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SOME NEW FAMILIES OF FILTRATION FOUR IN THE STABLE HOMOTOPY OF SPHERES

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Abstract

This paper proves the existence of 4 families of nontrivial homotopy elements in the stable homotopy of spheres which are represented by $\alpha_2 b_n$,$\alpha_2 k_n$, $\alpha_2 g_n$ and $\alpha_2 h_n h_m$ in the $E_2^{4,*}$-terms of the Adams spectral sequence respectively, where α_2, b_n, k_n, g_n and $h_n h_m$ are the known generators in the $E_2^{2,*}$-terms whose internal degree are $4(p-1)+1$, $2p^{n+1}(p-1)$, $(4p^{n+1}+2p^n)(p-1)$,$(2p^{n+1}+4p^n)(p-1)$, $(2p^n+2p^m)(p-1)$ respectively and $p \geq 5$ is a prime, $m \geq n+2 \geq 4$.

Keywords Stable homotopy of spheres, Adams spectral sequence, Derivations of maps, M-module spectra

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§1. Introduction

Let A be the mod p Steenrod algebra and S the sphere spectrum localized at an odd prime p. To determine the stable homotopy groups of spheres $\pi_* S$ is one of the central problem in homotopy theory. One of the main tools to reach it is the Adams spectral sequence (ASS) $E_2^{s,t} = \mathrm{Ext}_A^{s,t}(Z_p, Z_p) \Longrightarrow \pi_{t-s} S$, where the $E_2^{s,t}$-term is the cohomology of the Steenrod algebra A. If a family of generators x_i in $E_2^{s,*}$ converges nontrivially in the ASS, then we get a family of homotopy elements f_i in $\pi_* S$ and f_i is represented by $x_i \in E_2^{s,*}$ and has filtration s in the ASS. So far, not so many families of homotopy elements in $\pi_* S$ have been detected. For example, a family $\zeta_{n-1} \in \pi_{p^n q+q-3} S$, which has filtration 3 and is represented by $h_0 b_{n-1} \in \mathrm{Ext}_A^{3,p^n q+q}(Z_p, Z_p)$, was detected in [2], where $q = 2(p-1)$. The main purpose of this paper is to detect some new families of homotopy elements in $\pi_* S$ of filtration 4 in the ASS.

From [3], $\mathrm{Ext}_A^{2,*}(Z_p, Z_p)$ has Z_p-base consisting of the generators $\alpha_2, a_0^2, a_0 h_n (n > 0)$, $g_n (n \geq 0)$, $k_n (n \geq 0), b_n (n \geq 0)$ and $h_n h_m (m \geq n+2, n \geq 0)$ whose internal degree are $2q+1, 2, p^n q+1, p^{n+1}q+2p^n q, 2p^{n+1}q+p^n q, p^{n+1}q$ and $p^n q+p^m q$ respectively. Our main result is the following theorem.

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Theorem A. *Let $p \geq 5$. Then the products*

$$\alpha_2 g_n \neq 0 \ \in \ \mathrm{Ext}_A^{4,p^{n+1}q+2p^nq+2q+1}(Z_p, Z_p), \quad n \geq 2,$$
$$\alpha_2 k_n \neq 0 \ \in \ \mathrm{Ext}_A^{4,2p^{n+1}q+p^nq+2q+1}(Z_p, Z_p), \quad n \geq 2,$$
$$\alpha_2 b_n \neq 0 \ \in \ \mathrm{Ext}_A^{4,p^{n+1}q+2q+1}(Z_p, Z_p), \quad n \geq 1,$$
$$\alpha_2 h_n h_m \neq 0 \ \in \ \mathrm{Ext}_A^{4,p^nq+p^mq+2q+1}(Z_p, Z_p), \quad m \geq n+2, n \geq 2,$$

and they all converge in the ASS *so that the corresponding homotopy elements in $\pi_* S$ are nontrivial and of order p.*

From [4, p.513, Corollary 9.6 (b)], $g_n, k_n (n \geq 2), h_n h_m (n \geq 2, m \geq n+2) \in \mathrm{Ext}_A^{2,*}(Z_p, Z_p)$ do not converge in the ASS. From [6], $b_n \in \mathrm{Ext}_A^{2,p^{n+1}q}(Z_p, Z_p)$ admits a nontrivial differential $d_{2p-1}(b_n) \neq 0$, i.e. b_n also do not converge in the ASS. So, the homotopy elements obtained in Theorem A are indecomposable elements in $\pi_* S$. They have filtration 4 in the ASS and, by degree reasons, they should have filtration 3 in the Adams-Novikov spectral sequence (ANSS)

$$E_2^{s,t} = \mathrm{Ext}_{BP_*BP}^{s,t}(BP_*, BP_*) \Longrightarrow \pi_{t-s} S.$$

Let M be the Moore spectrum modulo an odd prime p given by the cofibration

$$S \xrightarrow{p} S \xrightarrow{i} M \xrightarrow{j} \Sigma S. \tag{1.1}$$

Theorem A leads to the following conjecture.

Conjecture. (1) *There are generators $g_n, k_n, b_n (n \geq 0), h_n h_m (m \geq n+2 \geq 2)$ in $\mathrm{Ext}_{BP_*BP}^{2,*}(BP_*, BP_*M)$ so that their images under the Thom map $\Phi : \mathrm{Ext}_{BP_*BP}^{2,*}(BP_*, BP_*M) \longrightarrow \mathrm{Ext}_A^{2,*}(H^*M, Z_p)$* (see [4, p.511–512]) *are*

$$\Phi(g_n) = i_*(g_n) \ \in \ \mathrm{Ext}_A^{2,*}(H^*M, Z_p),$$
$$\Phi(k_n) = i_*(k_n) \ \in \ \mathrm{Ext}_A^{2,*}(H^*M, Z_p),$$
$$\Phi(b_n) = i_*(b_n) \ \in \ \mathrm{Ext}_A^{2,*}(H^*M, Z_p),$$
$$\Phi(h_n h_m) = i_*(h_n h_m) \ \in \ \mathrm{Ext}_A^{2,*}(H^*M, Z_p),$$

where $i_* : \mathrm{Ext}_A^{2,*}(Z_p, Z_p) \to \mathrm{Ext}_A^{2,*}(H^*M, Z_p)$ is a homomorphism induced by $i : S \longrightarrow M$.

