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RAPID EXACT CONTROLLABILITY OF THE WAVE EQUATION BY CONTROLS DISTRIBUTED ON A TIME-VARIANT SUBDOMAIN***

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Abstract

Consider the wave equation with distributed controls supported on a subdomain, called control subdomain, which is allowed to be variant in time. For any prescribed time duration, the authors work out a scheme for changing the control subdomain such that the wave equation is exactly controllable on this time duration, where the control subdomain at any time is allowed to have arbitrarily small measure and relatively simple shape.

Keywords Rapid exact controllability, Wave equation, Control subdomain, Unique continuation

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§1. Introduction and Result

Suppose you want to stop a vibrating drum with your hand in a time period as short as possible. What would you do? In common sense, the answer is "keep your hand on the surface of the drum and move it from side to side as fast as possible". In the present paper, we will explain the mathematical theory behind this phenomenon.

Consider the following wave equation with a locally distributed control:

$$\begin{cases} \Box y \triangleq y_{tt} - \Delta y = \chi_{G(t)}(x)u(x,t), & \text{in } \Omega \times (0,\infty), \\ y = 0, & \text{on } \partial\Omega \times [0,\infty), \\ y(x,0) = y_0(x), \quad y_t(x,0) = y_1(x), & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with the Lipschitz boundary $\partial\Omega$, $G(t)$ is a subdomain of Ω for each $t \in [0,\infty)$, $\chi_{G(t)}(x)$ is the characteristic function of the set $G(t)$, $y(x,t)$ is the state and $\chi_{G(t)}(x)u(x,t)$ is the control. We call $G(t)$ control subdomain at time t. It expresses geometric characteristics (such as location, measure and shape) of the controller.

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The main feature of our formulation is that the geometric characteristics of the controller is allowed to be variant in time. A vibrating membrane with distributed loading can be described by equation (1.1) with $n = 2$, where the loading is distributed only on the subdomain $G(t)$ at time t.

Let $\mathcal{U} = L^2_{\rm loc}(0, \infty; L^2(\Omega))$ and $\mathcal{G}$ be a family of set-valued functions $G(\cdot)$ defined on $[0, \infty)$ taking subdomains of Ω as its values.

Definition 1.1. (i) *For a given* $G(\cdot) \in \mathcal{G}$ *and* $T > 0$*, the system* (1.1) *is said to be exactly controllable on* $[0, T]$ *if for any* $(y_0, y_1) \in H^1_0(\Omega) \times L^2(\Omega)$*, there exists a control* $u(\cdot) \in \mathcal{U}$*, such that the corresponding solution* $y(x, t)$ *of* (1.1) *satisfies*

$$y(x, T) = y_t(x, T) = 0, \qquad a.e.\ x \in \Omega. \tag{1.2}$$

(ii) *The system* (1.1) *is said to be rapidly exactly controllable under* $\mathcal{G}$ *if for any* $T > 0$*, there exists a* $G(\cdot) \in \mathcal{G}$*, such that the system* (1.1) *is exactly controllable on* $[0, T]$*.*

We use the word "rapidly" because we might require that the time duration $T > 0$ be arbitrarily small. On the other hand, we expect that the control subdomain has a relatively small measure, more precisely, there exists constant $0 < \alpha << 1$ such that

$$\operatorname{mes} G(t) < \alpha \operatorname{mes} \Omega, \qquad \forall t \in [0, \infty), G(\cdot) \in \mathcal{G}. \tag{1.3}$$

The problem we are interested in is the following

Find a $\mathcal{G}$ satisfying (1.3), such that the system (1.1) is rapidly exactly controllable under $\mathcal{G}$.

Systems with changing controllers were studied by many authors in some quite different contexts, see [3, 4, 19] and references cited therein. Recently, Khapalov[10] discussed the exact controllability of the wave equation with moving point control using some duality method. The controls in [10] were taken to be some derivatives of the Dirac function. In the present paper, however, the controls are locally supported L^2 functions. Therefore, in control processes the states of the system (1.1) are kept in the finite energy state space, which is of special interest in physics and engineering.

It is not hard to see that the construction of $\mathcal{G}$ is crucial in our problem. Physically, any $G(\cdot) \in \mathcal{G}$ gives a way of changing the geometric characteristics of the controller.

We note that if $G(t) \equiv G$ is a fixed subdomain of Ω, then (1.1) becomes a system with the fixed control subdomain. For the case $\mathcal{G} = \{G\}$ (the singleton), the system is rapidly exactly controllable under $\mathcal{G}$. This is easy to prove (see [7,15], for example). For the case that G is a proper subdomain of Ω, the consideration of exact controllability of (1.1) seems to have been initiated by Lagnese[14] in 1983. He discussed this problem on one-dimensional, rectangular, and spherical domains. Ho[9] discussed variable coefficient case of the one-dimensional system. For general multi-dimensional domain, Zuazua[22] showed that if G is a subdomain of Ω such that $\overline{G}$ contains a portion Γ_0 of the boundary $\partial\Omega$ of Ω with Γ_0 satisfying certain geometric conditions, then, the system (1.1) is exactly controllable on $[0, T]$ for some $T > T_0$, where $T_0 > 0$ only depends on the domain Ω and Γ_0. In [1], under some sufficiently smooth conditions, by means of microlocal analysis, Bardos, Lebeau and Rauch proved that the system (1.1) is exactly controllable on $[0, T]$ if and only if every ray of geometric optics in Ω meets G within time T. In [18], the first author of the present paper using a different method characterized some cases that the exact controllability of (1.1) can be achieved, in which the domain Ω is not necessarily smooth.

