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(1.2) INVERSES OF OPERATORS BETWEEN BANACH SPACES AND LOCAL CONJUGACY THEOREM**

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Abstract

Let E and F be Banach spaces and f non-linear C^1 map from E into F. The main result is Theorem 2.2, in which a connection between local conjugacy problem of f at $x_0 \in E$ and a local fine property of $f'(x)$ at x_0(see the Definition 1.1 in this paper) are obtained. This theorem includes as special cases the two known theorems: the finite rank theorem and Berger's Theorem for non-linear Fredholm operators. Moreover, the theorem gives rise the further results for some non-linear semi-Fredholm maps and for all non-linear semi-Fredholm maps when E and F are Hilbert spaces. Thus Theorem 2.2 not only just unifies the above known theorems but also really generalizes them.

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§1. (1.2) Inverse and Local Fine Property of a Family of Operators T_x

Let E and F be both Banach spaces, and $B(E,F)$ the set of all bounded linear operators from E into F. An operator $T^+ \in B(F,E)$ is said to be a (1.2) inverse of T if $TT^+T = T$ and $T^+TT^+ = T^+$. If T^+ satisfies only the first condition, then T^+ is said to be an inner inverse of T. For any $T \in B(E,F)$ with an inner inverse $T^+ \in B(F,E)$, we have

(i) T^+T, TT^+ are bounded projectors with the properties that $R(TT^+) = R(T)$ and $R(I_E - T^+T) = N(T)$.

(ii) E and F have the direct sum decompositions as follows: $E = N(T)\dot{+}R(T^+T)$ and $F = R(T)\dot{+}R(I_F - TT^+)$ respectively, where $R(\cdot)$ and $N(\cdot)$ denote the range and null space of operator in the parenthesis and I_F denotes identity in F(see, e.g. [1]).

Throughout the paper we suppose that X is a topological space and $T_x : x \to B(E,F)$ is continuous. Now we introduce an interesting conception for T_x as follows:

Definition 1.1. *By saying that T_x is locally fine at $x_0 \in X$, we mean that $T_0 = T_{x_0}$ has a* (1.2) *inverse $T_0^+ \in B(F,E)$ and there exists a neighborhood U_0 at x_0 such that*

$$R(T_x) \cap R(I_F - T_0T_0^+) = \{0\}, \quad \forall x \in U_0. \tag{1.1}$$

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Example 1.1. Suppose that $f: U \subset E \to F$ is a non-linear C^1 Fredholm map. If for some $x_0 \in U$, there is a neighborhood U_0 at x_0 such that either dim $N(f'(x))$ or codim $R(f'(x))$ is constant in U_0, then $f'(x)$ is locally fine at x_0. Hereafter U is an open set in E.

Example 1.2. Suppose $f: U \subset E \to F$ is C^1. If for some $x_0 \in U$, there exists a neighborhood U_0 at x_0 such that dim $R(f'(x))$ =finite constant, $\forall x \in U_0$, then $f'(x)$ is locally fine at x_0.

Example 1.3. Suppose that E and F are Hilbert spaces and, $f: U \subset E \to F$ is semi-Fredholm map. If for some $x_0 \in U$, there is a neighborhood U_0 at x_0 such that either dim $N(f'(x))$ or codim $R(f'(x))$ is finite constant, $\forall x \in U_0$, then $f'(x)$ is locally fine at x_0.

Example 1.4. Suppose that $f: U \subset E \to F$ is C^1 semi-Fredholm map. If for some $x_0 \in U$, $f'(x_0)$ has a (1.2) inverse and, there is a neighborhood U_0 at x_0 such that either dim $N(f'(x))$ or codim $R(f'(x))$ is finite constant, $\forall x \in U_0$, then $f'(x)$ is locally fine at x_0.

In order to show the examples above, we need Lemma 1.1 below, which extends Lemma 3 in [2] to Banach spaces. For short, in what follows, write N_0, R_0 for $N(T_0), R(T_0)$, respectively.