(2) *Let v_1 be a known generator in $\mathrm{Ext}_{BP_*BP}^{0,q}(BP_*, BP_*M)$, then the products $v_1^2 g_n$, $v_1^2 k_n$, $v_1^2 b_n$, $v_1^2 h_n h_m \in \mathrm{Ext}_{BP_*BP}^{2,*}(BP_*, BP_*M)$ are permanent cycles in the* ANSS *and the homotopy elements obtained in Theorem A are represented in the* ANSS *by*

$$j_*(v_1^2 g_n), \quad j_*(v_1^2 k_n), \quad j_*(v_1^2 b_n), \quad j_*(v_1^2 h_n h_m) \in \mathrm{Ext}_{BP_*BP}^{3,*}(BP_*, BP_*)$$

respectively, where

$$j_* : \mathrm{Ext}_{BP_*BP}^{2,*}(BP_*, BP_*M) \longrightarrow \mathrm{Ext}_{BP_*BP}^{3,*}(BP_*, BP_*)$$

is the boundary homomorphism induced by $j : M \to \Sigma S$.

(3) *For some appropriate value of the integers r and n, m, the elements $j_*(v_1^r g_n)$, $j_*(v_1^r k_n)$, $j_*(v_1^r b_n)$, $j_*(v_1^r h_n h_m)$ are nontrivial in $\mathrm{Ext}_{BP_*BP}^{3,*}(BP_*, BP_*)$ and they all converge in the* ANSS.

Theorem A will be proved by some technique on derivations of maps between M- module spectra processing in the Adams resolution of certain spectra. The basic knowledge on this technique is given in section 2 and the proof of Theorem A is given in Section 3.

§2. Derivations of Maps between M-Module Spectra

In this section, we recall some basic knowledge on M-module spectra developed in [8]. Let M be the Moore spectrum modulo an odd prime p given by the cofibration (1.1). M is a commutative ring spectrum with multiplication $m_M : M \wedge M \to M$ such that $m_M(i \wedge 1_M) = 1_M$ and there is $\overline{m}_M : \Sigma M \to M \wedge M$ such that $(j \wedge 1_M)\overline{m}_M = 1_M$, $m_M \overline{m}_M = 0$, $\overline{m}_M(j \wedge 1_M) + (i \wedge 1_M)m_M = 1_{M\wedge M}$, $m_M T = -m_M$, $T\overline{m}_M = \overline{m}_M$, where $T : M \wedge M \to M \wedge M$ is the switching map.

A spectrum X is called an M-module spectrum if the map $p \wedge 1_X : X \to X$ is nulhomotopic, i.e. $p \wedge 1_X = 0 \in [X, X]$. If X is an M-module spectrum, then the cofibration

$$X \xrightarrow{p\wedge 1_X} X \xrightarrow{i\wedge 1_X} M \wedge X \xrightarrow{j\wedge 1_X} \Sigma X \tag{2.1}$$

splits, i.e. there is a homotopy equivalence $M \wedge X = X \vee \Sigma X$ and there are maps $m_X : M \wedge X \to X$, $\overline{m}_X : \Sigma X \to M \wedge X$ satisfying

$$m_X(i \wedge 1_X) = 1_X, \quad (j \wedge 1_X)\overline{m}_X = 1_X,$$
$$m_X \overline{m}_X = 0, \quad \overline{m}_X(j \wedge 1_X) + (i \wedge 1_X)m_X = 1_{M\wedge X}.$$

The maps $m_X, \overline{m}_X$ are called the M-module actions of the M-module spectrum X.

Let X and X' be M-module spectra. Then we define a homomorphism

$$d : [\Sigma^s X', X] \longrightarrow [\Sigma^{s+1} X', X]$$

by $d(f) = m_X(1_M \wedge f)\overline{m}_{X'}$ for $f \in [\Sigma^s X', X]$. This operation d is called a derivation (of maps between M-module spectra) which has the following properties.

Theorem 2.1.[8, p.210, Theorem 2.2] (i) *d is derivative : $d(fg) = fd(g)+(-1)^{\deg g}d(f)g$ for $f \in [\Sigma^s X', X]$, $g \in [\Sigma^t X'', X']$, where X, X', X'' are M-module spectra.*

(ii) *Let Y', Y be arbitrary finite spectra and $h \in [\Sigma^r Y', Y]$. Then $d(h \wedge f) = (-1)^{\deg h} h \wedge d(f)$ for $f \in [\Sigma^s X', X]$.*

(iii) *If X' and X are associative M-module spectra, then*

$$d^2 = 0 : [\Sigma^* X', X] \to [\Sigma^{*+2} X', X].$$

Consider the spectra $V(k)$ given in [9] such that the Z_p-cohomology

$$H^* V(k) \cong E(Q_0, Q_1, \cdots, Q_k),$$

the exterior algebra generated by Milnor basis elements $Q_0, Q_1, \cdots, Q_k$ in A. From [9] we know that $V(0) = M$ and $V(1)$ exists for $p \geq 3$, being a ring spectrum for $p \geq 5$ and it is the cofibre of the Adams map $\alpha : \Sigma^q M \to M$, where $q = 2(p-1)$.

We briefly write $V(1)$ as K which is the cofibre of $\alpha : \Sigma^q M \to M$ given by the cofibration

$$\Sigma^q M \xrightarrow{\alpha} M \xrightarrow{i'} K \xrightarrow{j'} \Sigma^{q+1} M. \tag{2.2}$$

Then, from [8, p.218 (3.7)], M and K are M-module spectra and $d(\alpha) = d(i') = d(j') = 0$, $d(ij) = -1_M$.

§3. Proof of the Main Theorem

The main theorem A will follow easily from the following general Theorem 3.1.

Theorem 3.1. *Let $p \geq 5$ and x be the unique generator of* $\mathrm{Ext}_A^{s,t}(Z_p, Z_p)$ *so that* $\alpha_2 x \neq 0 \in \mathrm{Ext}_A^{s+2,t+2q+1}(Z_p, Z_p)$ *and* $a_0 x \neq 0, h_0 x \neq 0$ *are the unique generator of* $\mathrm{Ext}_A^{s+1,t+1}(Z_p, Z_p)$, $\mathrm{Ext}_A^{s+1,t+q}(Z_p, Z_p)$ *respectively, where* α_2, a_0, h_0 *are the known generators in* $\mathrm{Ext}_A^{2,2q+1}(Z_p, Z_p)$, $\mathrm{Ext}_A^{1,1}(Z_p, Z_p)$, $\mathrm{Ext}_A^{1,q}(Z_p, Z_p)$ *respectively.*

Suppose that :

(1) $a_0^2 x \neq 0 \in \mathrm{Ext}_A^{s+2,t+2}(Z_p, Z_p)$.

(2) $\mathrm{Ext}_A^{s+1,t+q+r}(Z_p, Z_p) = 0$ *for* $r = 1, 2, 3$.

