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EXTENSIONS OF HARDY MODULES OVER THE POLYDISK ALGEBRA***

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Abstract

In this paper, the Ext groups of Hardy modules over the polydisk algebra $A(D^n)$ are calculated. Of particular importance is that the calculating of Ext-groups is closely related to harmonic analysis of polydisks. Finally, the authors point out that Ext-groups reveal rigidity of Hardy submodules over $A(D^n)$ for $n > 1$.

Keywords Ext-groups, Hardy module, Polydisk algebra, Hilbert module, Rigidity
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§1. Introduction

A Hilbert module is a Hilbert space H which is also a module over a function algebra A, i.e., there is an associative bilinear multiplication $A \times H \to H$ which is continuous in both variables. The first systematic study of Hilbert modules appeared in the monograph of Douglas and Poulsen[2]. This coordinate free approach to multivariable operator theory has some remarkable consequences. In [1], Carlson and Clark introduced one of the central concepts from homological algebra, Ext-functor, into the discussion of Hilbert modules. Basically, they considered the following problem of classifying extensions in the category $\mathcal{H}(A)$ of all Hilbert modules over A. Suppose that H and K are in $\mathcal{H}(A)$. Let $S(K,H)$ be the set of all short exact sequences $E: \ 0 \longrightarrow H \xrightarrow{\alpha} J \xrightarrow{\beta} K \longrightarrow 0$, where α, β are Hilbert-module maps. We call two elements E, E' to be equivalent if there exists a Hilbert module map θ such that the diagram

$$\begin{array}{ccccccccc}
E: & 0 & \longrightarrow & H & \xrightarrow{\alpha} & J & \xrightarrow{\beta} & K & \longrightarrow & 0 \\
 & & & \| & & \downarrow{\scriptstyle\theta} & & \| & & \\
E': & 0 & \longrightarrow & H & \xrightarrow{\alpha'} & J' & \xrightarrow{\beta'} & K & \longrightarrow & 0
\end{array}$$

commutes. The set of equivalence classes of $S(K,H)$ under this relation is defined to be the extension group, $\mathrm{Ext}(K,H)$. In fact, $\mathrm{Ext}(-,-)$ is a bifunctor from the category $\mathcal{H}(A)$ to

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the category of abelian groups, and contravariant in the first and covariant in the second variables (see [1]). Since the categories of Hilbert modules lack enough projective and injective objects, it is impossible to define the functor Ext as the derived functor of Hom as in [3]. However, because of the special structure of Hilbert modules, Carlson and Clark[1] showed that $\mathrm{Ext}(-,-)$ has a natural form:

Theorem 1.1.[1] $\mathrm{Ext}(K,H) \cong \mathcal{U}/\mathcal{B}$, *where* $\mathcal{U}$ *is the set of all continuous* (*in both variables*) *bilinear maps* $\sigma : A \times K \to H$ *such that* $a\sigma(b,k) + \sigma(a,bk) = \sigma(ab,k)$, *where* $a, b \in A$ *and* $k \in K$, *and* $\mathcal{B}$ *is the set of all* $\sigma \in \mathcal{U}$ *having the form* $\sigma(a,k) = aTk - Tak$, *where* $T : K \to H$ *is a bounded linear operator.*

With the aid of Theorem 1.1, Carlson and Clark studied the extensions of Hilbert modules over the disk algebra $A(D)$. Their methods seem to be valid only in the case of the disk algebra. In this paper, we will calculate Ext-groups of Hardy modules over the polydisk algebra $A(D^n)$. Of particular interest is that the calculation of Ext-groups is closely related to harmonic analysis of polydisks. Finally, we point out that Ext-groups reveal rigidity of Hardy submodules over $A(D^n)$ for $n > 1$.

At the end of this section, we give the following concept, which is basic for our analysis. Let G be a semigroup. An invariant mean of G is a state μ on $l^\infty(G)$ such that $\mu(F) = \mu(gF)$, where $gF(g') = F(gg')$ for all g in G and F in $l^\infty(G)$. A basic fact is that every Abelian semigroup has an invariant mean[4].

