^1:a^2:\cdots:a^n)=(x^1:x^2:\cdots:x^n)$ such that $\varphi\big(\exp\rho\sum\limits_{i=1}^{n}a^ie_i\big)=x$.

Now we calculate the metric of M. For any unit vector X on T_pM, $\gamma_X(\rho)$ is the geodesic which issues from p and $\dot\gamma_X(0)=X$. For any $a\dot\gamma_X(\rho)\in T_{\gamma_X(\rho)}M$, since $\langle\dot\gamma_X(\rho),\dot\gamma_X(\rho)\rangle=\langle\mathrm{grad}\rho,\mathrm{grad}\rho\rangle=1$, so $\langle a\dot\gamma_X(\rho),a\dot\gamma_X(\rho)\rangle=|a|^2$ naturally. Now we calculate the length of the tangent vector perpendicular to $\dot\gamma_X(\rho)$. Let W be a unit vector, which is perpendicular to X on T_pM. A vector field on T_pM is obtained by parallelly displacing W to every point of T_pM. We still use W to denote this vector field, and by Γ_X denote the ray in the direction X on T_pM. Now $\gamma_X=\exp_p(\Gamma_X)$ is a geodesic which issues from p in M and ρW is a vector field along the ray $\Gamma_X(\rho)$. We set $V(\rho):=d\exp_p(\rho W)$, then $V(\rho)$ is a Jacobi vector field along γ_X and $V(0)=0$; $\dfrac{dV}{d\rho}(0)=W$. By Gauss Lemma, this Jacobi field $V(\rho)$ is perpendicular to $\gamma_X(\rho)$, i.e.$\langle V(\rho),\dot\gamma(\rho)\rangle\equiv 0, 0\le\rho<+\infty$.

Now substituting $V(\rho)$ into the equation (2.1) gives

$$\frac{\partial^2\Phi}{\partial\rho^2}d\rho\otimes d\rho(V(\rho),V(\rho))+\frac{\partial\Phi}{\partial\rho}D^2\rho(V(\rho),V(\rho))=\Phi(\gamma_X(\rho))\langle V(\rho),V(\rho)\rangle.$$

So we have

$$\sinh\rho D^2\rho(V(\rho),V(\rho))=\cosh\rho\left\langle V(\rho),V(\rho)\right\rangle.$$

By Sygne formula, the above equality turns out

$$\sinh\rho\int_0^\rho[\langle\dot V(t),\dot V(t)\rangle-R(\dot\gamma\wedge V)|V|^2]dt=\cosh\rho\langle V(\rho),V(\rho)\rangle. \tag{3.8}$$

Since $V(t)$ is a Jacobi field, the above equality can turn out to be

$$\sinh\rho\langle V(\rho),\dot V(\rho)\rangle=\cosh\rho\langle V(\rho),V(\rho)\rangle,$$

i.e. $\frac{1}{2}\frac{d}{d\rho}\log\langle V(\rho),V(\rho)\rangle=\frac{\cosh\rho}{\sinh\rho}=\frac{d}{d\rho}\log\sinh\rho$. Therefor, we have

$$\langle V(\rho),V(\rho)\rangle=C\sinh^2\rho, \tag{3.9}$$

where C is an undetermined coefficient. Since $\langle V(\rho), V(\rho)\rangle = \langle d\exp_p(\rho W), d\exp_p(\rho W)\rangle$, by asymptotic expansion of length of this Jacobi field[5]

$$\langle V(\rho), V(\rho)\rangle = \rho^2 - \frac{1}{3}R(\dot\gamma \wedge V)\rho^4 + o(\rho^5) \quad \text{as } \rho \to 0.$$

But asymptotic expansion of $\sinh^2\rho$ in the neighbour of $\rho = 0$ is

$$\sinh^2\rho = \rho^2 + \frac{1}{3}\rho^4 + o(\rho^5). \tag{3.10}$$

Comparing (3.9) with (3.10), we obtain $C \equiv 1$, so that

$$\langle V(\rho), V(\rho)\rangle = \sinh^2\rho. \tag{3.11}$$

Now if we use the polar coordinate to represent the metric on T_pM, it turns out $ds^2 = d\rho^2 + \rho^2 d\theta^2$, where $d\theta^2$ is the standard metric of $\mathbf{S}^{n-1}(1)$. Now the length of the vector ρW which is perpendicular to the ray $\Gamma_X(\rho)$ is ρ^2 , so that $d\theta^2(\rho W, \rho W) = 1$. If we choose the geodesic polar coordinates on M, (3.11) is valid for all tangent vectors which are orthornal to $\gamma_X(\rho)$ at $\gamma_X(\rho)$ and its length equals 1 with respect to standard metric $d\theta^2$, so we know the metric of M with respect to geodesic polar coordinates is

$$ds^2 = d\rho^2 + \sinh^2\rho d\theta^2. \tag{3.12}$$

In fact, we have already proved that M is $\mathbf{H}^n(-1)$, since (3.12) is the metric under the representation of geodesic polar coordinates hyperbolic space form of $\mathbf{H}^n(-1)$[6].

Now we use a few words to prove that $\varphi : M \longrightarrow \mathbf{H}^n(-1)$ is isometric homeomorphism, $\mathbf{H}^n(-1)$ is a spacelike hypersurface in $\mathbf{L}^{n,1}$. So $i : \mathbf{H}^n(-1) \hookrightarrow \mathbf{L}^{n,1}$ is isometric, hence we only need to prove $(i\circ\varphi)^* ds_L^2 = ds_M^2$.

$$\begin{aligned}(i\circ\varphi)^* ds_L^2 &= \sum_{i=1}^n \left(da^i \sinh\rho + a^i\cosh\rho d\rho\right)^2 - (\sinh\rho d\rho)^2 \\ &= \sinh^2\rho\sum_{i=1}^n \left(da^i\right)^2 + \sum_{i=1}^n \left(a^i\right)^2\cosh^2\rho d\rho^2 + 2a^i da^i \sinh\rho\cosh\rho d\rho - \sinh^2\rho d\rho^2 \\ &= d\rho^2 + \sinh^2\rho d\theta^2, \end{aligned} \tag{3.13}$$

the last equality of (3.13) is provided by $\sum\limits_{i=1}^n \left(a^i\right)^2 = 1$, $\sum\limits_{i=1}^n 2a^i da^i = 0$, and $\sum\limits_{i=1}^n \left(da^i\right)^2$ is the standard metric of $\mathbf{S}^{n-1}(1)$ under the restriction of $\sum\limits_{i=1}^n \left(a^i\right)^2 = 1$, so we complete the proof of the theorem.

At last we shall mention that we have not found any example to elucidate the assumption that the function Φ taking the extreme value on M is necessary.

References

[1] Obata, M., Certain conditions for a Riemannian manifold to be isometric with a sphere, *J. Math. Soc. Japan,* **14:**3(1965), 333–339.

[2] Zheng Yongfan, On isometry of a complete Riemannian manifold to a sphere, *Tsukuba, J. Math.*, **18:**1(1994), 135-143.

[3] Richard, R. & Crittenden, R.,, Geometry of manifold, Academic Press, New York, 1964.

[4] Klingenberg, W., Riemannian geometry, Walter de Gruyter, Berlin, 1982.

[5] Cheeger, J. & Ebin, D., Comparison theorem in Riemannian geometry, North Holland Publish Co., Amsterdam, 1975.

[6] Green, R. E. & Wu, H., Functions theory on manifolds which possess a pole, *Lecture notes in Math.,* **699**, Springer-Verlag, Berlin, 1979.