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SINGLE BIRTH PROCESSES**

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Abstract

The single birth process is a Markov chain, either time-continuous or time-discrete, valued in the non-negative integers: the system jumps with positive rate from k to $k+1$ but not to $k+j$ for all $j \geq 2$ (this explains the meaning of "single birth"). However, there is no restriction for the jumps from k to $k-j$ $(1 \leq j \leq k)$. This note mainly deals with the uniqueness problem for the time-continuous processes with an extension: the jumps from k to $k+1$ may also be forbidden for at most finite number of k. In both cases (time-continuous or -discrete), the hitting probability and the first moment of the hitting time are also studied

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§1. Introduction

Let us start from the time-continuous case. Then, the Q-matrix of a single birth process is as follows: $q_{i,i+1} > 0$, $q_{i,i+j} = 0$ for all $i \geq 0$ and $j \geq 2$. Throughout the note, we consider only totally stable and conservative Q-matrix: $q_i := -q_{ii} = \sum_{j \neq i} q_{ij} < \infty$ for all $i \geq 0$. Due to the fact that the boundary of such Q-matrix is at most single exit, the single birth processes consist of one of the most general class of Markov chains for which the three classical problems (the uniqueness, the recurrence and the positive recurrence) are all have computable criteria (see [5], or [1, Theorem 3.2.10 and Theorem 12.1.30] or [2, Theorem 3.16 and Theorem 4.54]). Thus, the single birth processes are on the one hand typical and fundamental and on the other hand, they are often used as a tool to study more complex processes[1, 2, 5].

The three problems for single birth processes mentioned above were solved first in [6] by using a probabilistic approach and then in [5] by using an analytic approach. In the latter paper, an intention was made to include the uniqueness criterion for the processes with an absorbing state (that is, $q_{01} = 0$). Unfortunately, the conclusion given in [5, Theorem 3 (ii)] (also [1, Corollary 3.2.18]) is incorrect. A correction is given in [3].

In what follows, let $q_{ij} = 0$ for all $j \geq i+2$ and suppose that $N := \max\{i+1 : q_{i,i+1} = 0\} < \infty$. Note that for the single birth process, $q_{i,i+1} > 0$ for all $i \geq 0$ and so $N = \max \emptyset = 0$ by convention. The main results of this note are the following

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Theorem 1.1. *The Q-process is unique iff* $\sum\limits_{n=N}^{\infty} m_n = \infty$, *where*

$$m_n = q_{n,n+1}^{-1}\Big(1 + \sum_{j=0}^{N-1} q_{nj} + \sum_{k=N}^{n-1} m_k \sum_{j=0}^{k} q_{nj}\Big)$$

for $n \ge N$. *By convention,* $\sum\limits_{\emptyset} = 0$.

Theorem 1.2. *Let* $N \ge 1$. *Choose arbitrarily a positive sequence* $(b_i : i \le N-1)$. *Define* $\bar{q}_{i,i+1} = b_i$ *and* $\bar{q}_i = q_i + b_i$ *if* $q_{i,i+1} = 0$ *and* $\bar{q}_{ij} = q_{ij}$ *for other* $j \ne i$. *Then the* (q_{ij})*-process is unique iff so is the* $(\bar{q}_{ij})$*-process. In other words, the case of* $N \ge 1$ *can be reduced to the one of* $N = 0$.

From the author's knowledge, the presentation of Theorem 1.1 is still new. When $N = 0$, Theorem 1.1 simplifies the previous result [5, Theorem 3 (i)] (or [1, Theorem 3.2.10] or [2, Theorem 3.16]), where an extra sequence $\big(F_k^{(j)}\big)$ was used (see also (2.1) below. The last sequence is needed for the study of recurrence of the process). When $N \ge 1$, Theorem 1.1 simplifies much more the main result of [3] in which four different sequences are employed. Theorem 1.2 presents an alternative criterion for the uniqueness in the case of $N \ge 1$.