Lemma 1.1. *Suppose that $T_0 \in B(E,F)$ has an inner inverse $T_0^+ \in B(F,E)$ and, $\bigwedge : B(E,F) \to B(E, R_0 \times N_0)$ is defined by*

$$\big(\bigwedge T\big)x = (T_0T_0^+Tx, (I_E - T_0^+T_0)x), \quad \forall x \in E,$$

then we have

(i) $\bigwedge$ *is continuous,*

(ii) *there exists a neighborhood V_0 at T_0 such that*

$$R(T_0T_0^+T) = R_0, \quad \big(\bigwedge T\big) \in B^\times(E, R_0 \times N_0), \tag{1.2}$$

$$N(T_0T_0^+T) = \big(\bigwedge T\big)^{-1}(0, N_0), \quad \forall T \in V_0, \tag{1.3}$$

where $B^\times(E, R_0 \times N_0)$ denotes the set of all invertible operators in $B(E, R_0 \times N_0)$.

Proof. (i) By the definition of $\bigwedge$, for each x in E, $\big(\bigwedge(T-S)\big)x = (T_0T_0^+(T-S)x, 0)$ for arbitrary T and $S \in B(E,F)$. Then, obviously, $\big\|\bigwedge(T-S)\big\| \le \|T_0T_0^+\|\|T-S\|$, and so (i) follows.

(ii) First to show $\bigwedge T_0 \in B^\times(E, R_0 \times N_0)$. To this end, we will prove that $N(\bigwedge T_0) = \{0\}$ and $\bigwedge T_0$ is surjective. We see that if

$$0 = \big(\bigwedge T_0\big)x = (T_0T_0^+T_0x, (I_E - T_0^+T_0)x) = (T_0x, (I_E - T_0^+T_0)x),$$

then $x \in N_0$ and $(I_E - T_0^+T_0)x = 0$, so that $x = 0$. This proves $N(\bigwedge T_0) = \{0\}$.

For each $(x,y) \in R_0 \times N_0$, taking $u = T_0^+x + y$, we see that

$$\big(\bigwedge T_0\big)u = (T_0T_0^+T_0(T_0^+x + y), \quad (I_E - T_0^+T_0)y) = (T_0T_0^+x, y) = (x,y),$$

i.e., $\bigwedge T_0$ is surjective. Thus, we have $\big(\bigwedge T_0\big) \in B^\times(E, R_0 \times N_0)$ and $(\bigwedge T_0)^{-1}(x,y) = T_0^+x + y$.

Next we show (ii). Let $M_0 = \mathrm{Max}(1, \|T_0^+\|)$. Then it is evident that

$$\big\{T : \big\|T - \big(\bigwedge T_0\big)\big\| < M_0^{-1}\big\} \subset B^\times(E, R_0 \times N_0)$$

(note $||(\bigwedge T_0)^{-1}|| \leq M_0$). Set

$$V_0 = \{T \in B(E,F) : ||T - T_0|| < (M_0||T_0T_0^+||)^{-1}\}.$$

We will show that V_0 is the required neighborhood in (ii). Because

$$\|\bigwedge T - \bigwedge T_0\| \leq ||T_0T_0^+||||T - T_0|| < M_0^{-1}$$

for any $T \in V_0$, we conclude $\bigwedge T \in B^\times(E, R_0 \times N_0)$ for any $T \in V_0$. Given $T \in V_0$, by taking $u = (\bigwedge T)^{-1}(x,0)$ for each $x \in R_0$, it yields that $(\bigwedge T)u = (x,0)$, and so $R(T_0T_0^+T) = R_0, \forall T \in V_0$. Given $T \in V_0$ for each $x \in (\bigwedge T)^{-1}(0, N_0)$, obviously,

$$(\bigwedge T)x = (T_0T_0^+Tx, (I_E - T_0T_0^+)x) \in (0, N_0).$$

Therefore $x \in N(T_0T_0^+T), \quad \forall T \in V_0$, i.e., the right hand side of (1.3) contains the left one. On the other hand, given $T \in V_0$, for each $x \in N(T_0T_0^+T)$,

$$(\bigwedge T)x = (0, (I_E - T_0T_0^+)x) \in (0, N_0).$$

This shows the converse inclusion. Thus we obtain (1.3).

The proof of Lemma 1.1 is completed.