(3) $\mathrm{Ext}_A^{s+1,t}(Z_p, Z_p) = 0$,$\mathrm{Ext}_A^{s+1,t+2}(Z_p, Z_p) = 0$ *or has unique generator* $a_0^2 x'$ *with* $x' \in \mathrm{Ext}_A^{s-1,t}(Z_p, Z_p)$, $\mathrm{Ext}_A^{s,t+1}(Z_p, Z_p) = 0$ *or has unique generator* $a_0 x'$ *with* $x' \in \mathrm{Ext}_A^{s-1,t}(Z_p, Z_p)$. *Then the product* $\alpha_2 \cdot x \in \mathrm{Ext}_A^{s+2,t+2q+1}(Z_p, Z_p)$ *is a permanent cycle in the* ASS.

Before proving Theorem 3.1, we do some preliminaries. Let K' be the cofibre of jj' : $\Sigma^{-1}K \to \Sigma^{q+1}S$ given by the cofibration in the following commutative diagram of 3×3 lemma in stable homotopy category (see [7, p.292–293])

$$\begin{array}{ccccccc}
\Sigma^{-1}K & \xrightarrow{jj'} & \Sigma^{q+1}S & \xrightarrow{p} & \Sigma^{q+1}S & & \\
 & \searrow j' \quad \nearrow j & & \searrow z \quad \nearrow y & & & \\
 & \Sigma^q M & & K' & & & \\
 & \nearrow i \quad \searrow \alpha & & \nearrow v \quad \searrow x & & & \\
\Sigma^q S & \xrightarrow{\alpha i} & M & \xrightarrow{i'} & K & &
\end{array} \tag{3.1}$$

From the above diagram, we know that K' also is the cofibre of $\alpha i : \Sigma^q S \to M$, i.e. we have two cofibrations

$$\Sigma^q S \xrightarrow{\alpha i} M \xrightarrow{v} K' \xrightarrow{y} \Sigma^{q+1}S, \tag{3.2}$$

$$\Sigma^{-1}K \xrightarrow{jj'} \Sigma^{q+1}S \xrightarrow{z} K' \xrightarrow{x} K. \tag{3.3}$$

Proposition 3.1. *Let* $z : \Sigma^{q+1}S \to K'$ *and* $y : K' \to \Sigma^{q+1}S$ *be the maps in* (3.1), *then the composition* $z \cdot y = 1_{K'} \wedge p : K' \to K'$.

Proof. Since $y(z \cdot y - 1_{K'} \wedge p) = p \cdot y - p \cdot y = 0$, $(z \cdot y - 1_{K'} \wedge p) \in v_*[K', M] = 0$ by the following exact sequence induced by (3.2)

$$\cdots \xleftarrow{(\alpha i)^*} [M, M] \xleftarrow{v^*} [K', M] \xleftarrow{y^*} [\Sigma^{q+1}S, M] = 0,$$

where $(\alpha i)^*$ is monic since $[M, M]$ has unique generator 1_M. The proof is finished.

The cofibre of $1_{K'} \wedge p : K' \to K'$ is $K' \wedge M$ and $K' \wedge M$ also is the cofibre of $\alpha i j j'$: $\Sigma^{-1}K \to \Sigma M$, which can be seen by the following commutative diagram of 3×3 lemma in stable homotopy category

$$\begin{array}{ccccccc}
K' & \xrightarrow{1_{K'} \wedge p} & K' & \xrightarrow{x} & K & & \\
 & \searrow y \quad \nearrow z & & \searrow 1_{K'} \wedge i \quad \nearrow \rho & & & \\
 & \Sigma^{q+1}S & & K' \wedge M & & & \\
 & \nearrow jj' \quad \searrow \alpha i & & \nearrow \psi \quad \searrow 1_{K'} \wedge j & & & \\
\Sigma^{-1}K & \xrightarrow{\alpha i j j'} & \Sigma M & \xrightarrow{v} & \Sigma K' & &
\end{array} \tag{3.4}$$

That is, we have two cofibrations

$$K' \xrightarrow{1_{K'} \wedge p} K' \xrightarrow{1_{K'} \wedge i} K' \wedge M \xrightarrow{1_{K'} \wedge j} \Sigma K', \tag{3.5}$$

$$\Sigma^{-1}K \xrightarrow{\alpha i j j'} \Sigma M \xrightarrow{\psi} K' \wedge M \xrightarrow{\rho} K. \tag{3.6}$$

Since $d(i') = d(j') = 0$ and $d(ij) = -1_M$, $d(\alpha ijj') = 0$ by Theorem 2.1(i). Hence, by [8, p.211, Lemma 2.3], we have

Proposition 3.2. *$K' \wedge M$ is an M-module spectrum and $d(\psi) = 0 \in [\Sigma^2 M, K' \wedge M]$, $d(\rho) = 0 \in [\Sigma K' \wedge M, K]$, where d's are derivations of maps between M-module spectra.*

Now we proceed to prove Theorem 3.1. We first prove the following lemma.

Lemma 3.1. *On the supposition of Theorem* 3.1*, we have*

(1) *There is unique generator $\overline{h_0x} \in \mathrm{Ext}_A^{s+1,t+q+1}(H^*M, Z_p)$ so that its image under j_* : $\mathrm{Ext}_A^{s+1,t+q+1}(H^*M, Z_p) \to \mathrm{Ext}_A^{s+1,t+q}(Z_p, Z_p)$ is $j_*(\overline{h_0x}) = h_0x$ and $\overline{h_0x} = (\alpha i)_*(x)$, where $(\alpha i)_* : \mathrm{Ext}_A^{s,t}(Z_p, Z_p) \to \mathrm{Ext}_A^{s+1,t+q+1}(H^*M, Z_p)$ is the boundary homomorphism induced by $\alpha i : \Sigma^q S \to M$.*

(2) $\mathrm{Ext}_A^{s+1,t+q+1}(H^*K', Z_p) = 0$.

Proof. (1) Consider the following exact sequence

$$\mathrm{Ext}_A^{s+1,t+q+1}(Z_p, Z_p) \xrightarrow{i_*} \mathrm{Ext}_A^{s+1,t+q+1}(H^*M, Z_p) \xrightarrow{j_*} \mathrm{Ext}_A^{s+1,t+q}(Z_p, Z_p) \xrightarrow{p_*}$$

induced by (1.1). From the supposition, the left group is zero and the right group has unique generator h_0x. Since

$$p_*(h_0) = a_0h_0 = 0 \in \mathrm{Ext}_A^{2,q+1}(Z_p, Z_p),$$

$p_*(h_0x) = 0$ and so $\mathrm{Ext}_A^{s+1,t+q+1}(H^*M, Z_p)$ has unique generator $\overline{h_0x}$ so that $j_*(\overline{h_0x}) = h_0x$.