§2. Ext for Hardy Modules over the Polydisk Algebra $A(D^n)$

Let $A(D^n)$ be the polydisk algebra, and $H^2(D^n)$ be the usual Hardy module over $A(D^n)$. By P to denote the orthogonal projection from $L^2(T^n)$ onto $H^2(D^n)$, for $\varphi \in L^2(T^n)$, a Hankel operator H_φ: $H^2(D^n) \to H^2(D^n)^\perp$ is defined by $H_\varphi f = (I-P)(\varphi f)$ with the domain $H^\infty(D^n)$, and a Toeplitz operator T_φ: $H^2(D^n) \to H^2(D^n)$ is defined by $T_\varphi f = P(\varphi f)$ with the domain $H^\infty(D^n)$, i.e., H_φ, T_φ are densely defined operators in $H^2(D^n)$. It is well known that in the case $n = 1$, a Hankel operator H_φ ($\varphi \in L^2(T)$) is bounded if and only if

(1) there is a function $\varphi' \in L^\infty(T)$ such that $H_{\varphi'} = H_\varphi$ on the domain of H_φ and $\| H_\varphi \| = \| \varphi' \|_\infty$, if and only if

(2) there is a function $\varphi_0 \in \mathrm{BMO} \bigcap H^2(D)^\perp$ and some $h \in H^2(D)$ such that $\varphi = \varphi_0 + h$, and $\| H_\varphi \| = \| \varphi_0 \|_{\mathrm{BMO}}$, if and only if

(3) there is a function $\varphi_1 \in L^\infty(T)$ such that $(I-P)\varphi = (I-P)\varphi_1$.

However, in the case $n > 1$, the corresponding statement to (1) does not hold. For (2) and (3) , a new function space is need, and is called the coordinate BMO , denoted by CBMO. Let $f \in L^2(T^n)$, a system of functions $(\varphi_1, \varphi_2, \cdots \varphi_n)$ satisfying for each $m = (m_1, m_2, \cdots m_n) \in Z^n$, $\hat{f}(m) = \hat{\varphi_1}(m)$ if $m_1 < 0$, $\cdots$, $\hat{f}(m) = \hat{\varphi_n}(m)$ if $m_n < 0$ is called a coordinate for f, where"ˆ" denotes the Fourier transform. We say that f belongs to CBMO, if f has a L^∞-coordinate $(\varphi_1, \varphi_2, \cdots \varphi_n)$, i.e., every argument φ_i of the coordinate $(\varphi_1, \varphi_2, \cdots \varphi_n)$ of f belongs to $L^\infty(T^n)$ ($i = 1, 2, \cdots n$). For $f \in \mathrm{CBMO}$, we define

$$\|f\|_{\mathrm{CBMO}} = \inf\{\max(\|\varphi_1\|_\infty, \|\varphi_2\|_\infty, \cdots, \|\varphi_n\|_\infty) \mid \\ \text{all } L^\infty - \text{coordinate } (\varphi_1, \varphi_2, \cdots \varphi_n) \ of \ f\}.$$

Writing CBMO_d for all such f , f having a diagonal L^∞-coordinate, that is , there is a L^∞-coordinate $(\varphi_1, \varphi_2, \cdots \varphi_n)$ of f such that $\varphi_1 = \varphi_2 = \cdots = \varphi_n$.

Lemma 2.1.[5] *Let $\varphi \in L^2(T^n)$. A Hankel operator H_φ in Hardy space $H^2(D^n)$ is bounded if and only if $\varphi \in \mathrm{CBMO}$ and $\|\varphi\|_{\mathrm{CBMO}} \le \|H_\varphi\| \le \sqrt{n}\|\varphi\|_{\mathrm{CBMO}}$. In particular, a Hankel operator H_φ is equal to some H_f (where $f \in L^\infty(T^n)$) if and only if $\varphi \in \mathrm{CBMO}_d$.*

Lemma 2.2. *Let A be a densely defined operator in $H^2(D^n)$ with its domain $A(D^n)$ and $A: (A(D^n), ||\cdot||_\infty) \to H^2(D^n)$ be continuous, also*

$$T_{\bar{z_1}^{m_1}\bar{z_2}^{m_2}\cdots\bar{z_n}^{m_n}} A T_{z_1^{m_1} z_2^{m_2}\cdots z_n^{m_n}} = A, \qquad (m_1, m_2, \cdots, m_n) \in Z_+^n \ .$$

Then there exists a function $\varphi \in L^2(T^n)$ such that $A = T_\varphi$.

Proof. For simplicity, the proof is sketched for $n = 2$, while conclusion holds for all $n \ge 1$. Put $\mathcal{A} = \{\ \bar{z_1}^{m_1}\bar{z_2}^{m_2}h \mid (m_1, m_2) \in Z_+^2, h \in H^2(D^2)\}$. Clearly, $\mathcal{A}$ is a dense subspace of $L^2(T^2)$. We define a map Φ from $\mathcal{A}$ to all complex numbers C by

$$\Phi(\bar{z_1}^{m_1}\bar{z_2}^{m_2}h) = \langle h, A(z_1^{m_1}z_2^{m_2})\rangle.$$

