Chin. Ann. of Math.
20B: 1(1999),83-92.

SOME RESULTS ON THE MINIMAL PERIOD PROBLEM OF NONCONVEX SECOND ORDER HAMILTONIAN SYSTEMS

FEI Guihua* WANG Tixiang**

Abstract

The authors study the existence of periodic solutions with prescribed minimal period for superquadratic and asymptotically linear autonomous second order Hamiltonian systems without any convexity assumption. Using the variational methods, an estimate on the minimal period of the corresponding nonconstant periodic solution of the above-mentioned system is obtained.

Keywords Minimal period solutions, Second order Hamiltonian systems, Iteration inequality, Variational method

1991 MR Subject Classification 58E05, 58F05, 34C25

Chinese Library Classification O19, O175.12

§1. Introduction and Main Results

In this paper, we consider the minimal period problem for the following autonomous second order Hamiltonian systems

$$\ddot{x} + V'(x) = 0, \quad \forall x \in \mathbb{R}^N, \tag{1.1}$$

where N is a positive integer. $V : \mathbb{R}^N \to \mathbb{R}$ and V' denotes its gradient. In the text of this paper, we denote by $a \cdot b$ and $|a|$ the usual inner product and norm in $\mathbb{R}^N$ respectively, and by $\mathcal{L}_s(\mathbb{R}^N)$ the set of all $N \times N$ real symmetric matrices. We also denote $\mathcal{L}_s^+(\mathbb{R}^N) = \{h \in \mathcal{L}_s(\mathbb{R}^N) \mid h \text{ is semi-positive definite }\}$.

More precisely, we make the following assumptions on V.

(V1) $V \in C^2(\mathbb{R}^N, \mathbb{R})$ and there exists $h_0 \in \mathcal{L}_s^+(\mathbb{R}^N)$ such that

$$V(x) = \frac{1}{2} h_0 x \cdot x + \tilde{V}(x), \quad \forall x \in \mathbb{R}^N.$$

(V2) $\tilde{V}(x) = o(|x|^2)$ as $|x| \to 0$.

(V3) $\tilde{V}(x) \geq \tilde{V}(0) = 0, \quad \forall x \in \mathbb{R}^N$.

(V4) There exist constants $\mu > 2$ and $r_0 > 0$ such that

$$0 < \mu \tilde{V}(x) \leq \tilde{V}'(x) \cdot x, \quad \forall |x| \geq r_0.$$

In his pioneering work[18] of 1978, Rabinowitz proved that, under the conditions (V1)–(V4) with $h_0 = 0$, the system (1.1) possesses a nonconstant periodic solution with any

Manuscript received January 21, 1997. Revised September 10, 1997.

*Department of Mathematics, University of Connecticut Storrs, CT 06269, USA.
E-mail: guf96001@uconnvm.uconn.edu
**Department of Mathematics, University of Connecticut Groton, CT 06340, USA.

prescribed period $T > 0$. Moreover, Rabinowitz conjectured that the system (1.1) possesses a nonconstant periodic solution with any prescribed minimal period under his conditions. Since then, there are many papers on this minimal period problem[1−7,9−12,17,20]. Among these results, most of them deal with convex Hamiltonian systems[6,7].

In the recent paper[13], by using the natural $\mathbb{Z}_2$-symmetry possesses by the system (1.1), Long extended some ideas of Ekeland and Hofer[6,7] to the second order Hamiltonian systems without any convexity assumptions and proved that, under the conditions (V1)–(V4) with $h_0 = 0$, for every $T > 0$, the system (1.1) possesses a nonconstant $T-$periodic even solution with minimal period not smaller than $T/(N + 2)$. The same ideas had been developed in [14, 15] to study the case that V is even.

The goal of this paper is to establish an estimate on the minimal period of the corresponding nonconstant periodic solution of (1.1) in the case that $h_0 \neq 0$.

For any $\tau > 0$ and $h \in \mathcal{L}_s(\mathbb{R}^N)$, let

$$T_0(h) = -h, \quad T_m(h) = (2\pi m/\tau)^2 I_N - h \ \text{ for } \ m \geq 1, \tag{1.2}$$

where I_N is the identity matrix in $\mathcal{L}_s(\mathbb{R}^N)$. We define the indices of h by

$$i_\tau(h) = \sum_{m=0}^{\infty} M^-(T_m(h)), \quad \nu_\tau(h) = \sum_{m=0}^{\infty} M^0(T_m(h)), \tag{1.3}$$

where $M^+(\cdot), M^-(\cdot)$ and $M^0(\cdot)$ denote the positive definite, negative definite and null subspace of the selfadjoint linear operator defining it, respectively.

Using the ideas in [13], we obtain an estimate of the minimal period in terms of the indices of h_0.