Moreover, $j\alpha i : \Sigma^{q-1}S \to S$ induces zero homomorphism in Z_p-cohomology and it is a map represented by $h_0 \in \mathrm{Ext}_A^{1,q}(Z_p, Z_p)$ in the ASS, then $j\alpha i : \Sigma^{q-1}S \to S$ induces a boundary homomorphism $(j\alpha i)_* : \mathrm{Ext}_A^{s,t}(Z_p, Z_p) \to \mathrm{Ext}_A^{s+1,t+q}(Z_p, Z_p)$ which is a multiplication by $h_0 \in \mathrm{Ext}_A^{1,q}(Z_p, Z_p)$. Then $j_*(\overline{h_0x}) = h_0x = (j\alpha i)_*(x) = j_*(\alpha i)_*(x)$ and we have $\overline{h_0x} = (\alpha i)_*(x)$ since $\mathrm{Ext}_A^{s+1,t+q+1}(Z_p, Z_p) = 0$.

(2) Consider the following exact sequence

$$\xrightarrow{(\alpha i)_*} \mathrm{Ext}_A^{s+1,t+q+1}(H^*M, Z_p) \xrightarrow{v_*} \mathrm{Ext}_A^{s+1,t+q+1}(H^*K', Z_p) \xrightarrow{y_*} \mathrm{Ext}_A^{s+1,t}(Z_p, Z_p)$$

induced by (3.2). The left group has unique generator $\overline{h_0x} = (\alpha i)_*(x)$, then $\mathrm{im}\, v_* = 0$. From the supposition (3), the right group is zero, so the result follows.

Let

$$\begin{array}{ccccccc}
\cdots \xrightarrow{\bar a_2} & \Sigma^{-2}E_2 & \xrightarrow{\bar a_1} & \Sigma^{-1}E_1 & \xrightarrow{\bar a_0} & E_0 = S \\
 & \downarrow \bar b_2 & & \downarrow \bar b_1 & & \downarrow \bar b_0 \\
 & \Sigma^{-2}KG_2 & & \Sigma^{-1}KG_1 & & KG_0 = KZ_p
\end{array}$$

be the minimal Adams resolution of S satisfying

(1) $E_s \xrightarrow{\bar b_s} KG_s \xrightarrow{\bar c_s} E_{s+1} \xrightarrow{\bar a_s} \Sigma E_s$ are cofibrations for all $s \geq 0$ which induce short exact sequences $0 \to H^*E_{s+1} \xrightarrow{\bar c_s^*} H^*KG_s \xrightarrow{\bar b_s^*} H^*E_s \to 0$ in Z_p-cohomology.

(2) KG_s is a wedge sum of suspensions of Eilenberg-Maclane spectra of type KZ_p.

(3) $\pi_t KG_s$ are the $E_1^{s,t}$-terms, $(\bar b_s \bar c_{s-1})_* : \pi_t KG_{s-1} \to \pi_t KG_s$ are the $d_1^{s-1,t}$- differentials of the ASS and $\pi_t KG_s \cong \mathrm{Ext}_A^{s,t}(Z_p, Z_p)$. Then

$$\begin{array}{ccccccc}
\cdots \xrightarrow{\bar a_2 \wedge 1_Y} & \Sigma^{-2}E_2 \wedge Y & \xrightarrow{\bar a_1 \wedge 1_Y} & \Sigma^{-1}E_1 \wedge Y & \xrightarrow{\bar a_0 \wedge 1_Y} & Y \\
 & \downarrow \bar b_2 \wedge 1_Y & & \downarrow \bar b_1 \wedge 1_Y & & \downarrow \bar b_0 \wedge 1_Y \\
 & \Sigma^{-2}KG_2 \wedge Y & & \Sigma^{-1}KG_1 \wedge Y & & KG_0 \wedge Y
\end{array}$$

is an Adams resolution of arbitrary finite spectrum Y.

Proof of Theorem 3.1. By Lemma 3.1(1) and the following exact sequence induced by (1.1)

$$0=\mathrm{Ext}_A^{s+1,t+q+2}(H^*M,Z_p)\xrightarrow{j^*}\mathrm{Ext}_A^{s+1,t+q+1}(H^*M,H^*M)\xrightarrow{i^*}\mathrm{Ext}_A^{s+1,t+q+1}(H^*M,Z_p)\xrightarrow{p^*}$$

we know that $\mathrm{Ext}_A^{s+1,t+q+1}(H^*M,H^*M)$ has a unique generator $\widetilde{h_0x}$ so that $i^*(\widetilde{h_0x})=\overline{h_0x}\in\mathrm{Ext}_A^{s+1,t+q+1}(H^*M,Z_p)$, where the left group is zero by supposition (2) and $p^*(\overline{h_ox})=p^*(\alpha i)_*(x)=(\alpha i)_*p^*(x)=\alpha_*i_*p_*(x)=0$.

Recall that

$$\begin{array}{ccccccc}\cdots\xrightarrow{\bar a_2\wedge 1_M} & \Sigma^{-2}E_2\wedge M & \xrightarrow{\bar a_1\wedge 1_M} & \Sigma^{-1}E_1\wedge M & \xrightarrow{\bar a_0\wedge 1_M} & M\\ & \downarrow \bar b_2\wedge 1_M & & \downarrow \bar b_1\wedge 1_M & & \downarrow \bar b_0\wedge 1_M\\ & \Sigma^{-2}KG_2\wedge M & & \Sigma^{-1}KG_1\wedge M & & KG_0\wedge M\end{array}$$

is an Adams resolution of M, then there is a d_1-cycle $\overline{h_0x}\in\pi_{t+q+1}KG_{s+1}\wedge M$ which represents $\overline{h_0x}\in\mathrm{Ext}_A^{s+1,t+q+1}(H^*M,Z_p)$. Since $KG_{s+1}\wedge M$ is an M-module spectrum, there is a d_1-cycle $\widetilde{h_0x}\in[\Sigma^{t+q+1}M,KG_{s+1}\wedge M]$ which represents $\widetilde{h_0x}\in\mathrm{Ext}_A^{s+1,t+q+1}(H^*M,H^*M)$ and satisfies $\widetilde{h_0x}\cdot i=\overline{h_0x}$. Moreover,

$$d(\widetilde{h_0x})\in[\Sigma^{t+q+2}M,KG_{s+1}\wedge M]=0$$

by the following exact sequence induced by (1.1)

$$0=\pi_{t+q+3}(KG_{s+1}\wedge M)\xrightarrow{j^*}[\Sigma^{t+q+2}M,KG_{s+1}\wedge M]\xrightarrow{i^*}\pi_{t+q+2}(KG_{s+1}\wedge M)=0,$$

where both side of groups are zero since $\pi_{t+q+r}KG_{s+1}\cong\mathrm{Ext}_A^{s+1,t+q+r}(Z_p,Z_p)=0$ for $r=1,2,3$ by supposition (2).