In what follows, for short, write T_x, N_x and R_x for $f'(x)$, $N(f'(x))$ and $R(f'(x))$, respectively.

Now we return to show the examples above.

Proof of Example 1.1. By the assumption of this example, dim N_x and codim R_x are both finite for all $x \in U$. Therefore, E and F have the following direct sum decompositions $E = N_x \dot{+} N_x^-$ and $F = R_x \dot{+} R_x^-, \forall x \in U$. Clearly, $T_x|_{N_x^-} : N_x^- \to R_x$ is onto and one-to-one, which gives rise a (1.2) inverse T_x^+ of T_x as follows

$$T_x^+ h = \begin{cases} (T_x|N_x^-)^{-1}, & h \in R_x, \\ 0, & h \in R_x^- \end{cases}$$

(see [1] for details). Thus, by Lemma 1.1, there is a neighborhood at x_0 contained in U_0, without loss of generality, still written as U_0, in which (1.2) and (1.3) are satisfied.

If dim $N_x = \dim N_0 < \infty$, $\forall x \in U_0$, then it follows from (1.3) and $N(T_0T_0^+T_x) \supset N_x$ that $N(T_0T_0^+T_x) = N_x$. Note that $N(T_0T_0^+T_x) = N_x + \{u \in E : \bar{T}_x[u] \in R_0^-\}$, where $[u] \in E/N_x, \bar{T}_x[u] = T_xu$ and $R_0^- = R(I_F - T_0T_0^+)$, we conclude $R_x \cap R_0^- = \{0\}$.

If codim $R_x = \text{codim} R_0 < \infty$, $\forall x \in U_0$, then, since $R(T_0T_0^+T_x) = R_0$, we see that

$$F = R_0 \dot{+} R_0^- = R_x + R_0^- = R_x \dot{+} R,$$

where R satisfies $R_0^- = R \dot{+} (R_x \cap R_o^-)$. If $R_0^- \cap R_x \not\supseteq \{0\}$, it could lead to the conclusion that codim $R_x = \dim R < \dim R_o^- = \text{codim} R_o$, a contradition. Hence $R_0^- \cap R_x = \{0\}$ and this shows Example 1.1.

Proof of Example 1.2. Suppose dim $R_x =$ dim $R_0 < \infty, \quad \forall x \in U_0$. It is easy to see that in this case, both of codim N_x and dim R_x are finite. Therefore E and F can be decomposed into the next direct sums $E = N_x \dot{+} N_x^-$ and $F = R_x \dot{+} R_x^-$.

Thus, by virtue of the proof of Example 1.1, there is a neighborhood at x_0, still written as U_0 in which (1.2) holds. So we have $F = R_0 \dot{+} R_0^- = R_x + R_0^- = R \dot{+} R_0^-$, where $R_x = R \dot{+} (R_x \cap R_o^-)$. It yields that $R_x \cap R_0^- = \{0\}$. Thus Example 1.2 is proved.

Proof of Example 1.3. In view of the fact that each of semi-Fredholm operators in Hilbert space possesses M. P. generalized inverse, which is also a (1.2) inverse, we can follow

Example 1.1 to complete the proof.

Proof of Example 1.4. Since the existence of T_0^+ is assumed there, the proof is entirely similar to that in Example 1.3.

§2. Local Linearzation Theorem

We first prove the next Lemma 2.1 and Theorem 2.1, which will be needed in the sequel.

Lemma 2.1. *If T_x satisfies that $R_x = R_0, \forall x \in X$ and $T_0 = T_{x_0}$ possesses an inner inverse $T_0^+ \in B(F, E)$, then there exists a neighborhood U_0 at x_0 and a family of bounded projectors P_x with $R(P_x) = N_x, \forall x \in U_0$, such that $\lim\limits_{x \to x_0} P_x = (I_E - T_0^+ T_0)$.*