On the other hand, if $\Omega = (0,1)^n \subset \mathbb{R}^n$ and

$$G(t) \equiv G = (0, 1-\varepsilon) \times (0,1)^{n-1} \quad \text{with} \quad 0 < \varepsilon < 1,$$

then, the system (1.1) is not exactly controllable on any $[0,T]$ (see [18], for example). We note that the control subdomain above can be arbitrarily close to the whole domain

$$\Omega = (0,1)^n.$$

It is possible that for any $T > 0$ there exists constant set-valued G satisfying (1.3) such that the system (1.1) is exactly controllable on $[0,T]$. In this case, however, the shorter control time duration T will require the more complicated shape of the control subdomain G. This is because by the well-known "domain of dependence" argument for the wave equation, the system (1.1) is not rapidly exactly controllable under $\mathcal{G}$ when $\mathcal{G}$ only consists of constant set-valued functions and $\Omega \setminus G$ contains a ball with the same radius for every $G \in \mathcal{G}$. In this paper, we will consider the case that the class $\mathcal{G}$ contains non-constant set-valued functions.

Let us now, make assumptions on $\mathcal{G}$ and give our main result. Without loss of generality, we assume the following

$$\begin{cases} \inf\{x_1 \in \mathbb{R} \big| \exists x' \in \mathbb{R}^{n-1}, \text{ such that } (x_1, x') \in \Omega\} = 0, \\ \sup\{x_1 \in \mathbb{R} \big| \exists x' \in \mathbb{R}^{n-1}, \text{ such that } (x_1, x') \in \Omega\} = \beta > 0. \end{cases} \tag{1.4}$$

Let $T > 0$ and $0 < \sigma < T$ be given. Define $a = \frac{T-\sigma}{\beta}$. Set

$$K_\sigma = \{(x_1, t) \in [0, \beta] \times [0, T] \bigm| ax_1 < t < ax_1 + \sigma\}, \tag{1.5}$$

$$D_\sigma = \Big(K_\sigma \times \mathbb{R}^{n-1}\Big) \bigcap \Omega_T \qquad (\Omega_T = \Omega \times (0,T)). \tag{1.6}$$

Assumption 1.1. *Let class $\mathcal{G}$ be a family of set-valued functions defined on $[0,\infty)$ taking subdomains of Ω as its values. Assume that the following properties hold:*

(i) *Any $G(\cdot) \in \mathcal{G}$ is continuous with respect to the Lebesgue measure.*

(ii) *For any $T > 0$, there exists a $G(\cdot) \in \mathcal{G}$ and $\sigma \in (0,T)$, such that*

$$\widetilde{G}_T \triangleq \{(x,t) \in \Omega_T \bigm| x \in G(t),\ t \in [0,T]\} \supseteq D_\sigma, \tag{1.7}$$

with D_σ being defined by (1.6).

Our main result can be stated as follows:

Theorem 1.1. *If $G(\cdot)$ is continuous with respect to the Lebesgue measure and satisfies* (1.7), *then the system* (1.1) *is exactly controllable on* $[0,T]$. *Consequently, the system* (1.1) *is rapidly exactly controllable under $\mathcal{G}$ if the class $\mathcal{G}$ satisfies Assumption* 1.1.

We point out that we can choose the class $\mathcal{G}$ such that it satisfies both Assumption 1.1 and condition that for every $G(\cdot) \in \mathcal{G}$ and $t \in [0,T]$, $G(t)$ has arbitrarily small measure and very simple shape, comparing with that of Ω. We explain this claim and some physical senses of our result with the following two examples.

Example 1.1. For any $\delta, T > 0$, set

$$\alpha(t) = \frac{\beta + \delta}{T} t - \frac{\delta}{2}, \tag{1.8}$$

$$R_\delta(t) = \Big\{ x_1 \in [0, \beta] \bigm| |x_1 - \alpha(t)| < \frac{\delta}{2} \Big\}, \tag{1.9}$$

$$G(\delta, T, t) = \Big(R_\delta(t) \times \mathbb{R}^{n-1} \Big) \bigcap \Omega_T. \tag{1.10}$$

Note that the set $\widetilde{G}_T$ defined in (1.7) is nothing but the graph of the set-valued function $G(\cdot)$. We see that $G(\cdot) = G(\delta, T, \cdot)$ satisfies (1.7) if σ is so small that $\frac{\sigma\beta}{T-\sigma} < \delta$. Therefore, the class $\mathcal{G}_\delta = \{G(\delta, T, \cdot) \bigm| T \in (0, \infty)\}$ satisfies Assumption 1.1. By Theorem 1.1, the system (1.1) is rapidly exactly controllable under $\mathcal{G}_\delta$.