The proofs of the theorems are given in the next section. Note that when $N \ge 1$, the recurrence and positive recurrence problems become trivial. Instead, we will study in Section 3 the hitting probability of τ_N (= the first time of the process hits the set $\{0, 1, \cdots, N-1\}$) and its first moment.

The study on the time-discrete analogs of the last two problems about the hitting time as well as the recurrence and positive recurrence in the irreducible case is delayed to Section 4. Thus, in the next two sections, we restrict ourselves to the time-continuous case only without further mentioning.

§2. Proof of Theorem 1.1 and Theorem 1.2

Proof of Theorem 1.1. The proof is simpler than but similar to the original one for the case of $N = 0$. Here, we need only to sketch the main steps.

(a) It suffices to show that the maximal solution (u_i^*) to the equation

$$u_i = \sum_{j \ne i} q_{ij} u_j / (\lambda + q_i), \qquad 0 \le u_i \le 1, \quad i \ge 0$$

equals zero identically for some fixed $\lambda > 0$. When $N \ge 1$, the set $\{0, 1, \cdots, N-1\}$ consists a closed subclass of the chain and so $u_i^* = 0$ for all $i \le N-1$.

(b) Define $q_k^{(i)} = \sum\limits_{j=0}^{i} q_{kj}$ $(i < k, k \ge 1)$ and

$$\begin{cases} F_k^{(k)} = 1, \quad k \ge N, \\ F_k^{(i)} = \sum\limits_{j=i}^{k-1} q_k^{(j)} F_j^{(i)} / q_{k,k+1}, \quad k > i \ge N. \end{cases} \tag{2.1}$$

Then $m_n = q_{n,n+1}^{-1}\Big(1 + q_n^{(N-1)} + \sum\limits_{k=N}^{n-1} q_n^{(k)} m_k\Big)$, $n \ge N$. By induction, we have

$$F_n^{(N)} \le q_{N,N+1} m_n, \qquad n \ge N. \tag{2.2}$$

(c) Let (u_i) be a solution to the equation

$$(1+q_i)u_i = \sum_{j\neq i} q_{ij}u_j, \quad i\geq 0 \quad \text{with } u_k=0 \text{ for all } k\leq N-1 \text{ and } u_N=1. \tag{2.3}$$

In view of (a) with $\lambda=1$, we need to prove that (u_i) is unbounded iff $\sum\limits_{n=N}^{\infty} m_n=\infty$. From (2.3), it follows that

$$u_{n+1}-u_n = q_{n,n+1}^{-1}\Big[\sum_{k=0}^{n-1} q_n^{(k)}(u_{k+1}-u_k)+u_n\Big], \qquad n\geq 0, \tag{2.4}$$

and hence $u_i\uparrow$ as $i\uparrow$. The key of the proof is to show that

$$m_k \leq u_{k+1}-u_k \leq (u_{N+1}-u_N)F_k^{(N)} + u_k m_k, \qquad k\geq N. \tag{2.5}$$

To check (2.5), we use induction. Note that $m_N = q_{N,N+1}^{-1}\big[1+q_N^{(N-1)}\big] = u_{N+1}-u_N$ and by (2.4) we have

$$u_{n+1}-u_n = q_{n,n+1}^{-1}\Big[q_n^{(N-1)} + \sum_{k=N}^{n-1} q_n^{(k)}(u_{k+1}-u_k)+u_n\Big], \qquad n\geq N.$$

Suppose that (2.5) holds for all k: $N\leq k\leq n-1$ and we now consider the case that $k=n$. Then

$$u_{n+1}-u_n \geq q_{n,n+1}^{-1}\Big[q_n^{(N-1)} + \sum_{k=N}^{n-1} q_n^{(k)}m_k+u_n\Big] \geq m_n, \qquad n\geq N+1,$$

$$\begin{aligned}
u_{n+1}-u_n &\leq q_{n,n+1}^{-1}\Big[(u_{N+1}-u_N)\sum_{k=N}^{n-1} q_n^{(k)}F_k^{(N)} + q_n^{(N-1)} + \sum_{k=N}^{n-1} q_n^{(k)}m_k u_k + u_n\Big]\\
&\leq (u_{N+1}-u_N)F_n^{(N)} + u_n\Big[1+q_n^{(N-1)}+\sum_{k=N}^{n-1} q_n^{(k)}m_k\Big]\Big/q_{n,n+1}\\
&= (u_{N+1}-u_N)F_n^{(N)} + u_n m_n, \qquad n\geq N+1.
\end{aligned}$$