Then Φ is well-defined and linear. In fact, if $\bar{z_1}^{m_1}\bar{z_2}^{m_2}h = \bar{z_1}^{m_1'}\bar{z_2}^{m_2'}h'$, we notice that

$$\Phi(\bar{z_1}^{m_1}\bar{z_2}^{m_2}h) = \langle h, A(z_1^{m_1}z_2^{m_2})\rangle = \langle h, T_{\bar{z_1}^{m_1'}\bar{z_2}^{m_2'}} A T_{z_1^{m_1'} z_2^{m_2'}}(z_1^{m_1}z_2^{m_2})\rangle$$

$$= \langle z_1^{m_1'}z_2^{m_2'}h, A(z_1^{m_1'+m_1}z_2^{m_2'+m_2})\rangle = \langle z_1^{m_1}z_2^{m_2}h', AT_{z_1^{m_1}z_2^{m_2}}(z_1^{m_1'}z_2^{m_2'})\rangle$$

$$= \langle h', T_{\bar{z_1}^{m_1}\bar{z_2}^{m_2}}AT_{z_1^{m_1}z_2^{m_2}}(z_1^{m_1'}z_2^{m_2'})\rangle = \langle h', A(z_1^{m_1'}z_2^{m_2'})\rangle = \Phi(\bar{z_1}^{m_1'}\bar{z_2}^{m_2'}h').$$

Φ is thus well-defined. It can be obtained as the above proof that Φ is linear. According to the definition of Φ , we see that

$$|\Phi(\bar{z_1}^{m_1}\bar{z_2}^{m_2}h)| \le ||\,A||_\infty||\ \bar{z_1}^{m_1}\bar{z_2}^{m_2}h\,||, \qquad (m_1, m_2) \in Z_+^2, \quad h \in H^2(D^2),$$

where $||\,A||_\infty$ denotes the norm of the map $A: (A(D^2), ||\cdot||_\infty) \to H^2(D^2)$. The Riesz representation theorem leads to the conclusion that there exists a function $\varphi \in L^2(T^2)$ such that

$$\Phi(\bar{z_1}^{m_1}\bar{z_2}^{m_2}h) = \langle h, A(z_1^{m_1}z_2^{m_2})\rangle = \langle \bar{z_1}^{m_1}\bar{z_2}^{m_2}h, \varphi\rangle\,.$$

Therefore $A(z_1^{m_1}z_2^{m_2}) = T_\varphi(z_1^{m_1}z_2^{m_2})$, $(m_1, m_2) \in Z_+^2$. Since all polynomials are dense in $A(D^2)$, the above equations imply that $A = T_\varphi$ on their domain $A(D^2)$. This completes the proof of Lemma 2.2.

Now we return to calculate $\mathrm{Ext}(H^2(D^n), H^2(D^n))$. According to Theorem 1.1, we must determine continuous bilinear map σ: $A(D^n) \times H^2(D^n) \to H^2(D^n)$ with the form

$$\sigma(f_1f_2, h) = f_1\sigma(f_2, h) + \sigma(f_1, f_2h), \quad f_1,\ f_2 \in A(D^n),\ h \in H^2(D^n)\,. \tag{2.1}$$

If we define $\delta: A(D^n) \to B(H^2(D^n))$ by $\delta(f) = \sigma(f, \cdot)$, $f \in A(D^n)$, then δ is a bounded linear map and satisfies

$$\delta(f_1f_2) = T_{f_1}\delta(f_2) + \delta(f_1)T_{f_2}, \quad f_1, f_2 \in A(D^n). \tag{2.2}$$

A bounded linear map $\delta: A(D^n) \to B(H^2(D^n))$ is called a derivation, if it satisfies (2.2), where $B(H^2(D^n))$ denotes the set of all linear bounded operators on $H^2(D^n)$. We say that a bounded derivation δ is inner, if there is a bounded linear operator T on $H^2(D^n)$ such that $\delta(f) = T_fT - TT_f$, $f \in A(D^n)$. Thus, a map σ with the form (2.1) determines a bounded derivation δ from $A(D^n)$ to $B(H^2(D^n))$. In another words, a bounded derivation

δ from $A(D^n)$ to $B(H^2(D^n))$ corresponds a map σ: $A(D^n)\times H^2(D^n)\to H^2(D^n)$ defined by $\sigma(f,h)=\delta(f)h,\ f\in A(D^n),\ h\in H^2(D^n)$. It is to see that σ is clearly a continuous bilinear map and satisfies (2.1). In particular, δ is inner if and only if the corresponding σ belongs to $\mathcal{B}$ in Theorem 1.1 in §1.