In Example 1.1, the controller is required to move in the positive x_1-direction at a constant speed $v = \frac{\beta+\delta}{T}$. This is physically realizable. Moreover, if the required time duration $T > 0$ is smaller, the moving speed of the controller has to be faster. This matches perfectly with the common intuition stated at the beginning of this paper.

Example 1.2. Let $\Omega = (0,1)^n$ $(n \geq 2)$. For any $\delta, T > 0$, take

$$G(t) = \begin{cases} (0, \delta) \times (0,1)^{n-1}, & 0 \leq t \leq t_0 \triangleq \frac{\delta T}{\delta+1}, \\ \Big(\frac{(\delta+1)t}{T} - \delta, \frac{(\delta+1)t}{T} \Big) \times (0,1)^{n-1}, & t_0 \leq t \leq T - t_0, \\ (1-\delta, 1) \times (0,1)^{n-1}, & T - t_0 \leq t \leq T. \end{cases} \tag{1.11}$$

The above amounts to say that the controller has fixed measure and shape. It stays at $(0,\delta) \times (0,1)^{n-1}$ until $t = t_0$, then, moves at a constant speed $v = \frac{\delta+1}{T}$ to location $(1-\delta, 1) \times (0,1)^{n-1}$ at $t = T - t_0$, and stays there until $t = T$. It is clear that the $G(\cdot)$ satisfies (1.7) for σ small enough. Hence, by Theorem 1.1, the system (1.1) can be steered to rest at time T.

It is known that in Example 1.2, if we fix the control subdomain at any place, say at the initial location, i.e., $G(t) \equiv (0, \delta) \times (0,1)^{n-1}$, then, the system is not even exactly controllable on any time duration. Whereas, by allowing the control subdomain to move, we can achieve the rapid exact controllability. We note that even in the one-dimensional case, moving the control subdomain can improve exact controllability of the system in the following sense. For every $G \in \mathcal{G}_1 = \{(a,b) \subset \Omega = (0,1) \bigm| b - a < \alpha << 1\}$, the system is exactly controllable on a time duration $[0, T_G]$, where it is necessary that $T_G \geq \frac{1-\alpha}{2}$ for all $G \in \mathcal{G}_1$. Thus, the system is not rapidly exactly controllable under $\mathcal{G}_1$. However, we have seen that by allowing the control subdomain to move from side to side, we are able to achieve the rapid exact controllability.

Concerning the proof of Theorem 1.1, we do not expect to make an immediate application of the microlocal analysis method because we consider the problem on a general nonsmooth domain. We basically use the multiplier method, which has been applied extensively to studies of exact controllability and exponential stability of PDEs (see [13] and references therein). Recently, a piecewise multiplier method was introduced in [18]. Since the problem in the present paper is sort of "non-cylindrical", we will introduce some (t,x)-dependent piecewise multipliers (see §3 for details). These will help us to obtain a suitable estimate with some undesirable lower-order term. Then, using the uniqueness-compactness argument (see [17], for example), we absorb the lower-order term and obtain the desired estimate.

At this moment, we would like to mention that there are many results concerning exact controllability for the wave equation with (fixed) Dirichlet or Neumann boundary controllers. We refer the readers to [2, 5, 11, 16, 21] and references therein. Some other related works are [6, 12, 23]. However, due to the finite speed of propagation of wave, we can not expect the rapid exact controllability for the wave equation by any boundary controller.

§2. Some Preliminaries

Let $H = L^2(\Omega)$, $V = H_0^1(\Omega)$ (over the field $\mathbb{C}$ of complex numbers), and $\mathcal{H} = H_0^1(\Omega) \times L^2(\Omega)$. Take the usual $L^2(\Omega)$ inner product in H and the following as the inner product in V:

$$\langle y, z\rangle_V = \int_\Omega \nabla y \cdot \nabla \overline{z} dx, \qquad \forall y, z \in V, \tag{2.1}$$

with $\overline{z}$ standing for the complex conjugate of z. Then, both H and V are (complex) Hilbert spaces and so is $\mathcal{H}$ under the natural induced inner product. Usually, $\mathcal{H}$ is regarded as the finite energy state space of the system. Let

$$\begin{cases} \mathcal{D}(A) \triangleq \{y \in V \bigm| \Delta y \in H\}, \\ Ay = -\Delta y, \qquad \forall y \in \mathcal{D}(A). \end{cases} \tag{2.2}$$

Then A is a self-adjoint positive definite unbounded operator on H. Define in $\mathcal{H}$

$$\begin{cases} \mathcal{D}(\mathcal{A}) = \mathcal{D}(A) \times V, \\ \mathcal{A}\begin{pmatrix} y \\ z \end{pmatrix} \triangleq \begin{pmatrix} z \\ -Ay \end{pmatrix} \equiv \begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix}, \quad \forall \begin{pmatrix} y \\ z \end{pmatrix} \in \mathcal{D}(\mathcal{A}). \end{cases} \tag{2.3}$$

It is easy to see that $\mathcal{A}^* = -\mathcal{A}$ (skew-symmetric). Thus, $\mathcal{A}$ generates a C_0 group $e^{\mathcal{A}t}$ of unitary operators on $\mathcal{H}$ (see [20]).