Theorem 1.1. *Suppose V satisfies* (V1)–(V4)*. Then for every $T > 0$, the system* (1.1) *possesses a nonconstant T-periodic even solution with minimal period not smaller than $T/(i_T(h_0) + \nu_T(h_0) + 2)$.*

Let w_0 be the greatest eigenvalue of h_0. By a straightforward computation (see Corollary 2.3), $i_T(h_0) + \nu_T(h_0) = N$ for every $T \in (0, 2\pi/\sqrt{w_0})$. So we have

Corollary 1.1. *Suppose that V satisfies* (V1)–(V4)*. Then for every $T \in (0, 2\pi/\sqrt{w_0})$, the system* (1.1) *possesses a nonconstant T-periodic even solution with minimal period not smaller than $T/(N + 2)$.*

If $h_0 = 0$, we have $w_0 = 0$ and $i_T(h_0) + \nu_T(h_0) = N$ for any $T > 0$. We go back to the results due to Long[13, Theorem 1.1]. See Remark 3.1 for further comparision.

Next we consider the asymptotically linear Hamiltonian systems, i.e., the potential function V satisfies

(V5) There exists $h_\infty \in \mathcal{L}_s^+(\mathbb{R}^N)$ such that

$$G(x) = V(x) - \frac{1}{2} h_\infty x \cdot x = o(|x|^2) \text{ as } \ |x| \to \infty.$$

(V6) $|G'(x)|$ is bounded and $\quad G(x) \to +\infty \quad$ as $\quad |x| \to \infty$.

Theorem 1.2. *Suppose that V satisfies* (V1), (V2), (V5), (V6) *and the following*

$$\{x \in \mathbb{R}^N \mid V'(x) = 0\} = \{0\}. \tag{1.4}$$

Then for every $T > 0$ satisfying

$$i_T(h_\infty) + \nu_T(h_\infty) \notin [i_T(h_0), i_T(h_0) + \nu_T(h_0)], \tag{1.5}$$

the system (1.1) *possesses a nonconstant $T-$periodic even solution with minimal period not smaller than* $T/(i_T(h_\infty)+\nu_T(h_\infty)+1)$.

Theorem 1.3. *Suppose that V satisfies* (V1)–(V3), (V5), (V6) *and the following*

(V7) $h_\infty - h_0 \in \mathcal{L}_s^+(\mathbb{R}^N)$ *and* $h_\infty h_0 = h_0 h_\infty$.

Then for every $T>0$ satisfying

$$i_T(h_\infty)+\nu_T(h_\infty) > i_T(h_0)+\nu_T(h_0), \tag{1.6}$$

the conclusion of Theorem 1.1 *holds.*

Remark 1.1. (i) In Theorem 1.2 and Theorem 1.3, if $\nu_T(h_\infty)=0$, then the assumption $(V6)$ can be dropped out.

(ii) The conditions (1.5) and (1.6) can be satisfied by many matrices. See Corollary 2.3, Corollary 3.1 and Corollary 3.2.

§2. Computation of the Symmetric Morse Indices

For $T>0$, let $S_T=\mathbb{R}/(T\mathbb{Z})$ and $E_T=W^{1,2}(S_T,\mathbb{R}^N)$. Recall that E_T consists of those $z\in\mathcal{L}^2(S_T,\mathbb{R}^N)$ whose Fourier series

$$z(t)=a_0+\sum_{k=1}^{\infty}\Big(a_k\cos\Big(\frac{2\pi}{T}kt\Big)+b_k\sin\Big(\frac{2\pi}{T}kt\Big)\Big), \tag{2.1}$$

where $a_0,a_k,b_k\in\mathbb{R}^N$ satisfies

$$\|z\|^2=T|a_0|^2+\frac{T}{2}\sum_{k=1}^{\infty}\Big(\frac{2\pi}{T}k\Big)^2(|a_k|^2+|b_k|^2)<\infty.$$

The inner product in E_T is given by

$$\langle z,z'\rangle = Ta_0\cdot a_0'+\frac{T}{2}\sum_{k=1}^{\infty}\Big(\frac{2\pi}{T}k\Big)^2(a_k\cdot a_k'+b_k\cdot b_k'). \tag{2.2}$$

Notice that the norm $\|\cdot\|$ is equivalent to the usual $W^{1,2}-$norm.

We define the mentioned $\mathbb{Z}_2$-action on continuous functions with $\mathbb{Z}_2=\{\delta_0,\delta_1\}$ by

$$\delta_0 x = x,\quad \delta_1 x(t)=x(-t),\quad \forall x\in C(S_T,\mathbb{R}^N). \tag{2.3}$$

Define $SE_T=\{z\in E_T\mid \delta_1 z = z\}$. Then by easy computation we have

$$SE_T=\Big\{z(t)\in E_T\,\Big|\, z(t)=a_0+\sum_{k=1}^{\infty}a_k\cos\Big(\frac{2\pi}{T}kt\Big)\Big\}. \tag{2.4}$$

For $A(t)\in C(S_T,\mathbb{R}^N)$ and $A(t)$ being even about $t=0$, we define an operator $A_T : SE_T\to SE_T$ by

$$\langle A_T z,z'\rangle = Ta_0\cdot a_0'+\int_0^T (A(t)z\cdot z')\,dt,\quad \forall z,z'\in SE_T. \tag{2.5}$$