Lemma 2.3. *Let* $\varphi\in$ CBMO. *Define* $\delta_\varphi : A(D^n)\to B(H^2(D^n))$ *by* $\delta_\varphi(f)=H^*_{\bar f}H_\varphi$, $f\in A(D^n)$. *Then* δ_φ *is a bounded derivation. In particular,* δ_φ *is inner if and only if* $\varphi\in\mathrm{CBMO}_d$.

Proof. It is easy to check that δ_φ satisfies (2.2). δ_φ is thus a bounded derivation. If $\varphi\in\mathrm{CBMO}_d$, Lemma 2.1 implies that δ_φ is inner. In another words, if δ_φ is inner, that is, there exists a bounded linear operator T such that $\delta_\varphi(f)=T_fT-TT_f,\ f\in A(D^n)$, then the definition of δ_φ immediately leads to that the following is true for any $f\in A(D^n)$, $T_\varphi T_f-T_fT_\varphi=T_fT-TT_f$, where two sides of the above equation are defined on $A(D^n)$. That is, we have $(T+T_\varphi)T_f=T_f(T+T_\varphi),\ \forall f\in A(D^n)$. Set

$$g=(T+T_\varphi)1\in H^2(D^n).$$

Then for any $f\in A(D^n)$, $Tf=T_{g-\varphi}f$. We claim that $g-\varphi$ belongs to $L^\infty(T^n)$. In fact, if we use k_z to denote the normalized Hardy reproducing kernel at $z\in D^n$, then

$$k_z(\theta)=\prod_{j=1}^n\frac{(1-|z_j|^2)^{\frac12}}{1-e^{i\theta_j}\bar z_j},\qquad \theta=(\theta_1,\theta_2,\cdots,\theta_n)\in T^n .$$

Thus we get $|k_z(\theta)|^2=P(z,\theta)$ by [6, p.17], where $P(z,\theta)$ is the Poisson kernel at $z\in D^n$. Since $k_z\in A(D^n)$, we see that for every $z\in D^n$,

$$\begin{aligned}|\langle Tk_z,k_z\rangle|=|\langle T_{g-\varphi}k_z,k_z\rangle|&=\frac{1}{(2\pi)^n}\Big|\int_{T^n}(g-\varphi)|k_z|^2d\theta\Big|\\&=\frac{1}{(2\pi)^n}\Big|\int_{T^n}(g-\varphi)(\theta)P(z,\theta)d\theta\Big|\le\|T\|.\end{aligned}$$

Furthermore, for almost every $\theta\in T^n$, the following is true by [6, Theorem 2.3.1]

$$(g-\varphi)(\theta)=\lim_{r\to1}\frac{1}{(2\pi)^n}\int_{T^n}(g-\varphi)(\theta')P(r\theta,\theta')d\theta'.$$

The function $g-\varphi$ is thus in $L^\infty(T^n)$. That is, φ belongs to BMO_d. The proof of Lemma 2.3 is completed.

According to Lemma 2.3, we can establish an injective homomorphism of groups

$$\tau:\ \mathrm{CBMO}/\mathrm{CBMO}_d\to\mathrm{Ext}(H^2(D^n),H^2(D^n))$$

by $\tau([\varphi])=[\,\delta_\varphi]$, where $[\,\delta_\varphi]$ denotes the coset $\delta_\varphi+\mathcal{D}$, $\mathcal{D}$ is the space of all inner derivations.

Theorem 2.1. *The group* $\mathrm{Ext}(H^2(D^n),H^2(D^n))$ *is isomorphic to* $\mathrm{CBMO}/\mathrm{CBMO}_d$, *and the correspondence is given by* $\tau([\varphi])=[\delta_\varphi],\quad \varphi\in$ CBMO.

Proof. For simplicity, the proof is sketched for $n=2$, while conclusion holds for all $n\geqslant1$. According to the preceding statement, we only need to prove that τ is surjective. Let δ be a bounded derivation from $A(D^2)$ to $B(H^2(D^2))$. For any C in the trace class $B_1(H^2(D^2))$, a function F_C on Z^2_+ is defined by

$$F_C(m,n)=\langle T_{\bar z^m\bar w^n}\delta(z^mw^n),C\rangle=\mathrm{tr}(T_{\bar z^m\bar w^n}\delta(z^mw^n)C),\quad (m,n)\in Z^2_+.$$