It is standard that (see [20]) the following initial-boundary value problem

$$\begin{cases} \Box w \triangleq w_{tt} - \Delta w = f(x,t), & \text{in } \Omega \times [0,\infty), \\ w = 0, & \text{on } \partial\Omega \times [0,\infty), \\ w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x), & \text{in } \Omega \end{cases} \tag{2.4}$$

has a mild form of the following type

$$W(t) \triangleq \begin{pmatrix} w(t) \\ w_t(t) \end{pmatrix} = e^{\mathcal{A}t}\begin{pmatrix} w_0 \\ w_1 \end{pmatrix} + \int_0^t e^{\mathcal{A}(t-s)}\begin{pmatrix} 0 \\ f(s) \end{pmatrix} ds, \quad t \in [0,\infty). \tag{2.5}$$

For any $W_0 \equiv (w_0, w_1) \in \mathcal{H}$ and $f(\cdot) \in L^2(0,T; L^2(\Omega))$, (2.5) yields $W(\cdot) \in C([0,T];\mathcal{H})$. Thus, (2.4) admits a unique weak solution $w \in C([0,T]; H_0^1(\Omega)) \bigcap C^1([0,T]; L^2(\Omega))$.

Next, let $U = H \equiv L^2(\Omega)$, (thus, $\mathcal{U} = L^2_{\rm loc}(0,\infty; U)$) and for any $G(\cdot) \in \mathcal{G}$, define $B : [0,\infty) \to \mathcal{L}(U;\mathcal{H})$ by the following

$$B(t)u = \begin{pmatrix} 0 \\ \chi_{G(t)}u \end{pmatrix}, \qquad \forall u \in U, \quad t \in [0,\infty). \tag{2.6}$$

Clearly,

$$\|B(t)\|_{\mathcal{L}(U,\mathcal{H})} = 1, \qquad \forall t \in [0,\infty). \tag{2.7}$$

From (2.4)–(2.5), replacing $f(s)$ by $B(s)u(s)$, we know that (1.1) has the following mild form in the finite energy state space $\mathcal{H}$:

$$Y(t) \triangleq \begin{pmatrix} y(t) \\ y_t(t) \end{pmatrix} = e^{\mathcal{A}t} \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} + \int_0^t e^{\mathcal{A}(t-s)} B(s)u(s)ds, \quad t \in [0, \infty). \tag{2.8}$$

In what follows, we will not distinguish (1.1) and (2.8). To state some criteria for the exact controllability on $[0, T]$, we introduce some more notations. For any $T > 0$, define $L_T \in \mathcal{L}(L^2(0, T; U); \mathcal{H})$ by the following

$$L_T u(\cdot) = \int_0^T e^{-\mathcal{A}s} B(s)u(s)ds, \qquad \forall u(\cdot) \in L^2(0, T; U). \tag{2.9}$$

Clearly, the adjoint $L_T^* \in \mathcal{L}(\mathcal{H}; L^2(0, T; U))$ of L_T is given by

$$(L_T^* W_0)(t) = B^*(t) e^{\mathcal{A}t} W_0, \qquad \forall W_0 \in \mathcal{H}, \text{ a.e. } t \in [0, T]. \tag{2.10}$$

The following result is almost standard whose proof can follow from relevant results in [7,15].

Proposition 2.1. *Let $G(\cdot) \in \mathcal{G}$ and $T > 0$ be given. Then the following are equivalent*

(i) *System* (2.8) *is exactly controllable on* $[0, T]$*;*

(ii) *The range $\mathcal{R}(L_T) = \mathcal{H}$;*

(iii) *There exists a constant $c_0 > 0$, such that*

$$\|L_T^* W_0\|^2_{L^2(0,T;H)} \equiv \int_0^T \|B^*(s) e^{\mathcal{A}s} W_0\|_U^2 ds \geq c_0 \|W_0\|^2_{\mathcal{H}}, \qquad \forall W_0 \in \mathcal{H}. \tag{2.11}$$

For any $W_0 \equiv (w_0, w_1) \in \mathcal{H}$, denote $e^{\mathcal{A}\cdot} W_0 = (w(\cdot), \widehat{w}(\cdot))$. Then, $\widehat{w}(\cdot) = w_t(\cdot)$ and $w(\cdot)$ is the weak solution to the following problem

$$\begin{cases} \Box w \equiv w_{tt} - \Delta w = 0, & \text{in } \Omega_T, \\ w = 0, & \text{on } \partial\Omega \times [0, T], \\ w(x, 0) = w_0(x), \quad w_t(x, 0) = w_1(x), & \text{on } \Omega. \end{cases} \tag{2.12}$$

Consequently, by (2.10) and (2.6), we see that

$$(L_T^* W_0)(x, t) = \chi_{G(t)}(x) w_t(x, t), \qquad \text{a.e. } (x, t) \in \Omega_T. \tag{2.13}$$

Thus, (2.11) can be rewritten as follows

$$\int_0^T \int_{G(t)} |w_t|^2 dxdt \geq c_0 \int_\Omega \big\{ |\nabla w_0|^2 + |w_1|^2 \big\} dx, \qquad \forall (w_0, w_1) \in V \times H, \tag{2.14}$$

where $w \in C([0, T]; H_0^1(\Omega)) \bigcap C^1([0, T]; L^2(\Omega))$ is the weak solution of (2.12). Thus, the rapid exact controllability problem is reduced to the following

Problem. For any $T > 0$, find a $G(\cdot) \in \mathcal{G}$, such that (2.14) holds for some $c_0 > 0$.