Proof. Let $U_0 = \{x \in X : \|T_x - T_0\| < \|T_0^+\|^{-1}\}$. Obviously, U_0 is an open set in X by the continuity of T_x. Noting that $T_0 T_0^+ T_x = T_x$ in the case $R_x = R_0$, we see that $T_x = T_0(I_E - T_0^+(T_0 - T_x)), \forall x \in U_0$. Thus we get

$$N_x = (I_E - T_0^+(T_0 - T_x))^{-1} N_0, \forall x \in U_0. \tag{2.1}$$

Put $S_x = (I_E - T_0^+(T_0 - T_x))^{-1}(I_E - T_0^+ T_0)$, $\widehat{S_x} = (I_E - T_0^+ T_0)(I_E - T_0^+(T_0 - T_x))$. Clearly, S_x and $\widehat{S_x}$ are both in $B(E)$ and depend continuously on x. It is also clear that $P_x = S_x \widehat{S_x}$ is projector in $B(E)$ and depends continuously on x. It follows from (2.1) that $R(S_x \widehat{S_x}) = N_x$. So P_x is the required. The proof is completed.

Theorem 2.1. *If T_x is locally fine at $x_0 \in X$, then there exist a neighborhood U_0 at x_0 and a family of bounded projectors P_x with $R(P_x) = N_x$ such that $\lim\limits_{x \to x_0} P_x = I_E - T_0^+ T_0$.*

Proof. The local fine property of T_x at x_0 implies that T_0 has an inner inverse $T_0^+ \in B(F, E)$. Thus,by Lemma 1.1, there is a neighborhood U_0 at x_0 such that $R(T_0 T_0^+ T_x) = R_0, \forall x \in U_0$. Let $S_x = T_0 T_0^+ T_x$. Obviously, $S_{x_0} = T_0$ and S_x depend continuously on x. By Lemma 2.1, in order to complete our proof, we only need to show $N(S_x) = N_x$. For this, let us observe that $N(S_x) = N_x + \{u \in E : \tilde{T}_x[u] \in R_0^-\}$, where $\tilde{T}_x$ and $[u]$ are the same as in the proof of Example 1.1. Since $R_x \cap R_0^- = \{0\}$, we then conclude that $N(S_x) = N_x$. Hence, Theorem 2.1 is proved.

Now we shall give our local linearization theorem. First recall the following

Definition 2.1.[3] *Suppose that $f : U \subset E \to F$ is C^1. By saying that f can be locally linearizated at x_0 or f is locally conjugate to $f'(x_0)$ near x_0, we mean that there exist two neighborhoods U_0 at x_0 and V_0 at 0, with two maps u and v, such that*

(i) *$u : U_0 \to u(U_0)$ and $v : V_0 \to v(V_0)$ with $v(0) = f(x_0)$ are both diffeomorphisms.*

(ii) *$f(x) = (v \circ f'(x_0) \circ u)(x), \forall x \in U_0$.*

Theorem 2.2. *Suppose that $f : U \subset E \to F$ is C^1. If $T_x = f'(x)$ is locally fine at $x_0 \in U$, then f can be locally linearized at x_0.*

Proof. By Theorem 2.1, there are a neighborhood U_0 at x_0 and a family of bounded projectors P_x with $R(P_x) = N_x$ for any $x \in U_0$, such that

$$\lim_{x \to x_0} P_x = I_E - T_0^+ T_0, \tag{2.2}$$

where $T_0 = f'(x_0)$.

Let $u(x) = T_0^+(f(x) - f(x_0)) + (I_E - T_0^+ T_0)(x - x_0)$. Obviously, $u(x_0) = 0$ and $u'(x_0) = I_E$. Moreover, we shall show the following results:

(i) There exists an open disk $D_r^E(0)$ such that

$$u : u^{-1}(D_r^E(0)) \to D_r^E(0) \text{ is a diffeomorphism,} \tag{2.3}$$

$$N_y = (u'(y))^{-1} N_0, \quad \forall y \in u^{-1}(D_r^E(0)). \tag{2.4}$$

(ii) There exists an open disk $D_\rho^E(x_0)$ in $u^{-1}(D_r^E(0))$ such that

$$T_0^+(f(x) - f(x_0)) \in D_r^E(0), \quad \forall x \in D_\rho^E(x_0), \tag{2.5}$$

$$u : D_\rho^E(x_0) \to u(D_\rho^E(x_0)) \quad \text{is a diffeomorphism,} \tag{2.6}$$

$$\begin{aligned}&(f \circ u^{-1})(T_0^+(f(x) - f(x_0)) + (I_E - T_0^+T_0)(x - x_0)) \\ = {}&(f \circ u^{-1})(T_0^+(f(x) - f(x_0))), \quad \forall x \in D_\rho^E(x_0).\end{aligned} \tag{2.7}$$