Then A_T is a linear compact selfadjoint operator on SE_T satisfying

$$\dim M^-(\mathrm{id}-A_T)<+\infty,\quad \dim M^0(\mathrm{id}-A_T)<+\infty. \tag{2.6}$$

Set

$$\phi_T(y,z)=\int_0^T[\dot y\cdot\dot z - A(t)y\cdot z]\,dt,\quad \forall y,z\in SE_T. \tag{2.7}$$

Then by (2.5) we have

$$\phi_T(y,z) = \frac{1}{2}\langle(\mathrm{id} - A_T)y, z\rangle, \quad \forall y, z \in SE_T. \tag{2.8}$$

Definition 2.1. *Define*

$$si_T(A(t)) = \dim M^-(\mathrm{id} - A_T), \ \ s\nu_T(A(t)) = \dim M^0(\mathrm{id} - A_T).$$

$si_T(A(t))$ and $s\nu_T(A(t))$ are called the symmetric Morse indices of $A(t)$.

Theorem 2.1. *For any $T > 0$ and $h \in \mathcal{L}_s(\mathbb{R}^N)$, there holds $si_T(h) = i_T(h)$, $\quad s\nu_T(h) = \nu_T(h)$.*

Proof. Let A_T be the operator defined by (2.5) on SE_T corresponding to h. By (2.2) and (2.4), the operator A_T has explicit expressions

$$A_T z = a_0 + ha_0 + \sum_{k=1}^{\infty}\Big(\frac{T}{2\pi k}\Big)^2 ha_k \cos\Big(\frac{2\pi}{T}kt\Big),$$

where $z(t) \in SE_T$. Thus

$$(\mathrm{id} - A_T)z = -ha_0 + \sum_{k=1}^{\infty}\Big(I_N - \Big(\frac{T}{2\pi k}\Big)^2 h\Big)a_k \cos\Big(\frac{2\pi}{T}kt\Big). \tag{2.9}$$

By a straightforward computation, we have

$$\dim M^\star(\mathrm{id} - A_T) = \sum_{m=0}^{\infty} \dim M^\star(T_m(h)), \quad \star = -, 0. \tag{2.10}$$

Combining this with (1.3) and Definition 2.1 yields the conclusions.

As a direct consequence, we have

Corollary 2.1. *For any $h \in \mathcal{L}_s(\mathbb{R}^N)$ and $T, T' > 0$, we have*

$$si_{T'}(h) = si_T((T'/T)^2 h), \qquad s\nu_{T'}(h) = s\nu_T((T'/T)^2 h).$$
$$\lim_{T\to 0^+} si_T(h) = \dim M^+(h), \quad \lim_{T\to 0^+} s\nu_T(h) = \dim M^0(h).$$

By (1.2), (1.3), and an elementary computation, we have

Corollary 2.2. (i) *For $h_1, h_2 \in \mathcal{L}_s(\mathbb{R}^N)$, if $h_1 - h_2 \in \mathcal{L}_s^+(\mathbb{R}^N)$, then*

$$si_T(h_1) + s\nu_T(h_1) \geq si_T(h_2) + s\nu_T(h_2), \quad \forall T > 0.$$

(ii) *If $h \in \mathcal{L}_s^+(\mathbb{R}^N)$, for any $T' \geq T > 0$, we have*

$$si_{T'}(h) + s\nu_{T'}(h) \geq si_T(h) + s\nu_T(h) \geq N.$$

Moreover, if $h = 0$, we have $si_T(h) + s\nu_T(h) = N$, $\quad \forall T > 0$.

Corollary 2.3. *Suppose $h \in \mathcal{L}_s(\mathbb{R}^N)$ and w is the largest eigenvalue of h.*

(i) *If $w \leq 0$, for $T > 0$, there holds $si_T(h) = 0$, $\quad s\nu_T(h) = \dim M^0(h)$.*

(ii) *If $w > 0$, for $0 < T < 2\pi/\sqrt{w}$, there holds*

$$si_T(h) = \dim M^+(h), \quad s\nu_T(h) = \dim M^0(h).$$

(iii) *If $w > 0$, for $T \geq 2\pi/\sqrt{w}$, there holds*

$$si_T(h) + s\nu_T(h) \geq \dim M^+(h) + \dim M^0(h) + 1.$$

Moreover, for $T > 2\pi/\sqrt{w}$, there holds $si_T(h) \geq \dim M^+(h) + 1$.

Proof. Let $\lambda_1 \le \lambda_2 \le \cdots \lambda_N = w$ be the eigenvalues of h. By (1.3) and Theorem 2.1 it is easy to show that

$$si_T(h) = \#\Big\{(k,m) \mid \Big(\frac{2\pi}{T}k\Big)^2 < \lambda_m, 1 \le m \le N, k = 0,1,2,\cdots\Big\}, \tag{2.11}$$

$$s\nu_T(h) = \#\Big\{(k,m) \mid \Big(\frac{2\pi}{T}k\Big)^2 = \lambda_m, 1 \le m \le N, k = 0,1,2,\cdots\Big\}. \tag{2.12}$$

By (2.11) and (2.12), we get (i). If $T < 2\pi/\sqrt{w}$, then $\lambda_N = w < (\frac{2\pi}{T})^2$. By (2.11) and (2.12) we obtain (ii). Similarly, (2.11) and (2.12) imply that the conclusion (iii) holds.

