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A REMARK ON THE HENSELIZATION OF A REGULAR LOCAL RING**

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Abstract

The author gives simple characterization of the Henselization of an excellent regular local ring.

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§1. Introduction

The notion of Henselization of integrally closed integrity domains was introduced by Nagata[3] from the concept of Henselian ring which was given firstly by Azumaya[1]. The Henselization of a local ring A is a couple $(\tilde{A}, i)$, where $\tilde{A}$ is a local Henselian ring and $i : A \to \tilde{A}$ a local morphism which satisfies the following universal property: For each local morphism $u : A \to B$, B being a local Henselian ring, there exists a unique local morphism $\tilde{u} : \tilde{A} \to B$ such that $u = \tilde{u}i$.

The purpose of this short note is to provide a very simple characterization of the Henselization of an excellent regular local ring. Note that most of the rings arising from natural constructions in Algebra and Geometry are excellent ones. The precise statement is the following

Theorem 1.1. *Let (R, m) be an excellent regular local ring and $\widehat{R}$ its m-adic completion. Then $f \in \tilde{R}$ if and only if, there exists $g \in \widehat{R}$ such that $fg \in R$, up to multiplication by a unit in $\widehat{R}$.*

§2. Proof of Theorem 1.1

This section is devoted to prove the theorem. First we shall give a Lemma.

Let D be an excellent integrally closed local domain. Consider the following multiplicative submonoid of the completion $\widehat{D}$ of D:

$$D^* := \Big\{ e \in \widehat{D} \big| e = \frac{d}{e'}, \quad d \in D, \quad e' \in \widehat{D} \Big\},$$

where the equality must be in the quotient field of $\widehat{D}$.

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Lemma 2.1. *With assumptions and notations as above, the following inclusion holds,*

$$\widetilde{D} \subseteq D^*.$$

Proof. Since $\widetilde{D} = \lim\limits_{\to i \in I} D_i$, where $D_i = \left(\frac{D[t]}{(p)}\right)_Q$, p being a monic polynomial in $D[t]$ and Q a prime ideal of $\frac{D[t]}{(p)}$ whose contraction with respect to the natural homomorphism $D \to D_i$ is the maximal ideal of D (see [2, Chapter viii]), it suffices to prove that if $e \in D[t]$, then there exist $e' \in D[t]$ and $d \in D$ such that $ee' - d \in (p)$.

Indeed, we can asume that $\deg(p) \geq \deg(e)$, where deg means degree of the polynomial. Denote by K the quotient field of D, then there exist $g_1, c_1 \in K[t]$, such that

$$p = eg_1 + c_1 \quad \text{and} \quad \deg(c_1) < \deg(e),$$

as $K[t]$ is an Euclidean domain. Now, assume $\deg(c_1) = 0$, then $c_1 \in K$ and we can pick $u \in D$ such that

$$g_1 u = g \in D[t] \quad \text{and} \quad c = c_1 u \in D,$$

which establishes the result. Finally, assume $\deg(c_1) > 0$, pick $v \in D$ so that

$$vc_1 = c \in D[t], \quad vg_1 = g \in D[t],$$

since we can get $c' \in D[t]$ such that

$$\deg(c') < \deg(c) \quad \text{and} \quad cg' + c' \in (p)$$

by the above procedure, the proof is completed after finitely many steps.

Proof of Theorem 1.1. Denote by R_I^* the set of analytically irreducible elements of R^*, and by $\widetilde{R}_I^u$ the set of products (in $\widehat{R}$) of analytically irreducible elements in $\widehat{R}$ by units of $\widehat{R}$. Now, since prime ideals in a local Henselian excellent ring extend to prime ideals in the completion, we deduce that if $f \in \widehat{R}$ is an analytically irreducible element and there exists any nonzero element $h \in \widehat{R}$ such that $fh \in R$, then $fu \in R$ for some unit $u \in \widehat{R}$. This and Lemma 2.1 prove

$$R_I^* = \widetilde{R}_I^u.$$

As a consequence of the above equality we see that R^* is the free monoid on the basis $\widetilde{R}_I^u$. To show it, we shall see that $s \in R^*$ if and only if, each element of its factorization into analytically irreducible factors remains in R^*. In fact, let $s \in \widehat{R}$ and $s = s_1 s_2 \cdots s_r$ its irreducible factorization. If $s \in R^*$, then $s_1 s_2 \cdots s_r s' \in R$ for any $s' \in \widehat{R}$ and so $s_i \in R^*$ for all i, $1 \leq i \leq r$. The converse also holds since R^* is a multiplicative monoid. Finally, the above result and Lemma 2.1 prove the following set equality

$$R^* = \{uf \mid f \in \widetilde{R} \ \text{ and } u \text{ is a unit in } \ \widehat{R}\}$$

and therefore the theorem.

References

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