We note that (2.14) is stronger than the condition $\mathcal{N}(L_T^*) = \{0\}$. While the latter is usually referred to as the unique continuation property, by this we mean the following

$$\begin{cases} \text{If } w \in C([0, T]; H_0^1(\Omega)) \bigcap C^1([0, T]; L^2(\Omega)) \text{ is a weak solution of} \\ \Box w = 0 \text{ on } \Omega_T \text{ with } w_t(x, t) = 0, \quad \text{a.e. } (x, t) \in \widetilde{G}_T, \\ \text{then } w(x, t) = 0, \quad \text{a.e. } (x, t) \in \Omega_T, \end{cases} \tag{2.15}$$

where $\widetilde{G}_T$ is the graph of $G(\cdot)$ defined in (1.7).

§3. Some Estimates

In this section, we will establish an estimate which will play an important role below. The main tool is the piecewise multiplier technique which has been used in [18]. We will see that the multipliers that we choose are nonzero only on a part of Ω_T. Different multipliers will contribute differently to the estimate.

Theorem 3.1. *Let $T > 0$ and $\Phi : \Omega_T \to I\!R$ be continuous, such that*

$$\begin{cases} \Omega \times \{0\} \subset S_0 \triangleq \{(x,t) \in \Omega_T \bigm| \Phi(x,t) < 0\}, \\ \Omega \times \{T\} \subset S_T \triangleq \{(x,t) \in \Omega_T \bigm| \Phi(x,t) > 0\}. \end{cases} \tag{3.1}$$

Then, there exists an $\varepsilon_0 > 0$ having the following property: for any $0 < \delta \le \varepsilon_0$, there exists a constant $C = C(\delta, \Phi) > 0$, such that for any $w \in C([0,T]; H_0^1(\Omega)) \bigcap C^1([0,T]; L^2(\Omega))$, with $\Box w \in L^2(\Omega_T)$, it holds

$$\int_{\Omega_T} \Big\{|w_t|^2 + |\nabla w|^2\Big\} dxdt \le C \Big\{ \int_{S^\delta} \Big\{|w_t|^2 + |w|^2\Big\} dxdt + \int_{\Omega_T} |\Box w|^2 dxdt \Big\}, \tag{3.2}$$

where S^δ is given by

$$S^\delta \triangleq \{(x,t) \in \Omega_T \bigm| |\Phi(x,t)| \le \delta\}. \tag{3.3}$$

Proof. Let $f = \Box w$, $w_0(x) = w(x,0)$ and $w_1(x) = w_t(x,0)$. Then, $f \in L^2(\Omega_T)$, $(w_0, w_1) \in \mathcal{H}$ and w is the weak solution of (2.4). Hence, by the usual density argument, it suffices to prove (3.2) for $(w_0, w_1) \in \mathcal{D}(\mathcal{A})$ and $f \in C^1([0,T]; H)$. For this case, it follows from (2.5) that

$$w \in C([0,T]; \mathcal{D}(A)) \bigcap C^1([0,T]; V) \bigcap C^2([0,T]; H).$$

We now proceed this. Since Φ is continuous and (3.1) holds, we see that there exists an $\varepsilon_0 > 0$, such that for all $0 < \varepsilon \le \varepsilon_0$,

$$\begin{cases} \Omega \times \{0\} \subset S_0^\varepsilon \triangleq \{(x,t) \in \Omega_T \bigm| \Phi(x,t) < -\varepsilon\}, \\ \Omega \times \{T\} \subset S_T^\varepsilon \triangleq \{(x,t) \in \Omega_T \bigm| \Phi(x,t) > \varepsilon\}. \end{cases} \tag{3.4}$$

Now, we let $0 < \varepsilon \le \varepsilon_0/2$ and let $\varphi^\varepsilon \in C_0^\infty(\mathbb{R}^{n+1}; [0,1])$, such that

$$\varphi^\varepsilon(x,t) = \begin{cases} 1, & (x,t) \in S_0, \\ 0, & (x,t) \in S_T^\varepsilon. \end{cases} \tag{3.5}$$

Note that by (3.4), we have

$$\varphi^\varepsilon(x,T) = 0, \qquad x \in \Omega. \tag{3.6}$$

Next, we observe the following

$$\begin{aligned} \int_{\Omega_T} t\varphi^\varepsilon \overline{w}_t w_{tt} dxdt &= \int_\Omega t\varphi^\varepsilon |w_t|^2 \Big|_0^T dx \\ &\quad - \int_{\Omega_T} w_t \big\{\varphi^\varepsilon \overline{w}_t + t\varphi_t^\varepsilon \overline{w}_t + t\varphi^\varepsilon \overline{w}_{tt}\big\} dxdt \\ &= - \int_{\Omega_T} \big\{\varphi^\varepsilon + t\varphi_t^\varepsilon\big\} |w_t|^2 dxdt - \int_{\Omega_T} t\varphi^\varepsilon w_t \overline{w}_{tt} dxdt, \end{aligned} \tag{3.7}$$

which yields

$$\mathrm{Re} \int_{\Omega_T} t\varphi^\varepsilon \overline{w}_t w_{tt} dxdt = -\frac{1}{2} \int_{\Omega_T} \big\{\varphi^\varepsilon + t\varphi_t^\varepsilon\big\} |w_t|^2 dxdt. \tag{3.8}$$