For every $T-$periodic solution x of (1.1) which is even about $t = 0$, let $A(t) = V''(x(t))$. Then the symmetric Morse indices of x, denoted by $si_T(x)$ and $s\nu_T(x)$, are defined to be symmetric Morse indices of $A(t)$, i.e., $si_T(x) = si_T(A(t)), \quad s\nu_T(x) = s\nu_T(A(t))$.

We also denote by $O(x)$ the greatest positive integer k such that x is T/k-periodic. The following theorem, which estimates $O(x)$ in term of $si_T(x)$, is given by Long[13,Theorem 4.2].

Theorem 2.2. *Suppose $V \in C^2(\mathbb{R}^N, \mathbb{R})$. For $T > 0$ and every non-constant even (about $t = 0$) $C^2(S_T, \mathbb{R}^N)$-solution x of (1.1), there holds $O(x) \le si_T(x) + 1$.*

§3. Minimal Period Problem for Hamiltonian Systems

In this section, we study the minimal period problem for the system (1.1). For $T > 0$, let $\mathbb{Z}_2 = \{\delta_0, \delta_1\}$, $E_T = W^{1,2}(S_T, \mathbb{R}^N)$ and SE_T be defined as in Section 2. For $z \in E_T$, we define

$$f(z) = \int_0^T \Big[\frac{1}{2}|\dot{z}|^2 - V(z)\Big]\,dt. \tag{3.1}$$

Then f is $\mathbb{Z}_2$-invariant, i.e., $f(\delta_1 z) = f(z)$, $\forall z \in E_T$. In [13], Long proved the following proposition.

Proposition 3.1.[13] *Suppose $V \in C^2(\mathbb{R}^N, \mathbb{R})$. Then for every $T > 0$, we have*

(1) *$f \in C^2(SE_T, \mathbb{R})$, and there hold*

$$\langle f'(x), y\rangle = \int_0^T [\dot{x}\cdot\dot{y} - V'(x)\cdot y]\,dt, \quad \forall x, y \in SE_T, \tag{3.2}$$

$$\langle f''(x)y, z\rangle = \int_0^T [\dot{y}\cdot\dot{z} - V''(x)y\cdot z]\,dt, \quad \forall x, y, z \in SE_T. \tag{3.3}$$

(2) *If $x \in SE_T$ is a critical point of f on SE_T, then x is a $C^2(S_T, \mathbb{R}^N)$-solution of (1.1), and is even about $t = 0$.*

(3) *Conversely, if $x \in C^2(S_T, \mathbb{R}^N)$ is a solution of (1.1), and is even about $t = 0$, then $x \in SE_T$ and x is a critical point of f on SE_T.*

In order to find $T-$periodic even solution of (1.1), we need the following saddle point theorem which was proved in [5, 8, 16, 19, 21].

Theorem 3.1. *Let E be the Hilbert space with orthogonal decomposition $E = X \oplus Y$, where $\dim X < \infty$. Suppose that $f \in C^2(E, \mathbb{R})$ satisfies the (PS) condition and the following conditions:*

(F1) *There exist ρ and $\alpha > 0$ such that $f(w) \ge \alpha$, $\forall w \in \partial B_\rho(0) \cap Y$.*

(F2) *There exist $e \in \partial B_1(0) \cap Y$ and $r_1 > \rho$ such that $f(w) \le 0$, $\forall w \in \partial Q$, where $Q = (B_{r_1}(0) \cap X) \oplus \{re \mid 0 \le r \le r_1\}$.*

Then

(1) f *possesses a critical value* $c \geq \alpha$, *which is given by* $c = \inf_{h\in\Gamma} \max_{w\in Q} f(h(w))$, *where* $\Gamma = \{h \in C(Q, E) \mid h = \mathrm{id}$ *on* $\partial Q\}$.

(2) *There exists an element* $w_0 \in \mathcal{K}_c \equiv \{w \in E \mid f'(w) = 0,\ f(w) = c\}$ *such that the negative Morse index* $i(w_0)$ *of* f *at* w_0 *satisfies* $i(w_0) \leq \dim X + 1$.

Proof of Theorem 1.1. For $T > 0$, let A_0 be the operator defined by (2.5) on SE_T corresponding to h_0. By (V1), (2.8) and (3.1) we have

$$f(z) = \frac{1}{2}\langle(\mathrm{id} - A_0)z, z\rangle - \int_0^T \tilde{V}(z)\,dt, \quad \forall z \in SE_T. \tag{3.4}$$

We carry out the proof in several steps.