(iii) There exists an open disk $D_l^F(0)$ such that

$$T_0^+ x \in u(D_\rho^E(x_0)), \quad \forall x \in D_l^F(0). \tag{2.8}$$

In fact, by the inverse map theorem, (2.3) is direct. Since $D_\rho^E(x_0) \subset u^{-1}(D_r^E(0))$, (2.6) is a direct result of (2.3).

Now we show (2.4). By differentation,

$$u'(y) = T_0^+ f'(y) + (I_E - T_0^+T_0) = T_0^+T_y + (I_E - T_0^+T_0),$$

which induces $u'(y)N_y = (I_E - T_0^+T_0)N_y, \forall y \in u^{-1}(D_r^E(0))$. Further, (2.2) bears a neighborhood at x_0 contained in $u^{-1}(D_r^E(0))$, without loss of generality, still written as $u^{-1}(D_r^E(0))$, such that $\|P_y - (I_E - T_0^+T_0)\| < 1$ for each $y \in u^{-1}(D_r^E(0))$. Hence, by [4, Section 4.6], the range of $(I_E - T_0^+T_0)P_y$ is N_0 (note $R(I_E - T_0^+T_0) = N_0$), and so $u'(y)N_y = N_0$, which gives (2.4).

According to the continuity of $T_0^+(f(x) - f(x_0))$ at x_0, (2.5) is immediate.

Since $T_0^+0 = 0 \in u(D_\rho^E(x_0))$ and $T_0^+ \in B(F, E)$, (2.8) is obvious.

Next we show (2.7). Let $y_1 = T_0^+(f(x) - f(x_0))$ and $y_2 = y_1 + (I_E - T_0^+T)(x - x_0)$ (so $y_2 = u(x)$). From (2.5) and (2.6) (note $D_\rho^E(x_0) \subset u^{-1}(D_r^E(0))$, we see that for any $x \in D_\rho^E(x_0)$, both of y_1 and y_2 belong to $D_r^E(0)$, so that

$$ty_1 + (1 - t)y_2 = y_1 + (1 - t)(I_E - T_0^+T_0)(x - x_0) \in D_r^E(0)$$

for any $x \in D_\rho^E(x_0)$ and each $t \in [0, 1]$.

Consider $\Phi(t) = (f \circ u^{-1})(y_1 + (1 - t)(I_E - T_0^+T_0)(x - x_0)) : [0, 1] \to F.$

By differentiation,

$$\frac{d}{dt}\Phi(t) = (f' \circ u^{-1})(ty_1 + (1 - t)y_2) \cdot ((u')^{-1} \circ u^{-1})(ty_1 + (1 - t)y_2) \cdot (T_0^+T_0 - I_E).$$

Noting that $R(T_0^+T_0 - I_E) = N_0$ and (2.4), we obtain $\frac{d\Phi(t)}{dt} = 0$ $\forall t \in [0, 1]$. Then (2.7) follows.

We now proceed to construct v required by Definition 1.1.

Because of (2.8) and (2.6), we can define $v(x) = (f \circ u^{-1} \circ T_0^+)(x) + (I_F - T_0T_0^+)x,\ \ \forall x \in D_l^F(0)$. Obviously, $v(0) = f(x_0)$ and

$$v'(0) = T_0 \cdot (u^{-1})'(0) \cdot T_0^+ + (I_F - T_0T_0^+) = T_0T_0^+ + (I_F - T_0T_0^+) = I_{F_0}.$$