(d) Having (2.2) and (2.5) in mind, we can easily complete the proof. Refer to [1] or the proof of (c) of either [1, Theorem 3.2.10] or [2, Theorem 3.16] for details.

Proof of Theorem 1.2. We adopt a probabilistic approach which goes back to [4]. Denote by (X_t) and $\big(\overline{X}_t\big)$ the minimal processes determined by (q_{ij}) and $(\bar q_{ij})$ respectively. We need to show that (X_t) has at most finite number of jumps in every finite time-interval iff so does $\big(\overline{X}_t\big)$. Let $\big(\widetilde{X}_t\big)$ denote the minimal process determined by the Q-matrix $(\tilde q_{ij})$: $\tilde q_i = 0$ for all $i\leq N-1$ and $\tilde q_{ij}=q_{ij}$ for all $i\geq N$. Then (X_t) has more jumps than $(\widetilde{X}_t)$. Note that for each $i\leq N-1$, $\big(\overline{X}_t\big)$ stays at i with the exponential law having parameter $\bar q_i$ and then jumps to other states. This is the only way $\big(\overline{X}_t\big)$ yields more jumps than $\big(\widetilde{X}_t\big)$. Due to the conditional independence and the fact $N<\infty$, such jumps can be happened at most finite times in a finite time-interval. Thus, $\big(\overline{X}_t\big)$ has at most finite number of jumps more than $\big(\widetilde{X}_t\big)$ in every finite time-interval. The same comparison holds for (X_t) and $\big(\widetilde{X}_t\big)$. We have thus proved the required assertion.

§3. Hitting Probability and the First Moment

Theorem 3.1. *Let $N\geq 1$. Assume that for each $i_0\geq N$, there exist some $i_1,\cdots,i_m$*

such that $i_m \le N-1$ *and* $q_{i_0 i_1} q_{i_1 i_2} \cdots q_{i_{m-1} i_m} > 0$. *Set* $\tau_N = \inf\{t \ge 0 : X_t \le N-1\}$. *Then* $\mathbb{P}^i[\tau_N < \infty] = 1$ *for every* $i \ge N$ *iff* $\sum\limits_{n=N}^{\infty} F_n^{(N)} = \infty$, *where* $\big(F_n^{(k)}\big)$ *is defined by* (2.1).

Theorem 3.2. *Under the hypothesis of Theorem* 3.1, $\mathbb{E}^i \tau_N < \infty$ *for every* $i \ge N$ *iff*

$$\sup_{k \ge N} \sum_{s=N}^{k} d_s \Big/ \sum_{s=N}^{k} F_s^{(N)} < \infty,$$

where $d_N = 0$, $d_n = \big(1 + \sum\limits_{s=N}^{n-1} d_s \sum\limits_{j=0}^{s} q_{nj}\big)/q_{n,n+1}$, $n > N$.

Proof of Theorem 3.1 and Theorem 3.2. (a) Without loss of generality, one may regard the set $\{0, \cdots, N-1\}$ as a single point 0. The resulting Markov chain is a single birth process with absorbing state 0. We have thus reduced the general case of $N \ge 2$ to the one of $N = 1$.

(b) Given an irreducible Markov chain, define a new Markov chain $\big(\widetilde{X}_t\big)$ by setting the origin as an absorbing state and set $\tilde{\tau}_0 = \inf\big\{t \ge 0 : \widetilde{X}_t = 0\big\}$. Then it is known (see [2, Proposition 4.21 (1), Theorem 4.30 (1) and Lemma 4.19]) that the original chain is recurrent (resp. positively recurrent) iff $\mathbb{P}^i[\tilde{\tau}_0 < \infty] = 1$ (resp. $\mathbb{E}^i \tilde{\tau}_0 < \infty$). We have thus reduced the case of $N = 1$ to the one of $N = 0$. Now, the assertions follow from the recurrence (positive recurrence) criteria for the single birth process, refer to [5], or [1, Theorem 12.1.30] or [2, Theorem 4.54]).