Step 1 Let $X = M^-(\mathrm{id} - A_0) \oplus M^0(\mathrm{id} - A_0)$, $\ Y = M^+(\mathrm{id} - A_0)$. Since $h_0 \in \mathcal{L}_s^+(\mathbb{R}^N)$, by (2.9) we have $\frac{1}{T}\int_0^T y(t)\,dt = 0$, $\ \forall y \in Y$. Thus Sobolev inequality implies that

$$\|y\|_\infty = \max_{t\in[0,T]} |y(t)| \leq \Big(\frac{T}{12}\Big)^{\frac{1}{2}} \|y\|, \quad \forall y \in Y. \tag{3.5}$$

Combining this with (V3) yields that, for $y \in Y$,

$$f(y) = \frac{1}{2}\langle(\mathrm{id} - A_0)y, y\rangle + o(\|y\|^2) \quad \text{as } y \to 0.$$

This implies that there exist $\rho, \alpha > 0$ such that

$$f(y) \geq \alpha, \quad \forall y \in \partial B_\rho(0) \cap Y. \tag{3.6}$$

Let $e \in \partial B_1(0) \cap Y$ and set $Q = \{re \mid 0 \leq r \leq r_1\} \oplus \{B_{r_1}(0) \cap X\}$, where r_1 is free for the moment. By (V4) we have

$$\tilde{V}(x) \geq c_1|x|^\mu - c_2, \quad \forall x \in \mathbb{R}^N, \tag{3.7}$$

where $c_1, c_2 > 0$. For $z \in z_- + z_0 \in X$, by (3.4), (3.7) and (2.6) we have

$$\begin{aligned} f(re + z) &= \frac{1}{2}\langle(\mathrm{id} - A_0)z_-, z_-\rangle + \frac{1}{2}r^2\langle(\mathrm{id} - A_0)e, e\rangle - \int_0^T \tilde{V}(z + re)\,dt \\ &\leq \frac{1}{2}r^2\|\mathrm{id} - A_0\| - \frac{1}{2}\|(\mathrm{id} - A_0)^{\#}\|^{-1}\|z_-\|^2 - c_1\int_0^T |z + re|^\mu\,dt + c_2T \\ &\leq c_3r^2 - c_4\|z_-\|^2 - c_5(\|z_0\|^\mu + r^\mu) + c_6. \end{aligned}$$

Then there exists $r_1 > 0$ such that

$$f(z) \leq 0, \qquad \forall z \in \partial Q. \tag{3.8}$$

Step 2 f satisfies the (PS) condition on SE_T, i.e. any sequence $\{u_k\} \subset SE_T$ satisfying $|f(u_k)| \leq M$ and

$$f'(u_k) \to 0 \ \text{ as } \ k \to \infty \tag{3.9}$$

possesses a subsequence convergent in SE_T.

In fact, for large k and $u = u_k$, by (V4), (3.4) and (3.7) we have

$$\begin{aligned} M + \|u\| &\geq f(u) - \frac{1}{2}\langle f'(u), u\rangle = \int_0^T \Big[\frac{1}{2}\tilde{V}'(u)\cdot u - \tilde{V}(u)\Big]\,dt \\ &\geq \Big(\frac{1}{2} - \frac{1}{\mu}\Big)\mu\int_0^T \tilde{V}(u)\,dt - M_1 \geq M_2\|u\|_{L^2}^\mu - M_3 \geq M_4|u^0|^\mu - M_3, \end{aligned} \tag{3.10}$$

where $u^0 = \frac{1}{T}\int_0^T u\,dt$. This implies

$$|u^0| \le M_5(1+\|u\|)^{\frac{1}{\mu}}, \quad \|u\|_{L^2} \le M_6(1+\|u\|)^{\frac{1}{\mu}}. \tag{3.11}$$

Now by (2.4), (2.5), (3.4), (3.9)-(3.11), we have

$$\begin{aligned}\|u\|^2 &= 2f(u) + \langle A_0 u, u\rangle + 2\int_0^T \tilde{V}(u)\,dt \\ &\le 2M + T|u^0|^2 + \|h_0\|\|u\|_{L^2}^2 + 2\int_0^T \tilde{V}(u)\,dt \\ &\le M_7 + M_8(1+\|u\|)^{2/\mu} + M_9\|u\|.\end{aligned}$$

This implies that $\{u_k\}$ is bounded. A standard argument shows that f satisfies (PS) condition.

Step 3 Now by Theorem 3.2 there exists a critical point $x \in SE_T$ of f with $f(x) = c \ge \alpha > 0$ and the Morse index $m^-(x)$ of f at x on SE_T satisfies

$$m^-(x) \le \dim M^-(\mathrm{id} - A_0) + \dim M^0(\mathrm{id} - A_0) + 1. \tag{3.12}$$

By (V2) and Proposition 3.1, x is a non-constant $C^2(S_T, \mathbb{R}^N)$- solution of (1.1) and is even about $t = 0$. Thus by Definition 2.1, Theorem 2.1, Theorem 2.2 and (3.12) we have

$$O(x) \le si_T(x) + 1 = m^-(x) + 1 \le i_T(h_0) + \nu_T(h_0) + 2.$$

This means that the minimal period of x is not smaller than $T/(i_T(h_0) + \nu_T(h_0) + 2)$. The proof is complete.