On the other hand, we note that

$$\begin{aligned}\int_{\Omega_T} t\varphi^\varepsilon \nabla w \cdot \nabla \overline{w}_t dxdt &= \int_\Omega t\varphi^\varepsilon |\nabla w|^2\Big|_0^T dx \\ &\quad - \int_{\Omega_T} \nabla \overline{w} \cdot \big\{\varphi^\varepsilon \nabla w + t\varphi_t^\varepsilon \nabla w + t\varphi^\varepsilon \nabla w_t\big\} dxdt \\ &= -\int_{\Omega_T} \big\{\varphi^\varepsilon + t\varphi_t^\varepsilon\big\} |\nabla w|^2 dxdt - \int_{\Omega_T} t\varphi^\varepsilon \nabla \overline{w} \cdot \nabla w_t dxdt.\end{aligned} \tag{3.9}$$

Hence

$$\mathrm{Re} \int_{\Omega_T} t\varphi^\varepsilon \nabla w \cdot \nabla \overline{w}_t dxdt = -\frac{1}{2} \int_{\Omega_T} \big\{\varphi^\varepsilon + t\varphi_t^\varepsilon\big\} |\nabla w|^2 dxdt. \tag{3.10}$$

Since

$$\int_{\Omega_T} t\varphi^\varepsilon \overline{w}_t \Delta w dxdt = -\int_{\Omega_T} t\varphi^\varepsilon \nabla \overline{w}_t \cdot \nabla w dxdt - \int_{\Omega_T} t\overline{w}_t \nabla \varphi^\varepsilon \cdot \nabla w dxdt, \tag{3.11}$$

we obtain by (3.10)

$$\mathrm{Re} \int_{\Omega_T} t\varphi^\varepsilon \overline{w}_t \Delta w dxdt = \frac{1}{2} \int_{\Omega_T} \big\{\varphi^\varepsilon + t\varphi_t^\varepsilon\big\} |\nabla w|^2 dxdt - \mathrm{Re} \int_{\Omega_T} t\overline{w}_t \nabla \varphi^\varepsilon \cdot \nabla w dxdt. \tag{3.12}$$

Now, combining (3.8) and (3.12), we have

$$\begin{aligned}\mathrm{Re} \int_{\Omega_T} t\varphi^\varepsilon \overline{w}_t \Box w dxdt &= \mathrm{Re} \int_{\Omega_T} t\varphi^\varepsilon \overline{w}_t w_{tt} dxdt - \mathrm{Re} \int_{\Omega_T} t\varphi^\varepsilon \overline{w}_t \Delta w dxdt \\ &= -\frac{1}{2} \int_{\Omega_T} \big\{\varphi^\varepsilon + t\varphi_t^\varepsilon\big\} \big\{|w_t|^2 + |\nabla w|^2\big\} dxdt \\ &\quad + \mathrm{Re} \int_{\Omega_T} t\overline{w}_t \nabla \varphi^\varepsilon \cdot \nabla w dxdt.\end{aligned} \tag{3.13}$$

This gives

$$\begin{aligned}&\int_{\Omega_T} \big\{\varphi^\varepsilon + t\varphi_t^\varepsilon\big\} \big\{|w_t|^2 + |\nabla w|^2\big\} dxdt \\ &= 2\mathrm{Re} \int_{\Omega_T} t\overline{w}_t \nabla \varphi^\varepsilon \cdot \nabla w dxdt - 2\mathrm{Re} \int_{\Omega_T} t\varphi^\varepsilon \overline{w}_t \Box w dxdt.\end{aligned} \tag{3.14}$$

Next, we let $\psi^\varepsilon \in C_0^\infty(\mathbb{R}^{n+1}; [0,1])$, such that

$$\psi^\varepsilon(x,t) = \begin{cases} 0, & (x,t) \in S_0^\varepsilon, \\ 1, & (x,t) \in S_T. \end{cases} \tag{3.15}$$

Similar to the above, by using $(T-t)\psi^\varepsilon \overline{w}_t$ as a multiplier, we obtain the following

$$\begin{aligned}&\int_{\Omega_T} \big\{\psi^\varepsilon - (T-t)\psi_t^\varepsilon\big\} \big\{|w_t|^2 + |\nabla w|^2\big\} dxdt \\ &= -2\mathrm{Re} \int_{\Omega_T} (T-t)\overline{w}_t \nabla \psi^\varepsilon \cdot \nabla w dxdt + 2\mathrm{Re} \int_{\Omega_T} (T-t)\psi^\varepsilon \overline{w}_t \Box w dxdt.\end{aligned} \tag{3.16}$$

We note that

$$\mathrm{supp}\, \nabla \varphi^\varepsilon \bigcup \mathrm{supp}\, \varphi_t^\varepsilon \bigcup \mathrm{supp}\, \nabla \psi^\varepsilon \bigcup \mathrm{supp}\, \psi_t^\varepsilon \subseteq S^\varepsilon. \tag{3.17}$$