Now we first prove that $(\bar c_{s+1}\wedge 1_{K'\wedge M})(1_{KG_{s+1}}\wedge\psi)(\overline{h_0x})=0\in[\Sigma^{t+q+2}S,\ E_{s+2}\wedge K'\wedge M]$, where $\psi:\Sigma M\to K'\wedge M$ is the map in (3.6). That is to say, we will prove that $\psi_*(\overline{h_0x})\in\mathrm{Ext}_A^{s+1,t+q+2}(H^*K'\wedge M,Z_p)$ is a permanent cycle in the ASS.

Recall that $v:M\to K'$ is the injection map in (3.2), then $(1_{KG_{s+1}}\wedge v)(\widetilde{h_0x})\cdot i\in\pi_{t+q+1}KG_{s+1}\wedge K'$ is a d_1-cycle which represents an element in $\mathrm{Ext}_A^{s+1,t+q+1}(H^*K',Z_p)$. However, this group is zero by Lemma (3.1)(2), i.e. $(1_{KG_{s+1}}\wedge v)(\widetilde{h_0x})\cdot i\in\pi_{t+q+1}KG_{s+1}\wedge K'$ is a d_1-boundary, so $(1_{KG_{s+1}}\wedge v)(\widetilde{h_0x})\cdot i=(\bar b_{s+1}\bar c_s\wedge 1_{K'})g'$ for some $g'\in\pi_{t+q+1}KG_s\wedge K'$ and we have

$$(\bar c_{s+1}\wedge 1_{K'})(1_{KG_{s+1}}\wedge v)(\widetilde{h_0x})\cdot ij=0. \tag{3.7}$$

Recall from (3.4) that there is a factorization $v=(1_{K'}\wedge j)\psi:\Sigma M\xrightarrow{\psi}K'\wedge M\xrightarrow{1_{K'}\wedge j}\Sigma K'$, then from (3.7) and (3.5) we have

$$(\bar c_{s+1}\wedge 1_{K'\wedge M})(1_{KG_{s+1}}\wedge\psi)(\widetilde{h_0x})ij=(1_{E_{s+2}}\wedge 1_{K'}\wedge i)f$$

for some $f\in[\Sigma^{t+q+1}M,E_{s+2}\wedge K']$. Hence $(\bar a_{s+1}\wedge 1_{K'\wedge M})(1_{E_{s+2}}\wedge 1_{K'}\wedge i)f=0$ and so by (3.5) we have

$$\begin{aligned}(\bar a_{s+1}\wedge 1_{K'})f&=(1_{E_{s+1}}\wedge 1_{K'}\wedge p)f_2\quad\text{for some}\quad f_2\in[\Sigma^{t+q}M,E_{s+1}\wedge K']\\&=f_2(1_M\wedge p)=0.\end{aligned}$$

Thus $f=(\bar c_{s+1}\wedge 1_{K'})g$ for some $g\in[\Sigma^{t+q+1}M,KG_{s+1}\wedge K']$ and we have

$$(\bar c_{s+1}\wedge 1_{K'\wedge M})(1_{KG_{s+1}}\wedge\psi)(\widetilde{h_0x})ij=(\bar c_{s+1}\wedge 1_{K'\wedge M})(1_{KG_{s+1}}\wedge 1_{K'}\wedge i)g. \tag{3.8}$$

It follows that $(\bar b_{s+2}\bar c_{s+1}\wedge 1_{K'\wedge M})(1_{KG_{s+1}}\wedge 1_{K'}\wedge i)g=0$, and $(\bar b_{s+2}\bar c_{s+1}\wedge 1_{K'})g=(1_{KG_{s+2}}\wedge 1_{K'}\wedge p)g_2=0$ (with $g_2\in[\Sigma^{t+q+1}M,KG_{s+2}\wedge K']$) since $1_{KG_{s+2}}\wedge 1_{K'}\wedge p$

$= 0$. That is, $g \in [\Sigma^{t+q+1}M, KG_{s+1} \wedge K']$ is a d_1-cycle which represents an element in $\mathrm{Ext}_A^{s+1,t+q+1}(H^*K', H^*M)$. We claim that this group has a unique generator $v_*(\widetilde{h_0x})$, this can be proved as follows.

Consider the following exact sequence induced by (3.2)

$$\begin{aligned}\mathrm{Ext}_A^{s,t}(Z_p, H^*M) &\xrightarrow{(\alpha i)_*} \mathrm{Ext}_A^{s+1,t+q+1}(H^*M, H^*M)\\ &\xrightarrow{v_*} \mathrm{Ext}_A^{s+1,t+q+1}(H^*K', H^*M) \xrightarrow{y_*} \mathrm{Ext}_A^{s+1,t}(Z_p, H^*M),\end{aligned}$$

where $\mathrm{Ext}_A^{s+1,t+q+1}(H^*M, H^*M)$ has a unique generator $\widetilde{h_0x}$. So, it suffices to prove $\mathrm{Ext}_A^{s,t}$ $(Z_p, H^*M) = 0 = \mathrm{Ext}_A^{s+1,t}(Z_p, H^*M)$. This follows from the following exact sequences

$$\xrightarrow{p^*} \mathrm{Ext}_A^{s+1,t+1}(Z_p, Z_p) \xrightarrow{j^*} \mathrm{Ext}_A^{s+1,t}(Z_p, H^*M) \xrightarrow{i^*} \mathrm{Ext}_A^{s+1,t}(Z_p, Z_p),$$

$$\mathrm{Ext}_A^{s,t+1}(Z_p, Z_p) \xrightarrow{j^*} \mathrm{Ext}_A^{s,t}(Z_p, H^*M) \xrightarrow{i^*} \mathrm{Ext}_A^{s,t}(Z_p, Z_p) \xrightarrow{p^*}$$

induced by (1.1), where the upper right group is zero by supposition (3) and the upper left group has a unique generator $a_0x = p^*(x)$ so that $\mathrm{im}\, j^* = 0$, the lower left group is zero or has a unique generator $a_0x' = p^*(x')$ by the supposition (3) so that $\mathrm{im}\, j^* = 0$, the lower right group has unique generator x satisfying $p^*x = a_0x \neq 0 \in \mathrm{Ext}_A^{s+1,t+1}(Z_p, Z_p)$ so that $\mathrm{im}\, i^* = 0$. This proves the claim.