Then

$$\|F_c\|\le\|\delta\|\,\|C\|_{\mathrm{tr}}. \tag{2.3}$$

Let μ be an invariant mean of Z_+^2. Defining a bounded functional $\Im$ on the trace class $B_1(H^2(D^2))$ by $\Im(C) = \mu(F_C)$, $C \in B_1(H^2(D^2))$, we see that there exists a bounded linear operator T such that $\Im(C) = \langle T, C\rangle = \mathrm{tr}(TC)$, $\quad C \in B_1(H^2(D^2))$. Therefore, for every $(m,n) \in Z_+^2$ and $C \in B_1(H^2(D^2))$, we have

$$\begin{aligned}&\langle T_{z^m w^n}T - TT_{z^m w^n}, C\rangle = \langle T, CT_{z^m w^n} - T_{z^m w^n}C\rangle\\ &= \Im(CT_{z^m w^n} - T_{z^m w^n}C) = \mu(F_{CT_{z^m w^n} - T_{z^m w^n}C}).\end{aligned}$$

Since, for $(k,l) \in Z_+^2$,

$$\begin{aligned}&F_{CT_{z^m w^n} - T_{z^m w^n}C}(k,l) = \langle T_{\bar z^k \bar w^l}\delta(z^k w^l), CT_{z^m w^n} - T_{z^m w^n}C\rangle\\ &= \langle T_{z^m w^n}T_{\bar z^k \bar w^l}\delta(z^k w^l), C\rangle - \langle T_{\bar z^k \bar w^l}\delta(z^k w^l)T_{z^m w^n}, C\rangle\\ &= \langle T_{z^m w^n}T_{\bar z^k \bar w^l}\delta(z^k w^l), C\rangle - \langle T_{\bar z^k \bar w^l}\delta(z^{k+m} w^{l+n}), C\rangle + \langle \delta(z^m w^n), C\rangle\\ &= \langle T_{z^m w^n}T_{\bar z^k \bar w^l}\delta(z^k w^l), C\rangle - \langle P_{(m,n)}T_{\bar z^k \bar w^l}\delta(z^{k+m} w^{l+n}), C\rangle\\ &\quad - \langle T_{z^m w^n}T_{\bar z^{k+m} \bar w^{l+n}}\delta(z^{k+m} w^{l+n}), C\rangle + \langle \delta(z^m w^n), C\rangle,\end{aligned}$$

where $P_{(m,n)}$ is the orthogonal projection from $H^2(D^2)$ onto $(z^m w^n H(D^2))^\perp$. For each $(m,n) \in Z_+^2$, and $C \in B_1(H^2(D^2))$, defining a function $F_C^{(m,n)}$ on Z_+^2 by

$$F_C^{(m,n)}(k,l) = \langle P_{(m,n)}T_{\bar z^k \bar w^l}\delta(z^{k+m} w^{l+n}), C\rangle, \quad (k,l) \in Z_+^2,$$

we have

$$\langle T_{z^m w^n}T - TT_{z^m w^n}, C\rangle = \mu(F_{CT_{z^m w^n} - T_{z^m w^n}C}) = \langle \delta(z^m w^n), C\rangle - \mu(F_C^{(m,n)}).$$

Denoting $T_f T - TT_f$ by $\delta_T(f), f \in A(D^2)$, we have

$$\langle (\delta - \delta_T)(z^m w^n), C\rangle = \mu(F_C^{(m,n)}), \quad (m,n) \in Z_+^2, \quad C \in B_1(H^2(D^2)).$$

Clearly, by the definition of $F_C^{(m,n)}$, the following

$$\langle (\delta - \delta_T)(z^m w^n), CT_{z^m w^n}T_{\bar z^m \bar w^n}\rangle = 0, \quad (m,n) \in Z_+^2, \quad C \in B_1(H^2(D^2))$$

holds. Thus, for all $(m,n) \in Z_+^2$, we have

$$T_{\bar z^m \bar w^n}(\delta - \delta_T)(z^m w^n) = 0. \tag{2.4}$$

Writing δ' for $\delta - \delta_T$, an operator A in $H^2(D^2)$ with its domain $A(D^2)$ is defined by $Af = \delta'(f)1$, $f \in A(D^2)$, that is, A is a densely defined operator in $H^2(D^2)$. For every $f \in A(D^2)$, it is easy to check that the following is true , where two sides of (2.5) are seen to be defined on $A(D^2)$.

$$\delta'(f) = AT_f - T_f A. \tag{2.5}$$

By (2.4) and (2.5), we see that on the domain $A(D^2)$ of A , the following holds

$$T_{\bar z^m \bar w^n}AT_{z^m w^n} = A, \qquad (m,n) \in Z_+^2.$$

Using Lemma 2.2, we see that there exists a function $\varphi \in L^2(T^2)$ such that $A = T_\varphi$. By (2.5), this implies that for any $f \in A(D^2)$, the following equation holds, where two sides of equation are seen to be defined on $A(D^2)$,

$$\delta'(f) = T_\varphi T_f - T_f T_\varphi = -H_{\bar f}^* H_\varphi.$$

What we shall do next is to prove that H_φ can be continuously extended onto $H^2(D^2)$. Let