Remark 3.1. (i) Corollary 1.1 is a direct consequenc of Theorem 1.1 and Corollary 2.3. We omit the proof.

(ii) In [13, Theorem 1.2], Long got a similar result as our Corollary 1.1. But he required that $0 < T < 1/\sqrt{w_0}$. So Corollary 1.1 extends Theorem 1.2 in [13].

In order to prove Theorem 1.2, we need the following the definition and the theorem introduced and proved in [8], respectively.

Definition 3.1.[8] *Let E be a C^2-Riemannian manifold, D is a closed subset of E. A family $\mathcal{F}(\alpha)$ is said to be a homological family of dimension q with boundary D if for some nontrivial class $\alpha \in H_q(E, D)$ the family $\mathcal{F}(\alpha)$ is defined by*

$$\mathcal{F}(\alpha) = \{G \subset E : \alpha \ \ \textit{is in the image of}\ i_* : H_q(G, D) \to H_q(E, D)\},$$

where i_* is the homomorphism induced by the immersion $i : G \to E$.

Theorem 3.2.[8] *As in the Definition 3.1, for given E, D and α, let $\mathcal{F}(\alpha)$ be a homological family of dimension q with boundary D. Suppose that $f \in C^2(E, R)$ satisfies (PS) condition. Define*

$$c \equiv c(f, \mathcal{F}(\alpha)) = \inf_{G\in\mathcal{F}(\alpha)} \sup_{w\in G} f(w). \tag{3.13}$$

Suppose that $\sup_{w\in D} f(w) < c$ and f' is Fredholm on

$$\mathcal{K}_c = \{x \in E : f'(x) = 0, f(x) = c\}. \tag{3.14}$$

Then there exists $x \in \mathcal{K}_c$ such that the Morse indices $m^-(x)$ and $m^0(x)$ of the functional f at x satisfy $q - m^0(x) \le m^-(x) \le q$.

Proof of Theorem 1.2. For $T > 0$ satisfying (1.5), let f be defined by (3.1) and A_∞ be the operator defined by (2.5) on SE_T corresponding to h_∞. Set $X = M^-(\mathrm{id} - A_\infty) \oplus M^0(\mathrm{id} - A_\infty)$, $Y = M^+(\mathrm{id} - A_\infty)$. For $z \in Y$, by (V5) and (V6), we have

$$f(z) = \frac{1}{2}\langle(\mathrm{id} - A_\infty)z, z\rangle - \int_0^T G(z)\,dt \geq \frac{1}{2}\|(\mathrm{id} - A_\infty)^\#\|^{-1}\|z\|^2 - c_1\|z\|$$
$$\geq \delta = -\frac{1}{2}\|(\mathrm{id} - A_\infty)^\#\|c_1^2. \tag{3.15}$$

For $z = z_- + z_0 \in X$, by (V5) and (V6), we have

$$f(z) = \frac{1}{2}\langle(\mathrm{id} - A_\infty)z_-, z_-\rangle - \int_0^T G(z)\,dt$$
$$\leq -\frac{1}{2}\|(\mathrm{id} - A_\infty)^\#\|^{-1}\|z_-\|^2 + c_1\|z_-\| - \int_0^T G(z_0)\,dt. \tag{3.16}$$

By (V6) and (2.6) we have

$$\int_0^T G(z_0)\,dt \to +\infty \quad \text{as} \quad \|z_0\| \to \infty. \tag{3.17}$$

Combining this with (3.16) yields that there exist $r_1 > 0$ and $\beta < \delta$ such that

$$f(z) \leq \beta, \quad \forall z \in \partial Q, \tag{3.18}$$

where $Q = \{z \in X \mid \|z\| \leq r_1\}$. It is well known that, under the conditions $(V5)$ and $(V6)$, f satisfies (PS) condition (see [5, 22]).

Let $S = Y$, then ∂Q and S are homologically link[5,8]. Let $D = \partial Q$ and $\alpha = [Q] \in H_k(SE_T, D)$, where $k = \dim X$. Then α is nontrivial and $\mathcal{F}(\alpha)$ defined by Definition 3.1 is a homological family of dimension k with boundary D (see [5, 8]). It is well known that f' is Fredholm on $\mathcal{K}_c$ defined by (3.13) and (3.14). By (3.15) and (3.18) we obtain

$$\sup_{z\in D} f(z) \leq \beta < \delta \leq c = c(f, \mathcal{F}(\alpha))$$

(see [5]). Thus by Theorem 3.2, there exists $x \in \mathcal{K}_c$ such that the Morse indices $m^-(x)$ and $m^0(x)$ of f at x satisfies $\dim X - m^0(x) \leq m^-(x) \leq \dim X$. Combining this with Proposition 3.1, (1.4), (1.5) and Theorem 2.1 yields that x is a nonconstant even (about $t = 0$) $C^2(S_T, \mathbb{R}^N)$- solution of (1.1) which satisfies $si_T(x) \leq i_T(h_\infty) + \nu_T(h_\infty)$. Thus by Theorem 2.2 we get the conclusion.