Then $g = \lambda \cdot (1_{KG_{s+1}} \wedge v)(\widetilde{h_0x})$ modulo a d_1- boundary with $\lambda \in Z_p$ and (3.8) becomes

$$\begin{aligned}&(\bar{c}_{s+1} \wedge 1_{K' \wedge M})(1_{KG_{s+1}} \wedge \psi)(\widetilde{h_0x})ij\\ &= (\bar{c}_{s+1} \wedge 1_{K' \wedge M})(1_{KG_{s+1}} \wedge 1_{K'} \wedge i)(1_{KG_{s+1}} \wedge v)(\widetilde{h_0x}),\end{aligned} \tag{3.9}$$

where we omitted the scalar $\lambda \in Z_p$ which is inessential in the argument below.

Now we will use some technique on derivations of maps between M-module spectra. The spectra $E_{s+2} \wedge K' \wedge M$, $KG_{s+1} \wedge K' \wedge M$ and $KG_{s+1} \wedge M$ are M-module spectra with M-module structure determined by the right M , then the derivations $d(\bar{c}_{s+1} \wedge 1_{K' \wedge M}) = \bar{c}_{s+1} \wedge d(1_{K' \wedge M}) = 0$, $d(1_{KG_{s+1}} \wedge \psi) = 1_{KG_{s+1}} \wedge d(\psi) = 0$ (see Theorem 2.1(ii) and Proposition 3.2). So, by Theorem 2.1(i), the derivation of the left-hand side of (3.9) equals to $-(\bar{c}_{s+1} \wedge 1_{K' \wedge M})(1_{KG_{s+1}} \wedge \psi)(\widetilde{h_0x})$ since $d(\widetilde{h_0x}) = 0$ and $d(ij) = -1_M$.

Moreover, we consider the derivation of the right-hand side of (3.9). Note that KG_{s+1} is an M-module spectrum and we write the M-module action as $m_G : M \wedge KG_{s+1} \to KG_{s+1}$, $\overline{m}_G : \Sigma KG_{s+1} \to M \wedge KG_{s+1}$. Then $KG_{s+1} \wedge K'$ also is an M- module spectrum with M-module action

$$\begin{aligned} m_G \wedge 1_{K'} : M \wedge KG_{s+1} \wedge K' &\longrightarrow KG_{s+1} \wedge K',\\ \overline{m}_G \wedge 1_{K'} : \Sigma KG_{s+1} \wedge K' &\longrightarrow M \wedge KG_{s+1} \wedge K'.\end{aligned}$$

By applying d to (3.9) we have

$$\begin{aligned}&(\bar{c}_{s+1} \wedge 1_{K' \wedge M})(1_{KG_{s+1}} \wedge \psi)(\widetilde{h_0x})\\ &= (\bar{c}_{s+1} \wedge 1_{K' \wedge M})d(1_{KG_{s+1}} \wedge 1_{K'} \wedge i) \cdot (1_{KG_{s+1}} \wedge v)(\widetilde{h_0x})\\ &\quad + (\bar{c}_{s+1} \wedge 1_{K' \wedge M})(1_{KG_{s+1}} \wedge 1_{K'} \wedge i)d(1_{KG_{s+1}} \wedge v) \cdot (\widetilde{h_0x}).\end{aligned} \tag{3.10}$$

Since $\widetilde{h_0x} \in [\Sigma^{t+q+1}M, KG_{s+1} \wedge M]$ is a d_1-cycle, i.e. $(\bar{b}_{s+2}\bar{c}_{s+1} \wedge 1_M)(\widetilde{h_0x}) = 0$, then $d(1_{KG_{s+1}} \wedge v) \cdot (\widetilde{h_0x}) \in [\Sigma^{t+q+2}M, KG_{s+1} \wedge K']$ also is a d_1-cycle. To check this, we need to prove the commutativity

$$(\bar{b}_{s+2}\bar{c}_{s+1}\wedge 1_{K'})\cdot d(1_{KG_{s+1}}\wedge v)=d(1_{KG_{s+2}}\wedge v)\cdot(\bar{b}_{s+2}\bar{c}_{s+1}\wedge 1_M).$$

Note that

$$d(1_{KG_{s+1}}\wedge v)=(m_G\wedge 1_{K'})(1_M\wedge 1_{KG_{s+1}}\wedge v)(T\wedge 1_M)(1_{KG_{s+1}}\wedge \overline{m}),$$

where $\overline{m}:\Sigma M\to M\wedge M$ is the M-module action of M. Then it suffices to prove the following diagram commutes (up to homotopy)

$$\begin{array}{ccc} M\wedge KG_{s+1} & \xrightarrow{m_G} & KG_{s+1} \\ \Big\downarrow 1_M\wedge \bar{b}_{s+2}\bar{c}_{s+1} & & \Big\downarrow \bar{b}_{s+2}\bar{c}_{s+1} \\ M\wedge KG_{s+2} & \xrightarrow{m_G} & KG_{s+2} \end{array}$$

Consider the induced homomorphism in Z_p-cohomology. Since there is a homotopy equivalence $M\wedge KG_{s+1}=KG_{s+1}\vee\Sigma KG_{s+1}$ and $m_G:M\wedge KG_{s+1}\to KG_{s+1}$ is the projection, $m_G^*:H^*(KG_{s+1})\to H^*(M\wedge KG_{s+1})=H^*(M)\otimes H^*KG_{s+1}$ is the injection, i.e. $m_G^*(c)=\tau\otimes c\in H^*M\otimes H^*KG_{s+1}$ for $\tau\in H^0M$ and any $c\in H^*KG_{s+1}, s\geq 0$. Hence, for any $a\in H^*KG_{s+2}$,

$$\begin{aligned} &m_G^*(\bar{b}_{s+2}\bar{c}_{s+1})^*(a)=\tau\otimes(\bar{b}_{s+2}\bar{c}_{s+1})^*(a)\\ &=(1_M\wedge\bar{b}_{s+2}\bar{c}_{s+1})^*(\tau\otimes a)=(1_M\wedge\bar{b}_{s+2}\bar{c}_{s+1})^*m_G^*(a).\end{aligned}$$