Next, we introduce a simple sufficient condition for $\mathbb{E}^i \tau_N < \infty$ for all $i \ge N$.

Corollary 3.1. *The condition of Theorem* 3.2 *holds if there exist constants* $c_1 \ge c_2 \ge 0$ *such that*

$$M_n := c_1 \sum_{j=0}^{N} q_{nj} + c_2 \Big[\sum_{k=N+1}^{n-1} \sum_{j=0}^{k} q_{nj} - q_{n,n+1} \Big] \ge 1, \qquad n \ge N+1. \tag{3.1}$$

Proof. Define $G_n = c_1 F_n^{(N)} - c_2$, $n \ge N$. From (2.1) and the definition of (d_n) given in Theorem 3.2, it follows that

$$\begin{cases} G_n = M_n / q_{n,n+1} + \sum\limits_{j=N+1}^{n-1} q_n^{(j)} G_j \big/ q_{n,n+1}, \\ d_n = 1/q_{n,n+1} + \sum\limits_{j=N+1}^{n-1} q_n^{(j)} d_j \big/ q_{n,n+1}, \qquad n \ge N+1. \end{cases} \tag{3.2}$$

Because $G_N = c_1 - c_2 \ge 0 = d_N$ and by (3.1),

$$G_{N+1} = M_{N+1}/q_{N+1,N+2} \ge 1/q_{N+1,N+2} = d_{N+1}.$$

By using (3.1) and (3.2) again and induction, it follows that

$$d_n \le G_n = c_1 F_n^{(N)} - c_2 \le c_1 F_n^{(N)} \quad \text{for all} \quad n \ge N$$

and hence

$$\sup_{n \ge N} \sum_{k=N}^{n} d_k \Big/ \sum_{k=N}^{n} F_k^{(N)} \le \sup_{n \ge N} d_n / F_n^{(N)} \le c_1 < \infty.$$

§4. Time-Discrete Case

We now study the time-discrete analog of the processes discussed above. That is, $P_{ij} = 0$

for all $j \ge i+2$ and $N := \max\{i+1 : P_{i,i+1} = 0\} < \infty$. When $N = 0$, the process is called the time-discrete single birth process. In that case, the assertion that $\mathbb{P}^i[\tau_N < \infty] = 1$ (resp. $\mathbb{E}^i \tau_N < \infty$) is equivalent to the recurrence (resp. positive recurrence) of the process.

Our goal of the study is reducing the present case to the time-continuous one by using the following Q-matrix

$$q_{ij} = P_{ij}, \quad j \ne i; \qquad q_i = 1 - P_{ii}, \quad i \in E. \tag{4.1}$$

Denote by (X_n) the Markov chain with transition probability (P_{ij}) and redefine

$$\tau_N = \inf\{n \ge 0 : X_n \le N-1\}.$$

Theorem 4.1. *The conclusions of Theorem* 3.1*, Theorem* 3.2 *and Corollary* 3.1 *for the new* τ_N *all hold when the* Q*-matrix is specified by* (4.1).

The following example was presented by Ya. G. Sinai in the author's talk in his seminar at Moscow State University (June, 1997). The question he asked and some physicists are also interested in, as he mentioned, is the existence of an invariant measure for this Markov chain.

Example. Let $P_{i,i+1} = 1/2$, $P_{ij} = 1/(2i)$ for all $i \ge 1$ and $j \le i-1$. Then the chain is ergodic and hence it has an invariant probability measure.