In the following, we denote by w_0 and w_∞ the greatest eigenvalue of h_0 and h_∞ respectively.

Corollary 3.1. *Suppose the assumptions in Theorem* 1.2 *hold. Moreover we assume* h_0 *is positive definite and* $w_0 > w_\infty \geq 0$. *Then for every* $T \in (2\pi/\sqrt{w_0}, 2\pi/\sqrt{w_\infty})$, *the system* (1.1) *possesses a nonconstant* $T-$*periodic even solution with minimal period not smaller than* $T/(N + 1)$.

Proof. For $T \in (2\pi/\sqrt{w_0}, 2\pi/\sqrt{w_\infty})$, by Theorem 2.1 and Corollary 2.3 we have

$$i_T(h_0) \geq N + 1 > N = i_T(h_\infty) + \nu_T(h_\infty).$$

Thus Corollary 3.1 follows from Theorem 1.2.

Proof of Theorem 1.3. For $T > 0$, let f be defined by (3.4). Let $X = M^-(\mathrm{id} - A_0) \oplus M^0(\mathrm{id} - A_0)$, $Y = M^+(\mathrm{id} - A_0)$. Using the same arguments as Step 1 in the proof of

Theorem 1.1, we get (3.6). Let A_∞ be the operator defined by (2.5) on SE_T corresponding to h_∞. By (1.6), Definition 2.1 and Theorem 2.1 we have

$$Z = (M^-(\mathrm{id} - A_\infty) \oplus M^0(\mathrm{id} - A_\infty)) \cap M^+(\mathrm{id} - A_0) \neq \{0\}.$$

Let $e \in \partial B_1(0) \cap Z$ and set $Q = \{re \ : \ 0 \le r \le r_1\} \oplus \{B_{r_1}(0) \cap X\}$. By (V7) we have that $A_\infty - A_0$ is semi-positive definite and

$$(\mathrm{id} - A_\infty)(\mathrm{id} - A_0) = (\mathrm{id} - A_0)(\mathrm{id} - A_\infty).$$

This implies that

$$(\mathrm{id} - A_\infty)(M^+(\mathrm{id} - A_0)) \subset M^+(\mathrm{id} - A_0).$$

Especially, we have $(\mathrm{id} - A_\infty)e \in M^+(\mathrm{id} - A_0)$. Hence for any $z \in X$, we have

$$\langle (A_\infty - A_0)e, z\rangle = -\langle (\mathrm{id} - A_\infty)e, z\rangle + \langle (\mathrm{id} - A_0)e, z\rangle = 0. \tag{3.19}$$

Thus for any $z = re + z_- + z_0 \in Q$, by (V5), (V6) and (3.19) we obtain

$$\begin{aligned} f(z) &= \frac{1}{2}\langle (\mathrm{id} - A_\infty)z, z\rangle - \int_0^T G(z)\,dt \\ &= \frac{1}{2}r^2\langle (\mathrm{id} - A_\infty)e, e\rangle + \frac{1}{2}\langle (\mathrm{id} - A_0)z_-, z_-\rangle \\ &\quad - \frac{1}{2}\langle (A_\infty - A_0)(z_- + z_0), z_- + z_0\rangle - \int_0^T G(z)\,dt \\ &\le -\frac{1}{2}\|(\mathrm{id} - A_0)^{\#}\|^{-1}\|z_-\|^2 + M\|z_-\| - \int_0^T G(z_0 + re)\,dt. \end{aligned} \tag{3.20}$$

Let $X' = M^0(\mathrm{id} - A_0) \oplus \mathrm{span}\{e\}$. By (2.6) we know that $\dim X' < +\infty$. Therefore by (V6) we have

$$\int_0^T G(z_0 + re)\,dt \quad \to +\infty \text{ as } \|z_0 + re\| \to +\infty. \tag{3.21}$$

So there exist $M_1, M_2 \ge 0$ such that

$$-\frac{1}{2}\|(\mathrm{id} - A_0)^{\#}\|^{-1}\|z_-\|^2 + M\|z_-\| \le M_1, \ \forall z_- \in M^-(\mathrm{id} - A_0),$$

$$\int_0^T G(z_0 + re)\,dt \ge -M_2, \quad \forall z_0 + re \in X'.$$

By (3.21) there exists $r_2 > 0$ such that

$$M_1 - \int_0^T G(z_0 + re)\,dt \le 0, \quad \text{if } \ \|z_0 + re\| \ge r_2. \tag{3.22}$$

There also exists $r_3 > 0$ such that

$$-\frac{1}{2}\|(\mathrm{id} - A_0)^{\#}\|^{-1}\|z_-\|^2 + M\|z_-\| + M_2 \le 0, \quad \text{if } \ \|z_-\| \ge r_3. \tag{3.23}$$

If $r = 0$, i.e., $z = z_- + z_0 \in Q$, by (V3) we have

$$f(z) = \frac{1}{2}\langle (\mathrm{id} - A_0)z, z\rangle - \int_0^T \tilde{V}(z)\,dt \le 0. \tag{3.24}$$

Now let $r_1 = r_2 + r_3$. By (3.20)–(3.24) we know that (3.8) holds. By a standard argument, (V5) and (V6) imply that f satisfies (PS) condition[5,17,22]. Using the same arguments as Step 3 in the proof of Theorem 1.1, we get the conclusion. The proof is complete.