This proves the above commutativity and so $d(1_{KG_{s+1}}\wedge v)(\widetilde{h_0x})\in[\Sigma^{t+q+2}M,KG_{s+1}\wedge K']$ is a d_1-cycle which represents an element in $Ext_A^{s+1,t+q+2}(H^*K',H^*M)$. However, this group is zero, this follows from the following exact sequence induced by (3.2)

$$\mathrm{Ext}_A^{s+1,t+q+2}(H^*M,H^*M)\xrightarrow{v_*}\mathrm{Ext}_A^{s+1,t+q+2}(H^*K',H^*M)\xrightarrow{y_*}\mathrm{Ext}_A^{s+1,t+1}(Z_p,H^*M),$$

where the left group is zero by $\mathrm{Ext}_A^{s+1,t+q+r}(Z_p,Z_p)=0$ for $r=1,2,3$ and the right group is also zero by $\mathrm{Ext}_A^{s+1,t+2}(Z_p,Z_p)=0$ or has a unique generator a_0^2x' (see the supposition (3)) and $\mathrm{Ext}_A^{s+1,t+1}(Z_p,Z_p)$ has a unique generator a_0x satisfying $a_0^2x\neq 0$. Hence, $d(1_{KG_{s+1}}\wedge v)\cdot(\widetilde{h_0x})$ is a d_1-boundary and so (3.10) becomes

$$\begin{aligned} &(\bar{c}_{s+1}\wedge 1_{K'\wedge M})(1_{KG_{s+1}}\wedge\psi)(\widetilde{h_0x})\\ &=(\bar{c}_{s+1}\wedge 1_{K'\wedge M})d(1_{KG_{s+1}}\wedge 1_{K'}\wedge i)\cdot(1_{KG_{s+1}}\wedge v)(\widetilde{h_0x}). \end{aligned}\tag{3.11}$$

Recall from (3.7) that we have $(1_{KG_{s+1}}\wedge v)(\widetilde{h_0x})ij=(\bar{b}_{s+1}\bar{c}_s\wedge 1_{K'})g'\cdot j$. Moreover, by the same reason as stated above, we have the commutativity

$$d(1_{KG_{s+1}}\wedge 1_{K'}\wedge i)\cdot(\bar{b}_{s+1}\bar{c}_s\wedge 1_{K'})=(\bar{b}_{s+1}\bar{c}_s\wedge 1_{K'\wedge M})\cdot d(1_{KG_s}\wedge 1_{K'}\wedge i).$$

(Note : Here, we need only to check the commutativity $\overline{m}_G(\bar{b}_{s+1}\bar{c}_s)=(1_M\wedge\bar{b}_{s+1}\bar{c}_s)\overline{m}_G$ and this can be checked by the induced homomorphism in Z_p-cohomology.)

Then, from (3.11) we have

$$\begin{aligned} &(\bar{c}_{s+1}\wedge 1_{K'\wedge M})(1_{KG_{s+1}}\wedge\psi)(\widetilde{h_0x})ij\\ &=(\bar{c}_{s+1}\wedge 1_{K'\wedge M})d(1_{KG_{s+1}}\wedge 1_{K'}\wedge i)\cdot(1_{KG_{s+1}}\wedge v)(\widetilde{h_0x})ij=0,\end{aligned}$$

and so

$$\begin{aligned} (\bar{c}_{s+1}\wedge 1_{K'\wedge M})(1_{KG_{s+1}}\wedge\psi)(\widetilde{h_0x})\cdot i=f_3\cdot p\quad &\text{for some}\quad f_3\in\pi_{t+q+2}E_{s+2}\wedge K'\wedge M\\ &=(1_{E_{s+2}}\wedge 1_{K'\wedge M}\wedge p)f_3=0. \end{aligned}\tag{3.12}$$

This shows that $\psi_*(\overline{h_0x}) \in \mathrm{Ext}_A^{s+1,t+q+2}(H^*K' \wedge M, Z_p)$ is a permanent cycle in the ASS.

Let L be the cofibre of $\alpha_1 = j\alpha i : \Sigma^{q-1}S \to S$ given by the cofibration

$$\Sigma^{q-1}S \xrightarrow{\alpha_1} S \xrightarrow{i''} L \xrightarrow{j''} \Sigma^q S \tag{3.13}$$

and consider the following commutative diagram of 3×3 lemma in stable homotopy category

$$\begin{array}{ccccccc} M & \longrightarrow & L\wedge K & \xrightarrow{j''\wedge 1_K} & \Sigma^q K \\ \searrow i' & \nearrow i''\wedge 1_K & \searrow r & \nearrow \rho & \\ & K & & \Sigma^q W & \\ \nearrow \alpha_1\wedge 1_K & \searrow j' & \nearrow \psi & \searrow \epsilon & \\ \Sigma^{q-1}K & \longrightarrow & \Sigma^{q+1}M & \xrightarrow{\alpha} & \Sigma M \end{array} \tag{3.14}$$

Note that $j'(\alpha_1 \wedge 1_K) = \alpha ijj'$, then $\Sigma^q W = \Sigma^q K' \wedge M$ which is the cofibre of $\alpha ijj' : \Sigma^{q-1}K \to \Sigma^{q+1}M$ (see (3.6)). Then we have a cofibration

$$M \xrightarrow{(i''\wedge 1_K)i'} L\wedge K \xrightarrow{r} \Sigma^q K' \wedge M \xrightarrow{\epsilon} \Sigma M. \tag{3.15}$$

Since $(i''\wedge 1_K)i' : M \to L\wedge K$ induces an epimorphism, then $\epsilon : \Sigma^q K'\wedge M \to \Sigma M$ induces zero homomorphism in Z_p-cohomology. Also from (3.14) we have $\epsilon\psi = \alpha : \Sigma^q M \xrightarrow{\psi} \Sigma^{q-1}K'\wedge M \xrightarrow{\epsilon} M$.