$h \in A(D^2)$. Since

$$\|\delta'(z^m w^n)h\|^2 = \|P(z^m w^n H_\varphi h)\|^2$$
$$= \sum_{(k,l)\in Z_+^2} |\langle P(z^m w^n H_\varphi h), z^k w^l\rangle|^2 = \sum_{(k,l)\in Z_+^2} |\langle H_\varphi h, z^{k-m} w^{l-n}\rangle|^2,$$

$$\|H_\varphi h\| = \lim_{\substack{m\to\infty \\ n\to\infty}} \|\delta'(z^m w^n)h\| \le \|\delta'\|\|h\|.$$

This implies that H_φ can be continuously extended onto $H^2(D^2)$. Therefore φ belongs to CBMO by Lemma 2.1. That is, we have shown that $\delta = \delta_T + \delta' = \delta_T + \delta_{-\varphi}$. The homomorphism τ is thus surjective. This finishs the proof of Theorem 2.1.

Remark 2.1. In the case $n = 1$, $\mathrm{Ext}(H^2(D), H^2(D)) = 0$. This was first proved by Carlson and Clark[1]. However, the techniques from [1] fail to work for higher dimensions. In the case $n > 1$, $\mathrm{Ext}(H^2(D^n), H^2(D^n)) \neq 0$.

Our next goal is to calculate $\mathrm{Ext}(H^2(D^n)/M, H^2(D^n))$ over the polydisk algebra $A(D^n)$, where M is a non-zero $A(D^n)-$submodule of $H^2(D^n)$. We may identify the quotient module $H^2(D^n)/M$ with $H^2(D^n)\ominus M$. The action of $A(D^n)$ on $H^2(D^n)\ominus M$ is given by the formula $f \circ h = P_{H^2(D^n)\ominus M}(fh)$, where $P_{H^2(D^n)\ominus M}$ is the orthogonal projection from $H^2(D^n)$ onto $H^2(D^n) \ominus M$.

According to Theorem 1.1 in §1, we must also determine the following continuous bilinear map $\sigma : A(D^n) \times (H^2(D^n) \ominus M) \to H^2(D^n)$ with the property

$$\sigma(f_1 f_2, h) = T_{f_1}\sigma(f_2, h) + \sigma(f_1, f_2 \circ h). \tag{2.6}$$

For all $f \in A(D^n)$ we define $\delta : A(D^n) \to B(H^2(D^n) \ominus M, H^2(D^n))$ by $\delta(f) = \sigma(f, \cdot)$. Then δ is a bounded linear map and satisfies

$$\delta(f_1 f_2) = T_{f_1}\delta(f_2) + \delta(f_1)P_{H^2(D^n)\ominus M}T_{f_2}, \quad f_1, f_2 \in A(D^n). \tag{2.7}$$

In another words, a bounded linear map δ from $A(D^n)$ to $B(H^2(D^n) \ominus M, H^2(D^n))$ with the property (2.7) determines a map $\sigma : A(D^n) \times (H^2(D^n) \ominus M) \to H^2(D^n)$ with the property (2.6) by $\sigma(f, h) = \delta(f)h$, $f \in A(D^n)$, $h \in H^2(D^n) \ominus M$. Let $\varphi \in$ CBMO, then $\ker H_\varphi$ is an $A(D^n)-$ submodule of $H^2(D^n)$. Denoting the set $\{\varphi \in \mathrm{CBMO} : \ker H_\varphi \supseteq M\}$ by $\mathcal{K}(M)$, $\mathcal{K}(M)$ is clearly an $A(D^n)-$module. For $\varphi \in \mathcal{K}(M)$, we define $\delta_\varphi : A(D^n) \to B(H^2(D^n) \ominus M, H^2(D^n))$ by $\delta_\varphi(f) = H^*_{\bar{f}} H_\varphi$. Then the following conclusions hold.

(I) δ_φ is a bounded linear map from $A(D^n)$ to $B(H^2(D^n) \ominus M, H^2(D^n))$, and satisfies (2.7).

(II) If there exists an operator $A \in B(H^2(D^n) \ominus M, H^2(D^n))$ such that $\delta_\varphi(f) = T_f A - AP_{H^2(D^n)\ominus M}T_f$ for all $f \in A(D^n)$, then $\delta_\varphi = 0$, i.e, $\varphi \in H^2(D^n)$.

In fact, for (I), it is immediate from the definition of δ_φ.