Corollary 3.2. *Suppose that* V *satisfies* (V1)–(V3), (V5)–(V7) *and* $w_\infty > w_0 \geq 0$. *Then for every* $T \in [2\pi/\sqrt{w_\infty}, 2\pi/\sqrt{w_0})$, *the system* (1.1) *possesses a nonconstant* $T-$*periodic even solution with minimal period not smaller than* $T/(N+2)$.

Proof. For any $T \in [2\pi/\sqrt{w_\infty}, 2\pi/\sqrt{w_0})$, by Theorem 2.1, Corollary 2.2 and Corollary 2.3, we have

$$i_T(h_\infty) + \nu_T(h_\infty) > N = i_T(h_0) + \nu_T(h_0).$$

The conclusion follows from Theorem 1.3.

Acknowledgements. The authors express their sincere thanks to Professor Yiming Long and Professor Wenzhao Lu for their help and useful comments. The authors also thank the referee for useful suggestions.

References

[1] Amann, H. & Zehnder, E., Nontrivial solutions for a class of nonresonance problem and applications to nonlinear differential equations, *Ann. Scuola Norm. Sup. Pisa Cl. Sci., Series 4*, **7**(1980), 539–603.

[2] Ambrosetti, A & Coti Zelati, V., Solutions with minimal period for Hamiltonian systems in a potential well, *Ann. Inst. H. Poincare Anal. non lineaire.,* **4**(1987), 275–296.

[3] Ambrosetti, A. & Mancini, G., Solutions of minimal period for a class of convex Hamiltonian systems , *Math. Ann.,* **255**(1981) 405–421.

[4] Clarke, F. & Ekeland, I., Hamiltonian trajectories having prescribed minimal period , *Comm. pure Appl. Math.,* **33**(1980), 103–116.

[5] Chang, K. C., Infinite dimensional Morse theory and multiple solution problems, Progress in nonlinear differential equations and their applications, **6**, 1993.

[6] Ekeland, I. & Hofer, H., Periodic solutions with prescribed period for convex autonomous Hamiltonian systems , *Invent. Math.,* **81**(1985), 155–188.

[7] Ekeland, I., Convexity Method in Hamiltonian Mechanics , Springer–Verlag, Berlin, 1990.

[8] Ghoussoub, N., Location, multiplicity and Morse indices of min-max critical points , *J. Reine Angew Math.,* **417**(1991), 27–76.

[9] Giradi. M. & Matzeu, M., Some results on solution of minimal period to superquadratic Hamiltonian equations , *Nonlinear Anal. T.M.A.,* **7**(1983), 475–482.

[10] Giradi, M. & Matzeu, M., Solution of minimal period for a class of nonconvex Hamiltonian systems and applications to the fixed energy problem, *Nonlinear Anal. T.M.A.,* **10**(1986), 371–382.

[11] Giradi, M. & Matzeu, M., Dual Morse index estimates for periodic solutions of Hamiltonian systems in some nonconvex superquadratic case , *Nonlinear Anal. T.M.A.,* **17**(1991), 481–497.

[12] Giradi, M. & Matzeu, M., Essential critical points of linking type and solution of minimal period to superquadratic Hamiltonian systems , Preprint, 1991.

[13] Long, Y., The minimal period problem of periodic solutions for autonomous superquadratic second order Hamiltonian systems , *J. Diff. Eq.,* **111**(1994), 147–174.

[14] Long, Y., The minimal period problem of classical Hamiltonian systems with even potentials , *Ann. IHP. Anal. non Lineaire,* **10:**6(1993), 605–626.

[15] Long, Y., Nonlinear oscillations for classical Hamiltonian systems with bi-even subquadratic potentials, *Nonlinear Analysis T.M.A.*, to appear.

[16] Lazer, A. & Solimini, S., Nontrivial solution of operator equations and Morse indices of critical points of min-max type , *Nonlinear Anal. T.M.A.,* **12**(1988), 761–775.

[17] Mawhin, J. & Willem, M., Critical point theory and Hamiltonian systems, Springer, 1989.

[18] Rabinowitz, P., Periodic solutions of Hamiltonian systems , *Comm. Pure Appl. Math.,* **31**(1978), 157–184.

[19] Rabinowitz, P., Minimax methods in critical point theory with applications to differential equations, *CBMS Regional Conf. Ser. in Math.,* A.M.S., **65**(1986).

[20] Rabinowitz, P., On the existence of periodic solutions for a class of symmetric Hamiltonian systems , *Nonlinear Anal. T. M. A.,* **11**(1987), 599–611.

[21] Solimini, S., Morse index estimates in min-max theorems, *Manus. Math.,* **63**(1989), 421–453.

[22] Szulkin, A., Cohomology and Morse theory for strongly indefinite functionals, *Math. Z.,* **209**(1992), 375–418.