Now it follows from (3.12) that there is an $f' \in \pi_{t+q+2}E_{s+1}\wedge K'\wedge M$ such that $(\bar{b}_{s+1}\wedge 1_{K'\wedge M})f' = (1_{KG_{s+1}}\wedge\psi)(\overline{h_0x})$. Then

$$(\bar{b}_{s+1}\wedge 1_M)(1_{E_{s+1}}\wedge\epsilon)f' = (1_{KG_{s+1}}\wedge\epsilon\psi)(\overline{h_0x}) = 0$$

since $(1_{KG_{s+1}}\wedge\epsilon) = 0$. Thus $(1_{E_{s+1}}\wedge\epsilon)f' = (\bar{a}_{s+1}\wedge 1_M)f''$ for some $f'' \in \pi_{t+2q+2}E_{s+2}\wedge M$ and

$$\bar{b}_{s+2}(1_{E_{s+2}}\wedge j)f'' = \alpha_2 x \in \pi_{t+2q+1}KG_{s+2} \cong \mathrm{Ext}_A^{s+2,t+2q+1}(Z_p, Z_p).$$

This is because $(\bar{a}_0\bar{a}_1\cdots\bar{a}_s\wedge 1_{K'\wedge M})f' \in \pi_*K'\wedge M$ is represented by

$$\psi_*(\overline{h_0x}) \in \mathrm{Ext}_A^{s+1,t+q+2}(H^*K'\wedge M, Z_p)$$

in the ASS. Then

$$\bar{a}_0\bar{a}_1\cdots\bar{a}_{s+1}(1_{E_{s+2}}\wedge j)f'' = (\bar{a}_0\bar{a}_1\cdots\bar{a}_s)(1_{E_{s+1}}\wedge j\epsilon)f' \in \pi_*S$$

must be represented by

$$j_*\epsilon_*\psi_*(\overline{h_0x}) = j_*\alpha_*(\overline{h_0x}) = j_*\alpha_*\alpha_*i_*(x) = \alpha_2 x \neq 0 \in \mathrm{Ext}_A^{s+2,t+2q+1}(Z_p, Z_p)$$

in the ASS. Here we use the fact that the following composition

$$\mathrm{Ext}_A^{s,t}(Z_p,Z_p) \xrightarrow{i_*} \mathrm{Ext}_A^{s,t}(H^*M,Z_p) \xrightarrow{\alpha_*} \mathrm{Ext}_A^{s+1,t+q+1}(H^*M,Z_p)$$
$$\xrightarrow{\alpha_*} \mathrm{Ext}_A^{s+2,t+2q+2}(H^*M,Z_p) \xrightarrow{j_*} \mathrm{Ext}_A^{s+2,t+2q+1}(Z_p,Z_p)$$

is a multiplication by $\alpha_2 \in \mathrm{Ext}_A^{2,2q+1}(Z_p,Z_p)$. (Note : this fact follows from that $j\alpha^2 i \in \pi_{2q-1}S$ is represented by $\alpha_2 \in \mathrm{Ext}_A^{2,2q+1}(Z_p,Z_p)$ in the ASS). Hence

$$\alpha_2 x \in \mathrm{Ext}_A^{s+2,t+2q+1}(Z_p,Z_p)$$

is a permanent cycle in the ASS and the proof of Theorem 3.1 finishes.

Proof of Theorem A. Consider $x = g_n, k_n (n\geq 2), b_n (n\geq 1)$ and $h_nh_m (n\geq 2, m\geq n+2)$ in $\mathrm{Ext}_A^{s,*}(Z_p,Z_p)$ with $s=2, t = p^{n+1}q+2p^nq, 2p^{n+1}q+p^nq, p^{n+1}q$ and p^nq+p^mq respectively. We need to check that these elements satisfy the conditions of Theorem 3.1.

From [1], the Z_p-base of $\mathrm{Ext}_A^{3,*}(Z_p, Z_p)$ has been completely determined. From [1, p.110, Table 8.1] we know that

(1) $a_0x \neq 0, h_0x \neq 0$ is the unique generator of $\mathrm{Ext}_A^{3,t+1}(Z_p, Z_p), \mathrm{Ext}_A^{3,t+q}(Z_p, Z_p)$ respectively. (Note : the name of a_0, h_0 in [1, Table 8.1] are h_{-1}, h_0. The names of g_n, k_n, b_n in [1, Table 8.1] are $h_{n,2,1}, h_{n,1,2}, \overline{\lambda}_n$ respectively. Moreover, in these degrees, there are no other generators).

(2) $\mathrm{Ext}_A^{3,t+q+r}(Z_p, Z_p) = 0$ for $r = 1, 2, 3$.

(3) $\mathrm{Ext}_A^{3,t}(Z_p, Z_p) = 0$, $\mathrm{Ext}_A^{3,t+2}(Z_p, Z_p) = 0$ or has a unique generator $a_0^2 h_{n+1}$ when $t = p^{n+1}q$. $\mathrm{Ext}_A^{2,t+1}(Z_p, Z_p) = 0$ or has a unique generator $a_0 h_{n+1}$ when $t = p^{n+1}q$, where $t = p^{n+1}q + 2p^n q, 2p^{n+1}q + p^n q (n \geq 2), p^{n+1}q (n \geq 1)$ or $p^n q + p^m q (n \geq 2, m \geq n+2)$.

From [10], the Z_p-base of $\mathrm{Ext}_A^{4,*}(Z_p, Z_p)$ has been completely determined. From the table listed in [10, Theorem 4.1], we know that

(1) $\alpha_2 x \neq 0 \in \mathrm{Ext}_A^{4,t+2q+1}(Z_p, Z_p)$, where $\alpha_2 g_n, \alpha_2 k_n, \alpha_2 b_n, \alpha_2 h_n h_m$ are corresponding to the generators in [10, Theorem 4.1] of number (31), (32), (23), (22) respectively.

(2) $a_0^2 x \neq 0 \in \mathrm{Ext}_A^{4,t+1}(Z_p, Z_p)$, where $a_0^2 g_n, a_0^2 k_n, a_0^2 b_n, a_0^2 h_n h_m$ are corresponding to the generators in [10, Theorem 4.1] of number (47), (48), (10), (16) respectively.

Then, all the conditions of Theorem 3.1 are satisfied for $x = g_n, k_n (n \geq 2), b_n (n \geq 1), h_n h_m (n \geq 2, m \geq n+2)$ in $\mathrm{Ext}_A^{2,*}(Z_p, Z_p)$ and so we conclude that $\alpha_2 g_n, \alpha_2 k_n (n \geq 2), \alpha_2 b_n (n \geq 1), \alpha_2 h_n h_m (n \geq 2, m \geq n+2)$ in $\mathrm{Ext}_A^{4,*}(Z_p, Z_p)$ are permanent cycles in the ASS.

From [3], $\mathrm{Ext}_A^{2,t+2q}(Z_p, Z_p) = 0$ for $t = p^{n+1}q + 2p^n q, 2p^{n+1}q + p^n q, p^{n+1}q$ and $p^n q + p^m q$, and then $\alpha_2 g_n, \alpha_2 k_n (n \geq 2), \alpha_2 b_n (n \geq 1), \alpha_2 h_n h_m (n \geq 2, m \geq n+2) \in \mathrm{Ext}_A^{4,*}(Z_p, Z_p)$ cannot be hit by differentials in the ASS. This completes the proof of Theorem A.

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