For (II), it is easy to check that $\varphi \in \mathrm{BCMO}_d$. Lemma 2.1 implies that there exists a function $\varphi' \in L^\infty(T^n)$ such that $H_\varphi = H_{\varphi'}$. It follows that

$$T_f T_{\varphi'} - T_{\varphi'} T_f = T_f A P_{H^2(D^n)\ominus M} - AP_{H^2(D^n)\ominus M}T_f, \ \ f \in A(D^n).$$

Then we see that there exists a function $h \in H^\infty(D^n)$ such that $T_{\varphi'} - AP_{H^2(D^n)\ominus M} = T_h$. Because φ' belongs to $\mathcal{K}(M)$ and $M \neq \{0\}$, the above equation implies that $\varphi' = h$. That is, $H_\varphi(= H_{\varphi'})$ is equal to zero. This leads to $\delta_\varphi = 0$.

From the previous discussion, we can establish an injective homomorphism of groups $\tau : \mathcal{K}(M)/H^2(D^n) \to \mathrm{Ext}(H^2(D^n)/M, H^2(D^n))$ by $\tau([\varphi]) = [\delta_\varphi]$ for $\varphi \in \mathcal{K}(M)$, where $[\delta_\varphi]$ denotes the coset $\delta_\varphi + \mathcal{D}$, $\mathcal{D}$ is the space of all such δ, for which there exists $A \in B(H^2(D^n) \ominus M, H^2(D^n))$ such that $\delta(f) = T_f A - AP_{H^2(D^n)\ominus M} T_f$, $f \in A(D^n)$. In fact, τ is also surjective, and its proof is similar to that of Theorem 2.1. Thus we get the following

Theorem 2.2. *Let M be a nonzero $A(D^n)-$ submodule of $H^2(D^n)$. Then the group $\mathrm{Ext}(H^2(D^n)/M, H^2(D^n))$ is isomorphic to $\mathcal{K}(M)/H^2(D^n)$. The correspondence is given by $\tau([\varphi]) = [\delta_\varphi]$, where $\varphi \in \mathcal{K}(M)$.*

Corollary 2.1. *Let M be of finite codimension, and $n > 1$. Then*

$$\mathrm{Ext}(H^2(D^n)/M, H^2(D^n)) = 0.$$

Proof. For $\varphi \in \mathcal{K}(M)$, since

$$H_\varphi(H^2(D^n)) = H_\varphi(M \oplus (H^2(D^n) \ominus M)) = H_\varphi(H^2(D^n) \ominus M),$$

we see that H_φ is of finite rank. What we shall next do is to prove that $\varphi \in H^2(D^n)$. In fact , for each $\theta \in T^n$, an operator U_θ on $L^2(T^n)$ is defined by

$$U_\theta : L^2(T^n) \to L^2(T^n) \ , (U_\theta f)(\theta') = f(\theta' - \theta), \quad \theta' \in T^n, \quad f \in L^2(T^n) \ .$$

Clearly $U_\theta^* H_\varphi U_\theta = H_{\varphi_\theta}$, $\theta \in T^n$, where φ_θ is the function $\varphi(\theta + \cdot)$. If $\varphi \notin H^2(D^n)$, then there exists some $m = (m_1 \cdots m_n) \notin Z_+^n$ such that $\hat{\varphi}(m) \neq 0$. Since H_{φ_θ} is weakly measurable on T^n, we can define a bounded linear operator $H^{(m)}$ by

$$H^{(m)} = \frac{1}{(2\pi)^n} \int_{T^n} e^{-im\theta} H_{\varphi_\theta} d\theta,$$

where the integral is taken in the sense that for $f \in H^2(D^n)$ and $g \in H^2(D^n)^\perp$,

$$\Big\langle \Big(\int_{T^n} e^{-im\theta} H_{\varphi_\theta} d\theta \Big) f, g \Big\rangle = \int_{T^n} \langle e^{-im\theta} H_{\varphi_\theta} f, g \rangle d\theta.$$

A simple computation shows that $H^{(m)} = \hat{\varphi}(m) H_{\varphi_m}$, where $\varphi_m(\theta') = e^{im\theta'}$, $\theta' \in T^n$. Since for every $\theta \in T^n$, H_{φ_θ} is compact and uniformly bounded, we get that H_{φ_m} is compact by [7, Lemma 12]. This is impossible. Thus φ belongs to $H^2(D^n)$. Immediately from Theorem 2.2, we get $\mathrm{Ext}(H^2(D^n)/M, H^2(D^n)) = 0$.

Remark 2.2 (1) Corollary 2.1 is true only for $n > 1$. The inverse of Corollary 2.1 does not hold in general. For example, taking $M = \overline{(z - w)H^2(D^2)}$, we can prove that $\mathrm{Ext}(H^2(D^2)/M, H^2(D^2)) = 0$, but M is of infinite codimension.