Finally, let $\theta^\varepsilon \in C_0^\infty(\mathbb{R}^{n+1}; [0,1])$, such that

$$\theta^\varepsilon(x,t) = \begin{cases} 1, & (x,t) \in S^\varepsilon, \\ 0, & (x,t) \in \Omega_T \setminus S^{2\varepsilon}. \end{cases} \tag{3.18}$$

From (3.4) and the limitation $0<\varepsilon\le\varepsilon_0/2$, we see that

$$\theta^\varepsilon(x,0)=\theta^\varepsilon(x,T)=0,\qquad \forall x\in\Omega. \tag{3.19}$$

Now, we use $\theta^\varepsilon\overline{w}$ as a multiplier. It follows that

$$\begin{aligned}\int_{\Omega_T}\theta^\varepsilon\overline{w}\Box w dxdt&=\int_{\Omega_T}\theta^\varepsilon\overline{w}\big\{w_{tt}-\Delta w\big\}dxdt\\&=\int_\Omega\theta^\varepsilon\overline{w}w_t\Big|_0^T dx-\int_{\Omega_T}\big\{\theta^\varepsilon|w_t|^2+\theta^\varepsilon_t\overline{w}w_t\big\}dxdt\\&\quad+\int_{\Omega_T}\big\{\theta^\varepsilon|\nabla w|^2+\overline{w}\nabla\theta^\varepsilon\cdot\nabla w\big\}dxdt\\&=\int_{\Omega_T}\big\{-\theta^\varepsilon|w_t|^2+\theta^\varepsilon_t\overline{w}w_t+\theta^\varepsilon|\nabla w|^2+\overline{w}\nabla\theta^\varepsilon\cdot\nabla w\big\}dxdt.\end{aligned} \tag{3.20}$$

Hence

$$\begin{aligned}\int_{\Omega_T}\theta^\varepsilon|\nabla w|^2dxdt&\le\int_{\Omega_T}\big\{\theta^\varepsilon\big|w\Box w\big|+\theta^\varepsilon|w_t|^2+|\theta^\varepsilon_t\overline{w}w_t|\\&\quad+2|\nabla(\sqrt{\theta^\varepsilon})|\,|w|\sqrt{\theta^\varepsilon}|\nabla w|\big\}dxdt\\&\le\int_{S^{2\varepsilon}}\Big\{\theta^\varepsilon\big|w\Box w\big|+\theta^\varepsilon|w_t|^2+|\theta^\varepsilon_t\overline{w}w_t|\\&\quad+2|\nabla(\sqrt{\theta^\varepsilon})|^2|w|^2\Big\}dxdt+\frac{1}{2}\int_{\Omega_T}\theta^\varepsilon|\nabla w|^2dxdt.\end{aligned} \tag{3.21}$$

By (3.5) and (3.15), we have

$$\varphi^\varepsilon(x,t)+\psi^\varepsilon(x,t)\ge 1,\qquad \text{a.e. } (x,t)\in\Omega_T. \tag{3.22}$$

Combining (3.14), (3.16)–(3.18) and (3.21)–(3.22), using the Hölder's inequality, we obtain (3.2) with $\delta=2\varepsilon$.

Corollary 3.2. *Let the assumptions of Theorem 3.1 hold. Let*

$$w\in C([0,T];H_0^1(\Omega))\bigcap C([0,T];L^2(\Omega))$$

be a weak solution of $\Box w=0$ in Ω_T. Then, there exists an $\varepsilon_0>0$, such that for any $0<\delta\le\varepsilon_0$, there exists a constant $C=C(\delta,T)>0$, such that

$$E(s)=E(0)\le C\int_{S^\delta}\big\{|w_t|^2+|w|^2\big\}dxdt,\qquad \forall s\in[0,T], \tag{3.23}$$

where

$$E(t)=\int_\Omega\big\{|w_t(x,t)|^2+|\nabla w(x,t)|^2\big\}dx,\qquad \forall t\in[0,T]. \tag{3.24}$$

The proof is finished.

In what follows, we will use the above results with

$$\Phi(x,t)=t-ax_1-\frac{\sigma}{2},\qquad (x,t)\in\Omega_T. \tag{3.25}$$

In this case, we have

$$S^{\sigma/2}=D_\sigma,\qquad 0<\sigma<T. \tag{3.26}$$

§4. A Proof of the Main Result

In this section, we are going to prove Theorem 1.1 by the uniqueness-compactness argument. Let us first prove the uniqueness (recall (1.5)–(1.6), the definition of D_σ).