Proof. We have $N = 0$ and

$$q_{n0} = P_{n0} = \frac{1}{2n},$$
$$\sum_{k=1}^{n-1}\sum_{j=0}^{k} q_{nj} - \frac{1}{2} = \sum_{k=1}^{n-1} \frac{k+1}{2n} - \frac{1}{2} = \frac{n}{2} - \frac{1}{n}, \qquad n \ge 1.$$

Thus, condition (3.1) is satisfied with $c_1 = 8$ and $c_2 = 1$ and so the assertion follows from Theorem 4.1 and Corollary 3.1. Actually, this chain is geometrically ergodic by using [2, Theorem 4.31 (2)] with $y_j \equiv j$.

To prove Theorem 4.1, we need a simple observation.

Lemma 4.1. *Given a general, irreducible Markov chain with transition probability matrix* (P_{ij})*. The chain is recurrent* (*resp. positively recurrent*) *iff so is the process with* Q*-matrix specified by* (4.1).

Proof. (a) Recurrence. Fix an arbitrary point, say $0 \in E$. Then we have

The chain (P_{ij}) is recurrent

$\Longleftrightarrow$ The minimal solution to the equation $x_i = \sum_{j \ne 0} P_{ij} x_j + P_{i0},\ i \in E$ equals one identically (by [2, Proposition 4.21 and (4.6)])

$\Longleftrightarrow$ The equation $x_i = \sum_{j \ne 0} P_{ij} x_j,\ 0 \le x_i \le 1,\ i \in E$ has only trivial solution zero (by [2, Theorem 2.21])

$\Longleftrightarrow$ The equation $x_i = \sum_{j \ne 0, i} \frac{P_{ij}}{1 - P_{ii}} x_j,\ 0 \le x_i \le 1,\ i \in E$ has only trivial solution zero

$\Longleftrightarrow$ The process with Q-matrix (3.1) is recurrent (by [2, Lemma 4.53]).

(b) Positive recurrence. By [2, Theorem 4.31 (1)], the chain (P_{ij}) is positively recurrent iff for an arbitrarily fixed finite $H \subset E$, the equation

$$\begin{cases} \sum_j P_{ij} y_j \le y_i - 1, \qquad i \notin H, \\ \sum_{i \in H} \sum_j P_{ij} y_j < \infty \end{cases} \tag{4.2}$$

has a finite non-negative solution. On the other hand, by [2, Theorem 4.45 (1)], a (q_{ij})-process is positively recurrent iff for an arbitrarily fixed finite $H \subset E$, the equation

$$\begin{cases} \sum_j q_{ij} y_j \le -1, \qquad i \notin H, \\ \sum_{i \in H} \sum_{j \ne i} q_{ij} y_j < \infty \end{cases} \tag{4.3}$$

has a finite non-negative solution. Now, for the Q-matrix given by (4.1), the equations (4.2) and (4.3) are equivalent.

Proof of Theorem 4.1. (a) When $N = 0$, the theorem follows from Lemma 4.1.

(b) When $N \ge 1$, the proof of Theorem 3.1 and Theorem 3.2 given in the last section enables us to return to the case of $N = 0$.

Remark. Theorem 4.1 can also be used to study some multi-dimensional Markov chains. See for instance [2, Theorem 4.58] for an analog of the time-continuous result.

References

[1] Chen, M. F., Jump Processes and Particle Systems, Beijing Normal University Press, 1986 (in Chinese).

[2] Chen, M. F., From Markov Chains to Non-Equilibrium Particle Systems, Singapore, World Scientific, 1992.

[3] Li, J. P., A remark on the paper "Multidimensional Q-processes", *Chin. Ann. of Math.,* **11A:**4(1990), 505–506 (in Chinese).

[4] Pakes, A. G. & Tavaré, S., Comments on the age distribution of Markov processes, *Adv. Appl. Prob.,* **13**(1981), 681–703.

[5] Yan, S. J. & Chen, M. F., Multidimensional Q-processes, *Chin. Ann. of Math.,* **7B:**1(1986), 90–110.

[6] Zhang, J. K., Generalized birth-death processes, *Acta Math. Sci.,* **4:**6(1984), 241–259 (in Chinese).