(2) In [1], Carlson and Clark calculated $\mathrm{Ext}(H^2(D)/M, H^2(D))$. It is easy to see that this is a corollary of Theorem 2.2 by Beurling's theorem. However, the techniques from [1] fail to work for $n > 1$.

§3. Application to Rigidity of Hardy Submodules over $A(D^n)$ ($n>1$)

In this section, we shall point out that the calculation of Ext-groups for Hardy modules over $A(D^n)$ ($n > 1$) can reveal the rigidity of Hardy submodules. Firstly, we give the following

Theorem 3.1. *For $n > 1$, let M_1 be of finite codimension in $H^2(D^n)$ and $M_1 \subseteq M_2 \subseteq H^2(D^n)$. Then* $\mathrm{Ext}(H^2(D^n) \ominus M_1, M_2) \cong H^2(D^n) \ominus M_2$.

Proof. For the exact sequence

$$E:\ 0 \longrightarrow M_2 \xrightarrow{i} H^2(D^n) \xrightarrow{\pi} H^2(D^n)/M_2 \longrightarrow 0,$$

where i is the inclusion map and π the quotient map. We use [1, Proposition 2.1.5] and Corollary 2.1. This gives the following exact sequence

$$\begin{aligned} &0 \longrightarrow \mathrm{Hom}(H^2(D^n)/M_1, M_2) \xrightarrow{i_*} \mathrm{Hom}(H^2(D^n)/M_1, H^2(D^n)) \\ &\xrightarrow{\pi_*} \mathrm{Hom}(H^2(D^n)/M_1, H^2(D^n)/M_2) \xrightarrow{\delta} \mathrm{Ext}(H^2(D^n)/M_1, M_2) \longrightarrow 0, \end{aligned}$$

where δ is the connecting homomorphism. Let θ be an $A(D^n)$-Hilbert module map from $H^2(D^n)/M_1$ to $H^2(D^n)$, i.e., $\theta \in \mathrm{Hom}(H^2(D^n)/M_1, H^2(D^n))$. It is easy to check that θ can be extended into a Hilbert module map $\tilde{\theta}$ from $H^2(D^n)$ to $H^2(D^n)$ by setting $\tilde{\theta}(h_1 + h_2) = \theta(h_1)$, $h_1 \in H^2(D^n) \ominus M_1, h_2 \in M_1$. This implies that there exists an $f \in H^\infty(D^n)$ such that $\tilde{\theta} = M_f$. Since $\tilde{\theta}(M_1) = 0$, we see that $f = 0$. Therefore $\mathrm{Hom}(H^2(D^n)/M_1, H^2(D^n)) = 0$. It follows that the above connecting homomorphism δ gives an isomorphism from $\mathrm{Hom}(H^2(D^n)/M_1, H^2(D^n)/M_2)$ to $\mathrm{Ext}(H^2(D^n)/M_1, M_2)$. Using methods in [8], we see that $H^2(D^n) \ominus M_1$ and $H^2(D^n) \ominus M_2$ are contained in $H^\infty(D^n)$. For $\theta \in \mathrm{Hom}(H^2(D^n)/M_1, H^2(D^n)/M_2)$, it is easy to see that θ can be extended into a Hilbert module map $\tilde{\theta}$ from $H^2(D^n)$ to $H^2(D^n)/M_2$ by setting $\tilde{\theta}(h_1 + h_2) = \theta(h_1)$, $h_1 \in H^2(D^n) \ominus M_1, h_2 \in M_1$. Notice that θ is completely determined by $\tilde{\theta}(1)$ and, $\tilde{\theta}(1) \in H^2(D^n) \ominus M_2\,(\subset H^\infty(D^n))$. In another words, for $\varphi \in H^2(D^n) \ominus M_2\,(\subset H^\infty(D^n)$, since $M_1 \subseteq M_2$, it follows that $\varphi M_1 \subseteq M_2$. Thus φ induces a Hilbert module map from $H^2(D^n) \ominus M_1$ to $H^2(D^n) \ominus M_2$. This completes the proof of Theorem 3.1.

From Theorem 3.1, one finds that for $n > 1$, a finite codimensional Hardy submodule $M(\neq H^2(D^n))$ is never similar to $H^2(D^n)$. The reason is that $\mathrm{Ext}(H^2(D^n) \ominus M, H^2(D^n)) = 0$, but $\mathrm{Ext}(H^2(D^n) \ominus M, M) \cong H^2(D^n) \ominus M$. Of course, this observation is not new, and we may compare it with that of [9].

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