Lemma 4.1. *Let $T > 0$ and $0 < \sigma < T$ be given. Let $w \in C([0,T]; H_0^1(\Omega)) \bigcap C^1([0,T]; L^2(\Omega))$ be a weak solution of $\Box w = 0$ in Ω_T, such that*

$$w_t(x,t) = 0, \qquad a.e.\ (x,t) \in D_\sigma. \tag{4.1}$$

Then

$$w(x,t) = 0, \qquad a.e.\ (x,t) \in \Omega_T. \tag{4.2}$$

Proof. Define

$$v(x,s) = w(x, s + ax_1), \qquad (x,s) \in \Omega \times (0,\sigma). \tag{4.3}$$

Then it follows from (4.1) that

$$v_s(x,s) = w_t(x, s + ax_1) = 0, \qquad \text{a.e. } (x,s) \in \Omega \times (0,\sigma). \tag{4.4}$$

Consequently, the function $v(x,s)$ is independent of the variable s, i.e.,

$$w(x, s + ax_1) = v(x,s) = v(x), \qquad \text{a.e. } (x,s) \in \Omega \times (0,\sigma). \tag{4.5}$$

Now, we take $\theta \in C_0^\infty(\mathbb{R})$ satisfying

$$\operatorname{supp}\theta \subset (0,\sigma), \qquad \int_0^\sigma \theta(x_1)dx_1 = 1. \tag{4.6}$$

Then, for any $\varphi \in C_0^\infty(\Omega)$, we set

$$\psi(x,t) = \varphi(x)\theta(t - ax_1) \in C_0^\infty(\Omega_T), \qquad \operatorname{supp}\psi \subset D_\sigma. \tag{4.7}$$

It follows that

$$\begin{aligned}
0 &= \langle \Box w, \psi \rangle = \int_{\Omega_T} w\Box\psi dxdt = \int_{D_\sigma} w\Box\psi dxdt \\
&= \int_{D_\sigma} w(x,t)(\varphi(x)\theta''(t-ax_1) - \Delta_{n-1}\varphi(x)\theta(t-ax_1) - \varphi_{x_1x_1}(x)\theta(t-ax_1) \\
&\quad + 2a\varphi_{x_1}(x)\theta'(t-ax_1) - a^2\varphi(x)\theta''(t-ax_1))dxdt \\
&= \int_{\Omega\times(0,\sigma)} v(x)(\varphi(x)\theta''(s) - \Delta_{n-1}\varphi(x)\theta(s) - \varphi_{x_1x_1}(x)\theta(s) \\
&\quad + 2a\varphi_{x_1}(x)\theta'(s) - a^2\varphi(x)\theta''(s))dxds \\
&= -\int_\Omega v(x)\Delta\varphi(x)dx,
\end{aligned} \tag{4.8}$$

where Δ_{n-1} is the Laplacian operator in $\mathbb{R}^{n-1}$. The above implies that v is the weak solution of

$$\begin{cases} \Delta v = 0, & \text{in } \Omega, \\ v = 0, & \text{on } \partial\Omega. \end{cases} \tag{4.9}$$

Thus, it is necessarily that $v = 0$ in Ω. This leads to $w = 0$ in D_σ. Finally, applying Corollary 3.2 for $\Phi(x,t)$ defined by (3.25) yields the conclusion.

Proof of Theorem 1.1. We will verify (2.14) by contradiction argument. If (2.14) did not hold, then there would exist a sequence $W_0^m \equiv (w_0^m, w_1^m) \in \mathcal{H}$ with

$$\|W_0^m\|_{\mathcal{H}}^2 \equiv \int_\Omega (|\nabla w_0(x)|^2 + |w_1(x)|^2)dx = 1, \qquad m \geq 1, \tag{4.10}$$

such that the corresponding weak solution $w^m(x,t)$ satisfies (note (2.13))

$$\lim_{m\to\infty} \|L_T^* W_0^m\|^2 \equiv \lim_{m\to\infty} \int_{\widetilde{G}_T} |w_t^m(x,t)|^2 dxdt = 0. \tag{4.11}$$

We may assume that

$$W_0^m \triangleq (w_0^m, w_1^m) \to (w_0, w_1) \triangleq W_0, \qquad \text{weakly in } \mathcal{H}. \tag{4.12}$$

Then, we have

$$\|L_T^* W_0\|^2 = \lim_{m\to\infty} \langle L_T^* W_0^m, L_T^* W_0 \rangle = 0. \tag{4.13}$$

Thus, observing (1.7) and (2.13), by Lemma 4.1 we obtain $W_0 = 0$. This together with (4.12) implies

$$e^{\mathcal{A}t} W_0^m \to 0, \qquad \text{weakly in } \mathcal{H},\ \forall t \in [0,T]. \tag{4.14}$$

In another word, we have

$$w_t^m(t,\cdot) \to 0, \quad \nabla w^m(t,\cdot) \to 0, \qquad \text{weakly in } L^2(\Omega),\ \forall t \in [0,T]. \tag{4.15}$$

Hence, it follows from the compact embedding theorem[8] that

$$\lim_{m\to\infty} \|w^m\|_{L^2(\Omega_T)} = 0. \tag{4.16}$$

We choose δ so that

$$0 < \delta \leq \max\{\varepsilon_0, \sigma/2\},$$

where ε_0 is determined in Corollary 3.2 and σ is the same as in (1.7). Then (3.23), (3.26), (4.11) and (4.16) imply

$$\begin{aligned} &\int_\Omega \big\{|\nabla w_0^m(x)|^2 + |w_1^m(x)|^2\big\} dx \\ &\leq C \int_{D_\sigma} \big\{|w_t^m(x,t)|^2 + |w^m(x,t)|^2\big\} dxdt \to 0, \end{aligned} \tag{4.17}$$

which contradicts (4.10).

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