By the inverse map theorem we assert: there is an open disk $D_m^F(0)$ with $0 < m < l$, such that

$$v : D_m^F(0) \to v(D_m^F(0)) \quad \text{is a diffeomorphism.} \tag{2.9}$$

Because of the boundedness of T_0, there is an open disk $D_q^E(x_0) \subset D_\rho^E(x_0)$ such that

$$T_0 x \in D_m^F(0), \quad \forall x \in u(D_q^E(x_0)). \tag{2.10}$$

Noting that (2.6) and (2.7) keep valid in $D_q^E(x_0)$ since $D_q^E(x_0) \subset D_\rho^E(x_0)$, we have

$$\begin{aligned} f(x) &= (f \circ u^{-1} \circ u)(x) = (f \circ u^{-1})(T_0^+(f(x) - f(x_0)) + (I_E - T_0^+T_0)(x - x_0)) \\ &= (f \circ u^{-1})(T_0^+(f(x) - f(x_0)), \quad \forall x \in D_q^E(x_0). \end{aligned}$$

Since (2.10), (2.9) and $D_m^F(0) \subset D_l^F(0)$, $(v \circ T_0 \circ u)(x)$ is determined for each $x \in D_q^E(x_0)$. Thus we have

$$\begin{aligned} (v \circ T_0 \circ u)(x) &= (v \circ T_0)(T_0^+(f(x) - f(x_0)) + (I_E - T_0^+T_0)(x - x_0)) \\ &= v(T_0T_0^+(f(x) - f(x_0))) \\ &= (f \circ u^{-1})(T_0^+T_0T_0^+(f(x) - f(x_0))) + (I_F - T_0T_0^+)(T_0T_0^+(f(x) - f(x_0))) \\ &= (f \circ u^{-1})(T_0^+(f(x) - f(x_0)) \end{aligned}$$

for any $x \in D_q^E(x_0)$. Combining the two results above, we see that

$$f(x) = (v \circ f'(x_0) \circ u)(x), \quad \forall x \in D_q^E(x_0).$$

By (2.9) and (2.6), $u : D_q^E \to u(D_q^E(x_0))$ and $\nu : D_m^F(0) \to v(D_m^F(0)$ are both diffeomorphisms. The proof is finished.

Because of Examples 1.1–1.4, the following are direct results of Theorem 2.2.

Theorem (Finite Rank).[3] *Suppose that $f : U \subset E \to F$ is C^1. If there is a neighborhood U_0 at $x_0 \in U$ such that* $\dim R(f'(x)) =$ *finite constant, $\forall x \in U_0$, then f can be locally linearized at x_0.*

Corollary 2.1. *Suppose that $f : U \subset E \to F$ is C^1 Fredholm map. If there is a neighborhood U_0 at $x_0 \in U$ such that either* $\dim N(f'(x))$ *or* codim $R(f'(x))$ *is finite constant for any $x \in U_0$, then f can be locally linearized at x_0.*

Corollary 2.2. *Suppose that $f : U \subset E \to F$ is C^1 semi–Fredholm map. If $f'(x_0)$ has a* (1.2) *inverse $T_0^+ \in B(F, E)$ and, there is a neighborhood U_0 at $x_0 \in U$ such that either* $\dim N(f'(x))$ *or* codim $R(f'(x))$ *is finite constant for any $x \in U_0$, then f can be locally linearized at x_0.*

Corollary 2.3. *Suppose that $f : U \subset H_1 \to H_2$ is C^1 semi-Fredholm map, where H_1 and H_2 are Hilbert space. If there is a neighborhood U_0 at $x_0 \in U$ such that either* $\dim N(f'(x))$ *or* codim $R(f'(x))$ *is finite constant, then f can be locally linearized at x_0.*

Remark. In [5], when $f : U \subset H \to H$ is C^1 and $f'(x)$ has close range, where H is a Hilbert space, we gave a theorem on local linearizations for f.

References

[1] Nashed, Z. M., Generalized inverses and applications, Academic press, New York, San-Francisco, London, 1976.

[2] Ma Jipu, Continuously sufficient and necessary conditions for Moore-Penrose inverses A_x^+, *Science in China (Series A),* **33:**11(1990), 1294–1302.

[3] Berger, M. S., Non-linearity and functional analysis, Academic press, New York, 1977.

[4] Katô, T., Perturbation theory for linear operators, Springer-Verlag, Berlin, Heidelberg, New York, 1984.

[5] Cao Weiping & Ma Jipu, The local linearization theorem of nonlinear maps, *Journal of Nanjing University Math., Biquaterly,* **13:**2(1996), 210–213.