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BOUNDEDNESS OF WEYL QUANTIZATION**

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Abstract

The authors consider the boundedness of Weyl quantization in an as broad as possible frame and take the results already obtained as just particular cases. The notions of almost-diagonalizable operators and boundedness modulo a regularizing operator are proposed.

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§1. Introduction

Quantization is a fundamental process in quantum physics. Among various quantizations, the one formulated by H. Weyl associates to a certain function $a(x,\xi)$ (the symbol) an operator

$$Au(x) = a^w(x,D)u = (2\pi)^{-n} \iint e^{i(x-y)\xi} a\Big(\frac{x+y}{2},\xi\Big)u(y)dyd\xi. \tag{1.1}$$

Since quantized physical quantities are represented as operators in the state space H, which is a Hilbert space, hence canonically $L^2(R_n)$, a basic problem is to consider its boundedness in H, i.e., its $L^2(R_n)$-boundedness. There has been quite a lot references on L^2-boundedness of pseudo-differential operators (PsDO for short)

$$a(x,D)u = (2\pi)^{-n} \iint e^{i(x-y)\xi} a(x,\xi)u(y)dyd\xi \tag{1.2}$$

(or the natural quantization of $a(x,\xi)$), and the problem is related to the symbol class and its differentiability. See [1] of Wang and Li and [2] of Hwang for details and a rather complete literature. N. Lerner[3] uses Wigner function

$$H(u,v)(x,\xi) = \int u\Big(x+\frac{y}{2}\Big)\overline{v}\Big(x-\frac{y}{2}\Big)e^{-2\pi iy\xi}dy \tag{1.3}$$

to give an elementary proof of classical results, which are optimal in the sense that counter-examples can be found when violating the assumptions in the results.

In this paper, we consider the boundedness of Weyl quantization in an as broad as possible frame and take the results already obtained as just particular cases. In section 2, we consider L^2-boundedness of Weyl quantization, here we use modified Wigner function as is done in [3]

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and obtain a generalization of Hwang's and Lerner's result which is very close to be optimal but we fail to find counter-examples as in the case of natural quantization. In section 3, we prove some general propositions on L_p^Ω spaces, which are the foundation for the scales of function spaces considered in section 4 where we consider two basic scales $B_{p,q}^s$ and $F_{p,q}^s$ (notation of H. Triebel[4] is used) which contain many significant spaces, e.g., Besov spaces. Thus our results are applicable to many different spaces. In the last section 5, we propose the notions of almost-diagonalizable operators and boundedness modulo a regularizing operator.

§2. L^2-Boundedness of Weyl Quantization

In Weyl quantization (1.1), $a(\frac{x+y}{2},\xi)$ is just an amplitude for a PsDO. In what follows, we go a little further to prove the L^2-boundedness of a PsDO

$$Au(x)=(2\pi)^{-n}\iint e^{i(x-y)\xi}a(x,y,\xi)u(y)dyd\xi. \tag{2.1}$$

First, we write (2.1) weakly as

$$\begin{aligned}(Au,v)&=(2\pi)^{-n}\iiint e^{i(x-y)\xi}a(x,y,\xi)u(y)\overline{v}(x)dydxd\xi\\&=(2\pi)^{-3n}\iiiint\!\!\int e^{i[(x-y)\xi+y\zeta-x\eta]}a(x,y,\xi)\widehat{u}(\zeta)\overline{\widehat{v}(\eta)}dydxd\xi d\eta d\zeta.\end{aligned} \tag{2.2}$$

For the moment, we assume u and v to be in $\mathcal{S}(R_n)$ and then extend it to the case $u,v\in L^2(R_n)$. Our main result in this section is

Theorem 2.1. *Assume that $a(x,y,\xi)\in S^0_{0,0,0}(R_{3n})$, the operator A in (2.1) can be extended to a bounded linear operator $L^2\to L^2$:*

$$|(Au,v)|\le C\|u\|_{L^2}\|v\|_{L^2}, \tag{2.3}$$

where C depends on $\partial_x^\alpha\partial_y^\beta\partial_\xi^\gamma a(x,y,\xi)$ with orders up to $|\alpha|,|\beta|\le 2([\frac{n}{4}]+1), |\gamma|\le 2([\frac{n}{2}]+1)$.

Proof. Let $u,v\in L^2(R_n)$. (2.2) is at present only a formal expression, since integrability in ξ is not guaranteed and a certain regularization is required. Since this is just a standard practice, we would omit the details.

It is readily seen that

$$\begin{aligned}&[(1+|x-y|^2)^{-1}(I-\Delta_\xi)]^\nu[(1+|\xi-\eta|^2)^{-1}(I-\Delta_x)]^\mu\\&\cdot[(1+|\zeta-\xi|^2)^{-1}(I-\Delta_y)]^\mu e^{i((x-y)\xi+y\zeta-x\eta)}=e^{i((x-y)\xi+y\zeta-x\eta)}.\end{aligned}$$

Substituting into (2.2) and integrating by parts gives

$$(Au,v)=\sum I_{j,l,s}, \tag{2.4}$$

where

$$\begin{aligned}I_{j,l,s}=&\iiiint\!\!\int e^{i((x-y)\xi+y\zeta-x\eta)}\partial_x^{\alpha_j}\partial_y^{\beta_l}\partial_\xi^{\gamma_s}a(x,y,\xi)\\&\cdot P_j(\xi-\eta)Q_l(\zeta-\xi)R_s(x-y)\widehat{u}(\zeta)\overline{\widehat{v}(\eta)}dydxd\xi d\eta d\zeta.\end{aligned} \tag{2.5}$$

Summation in (2.4) is extended to $|\alpha_j|\le 2\mu, |\beta_l|\le 2\mu, |\gamma_s|\le 2\nu$, and it is easy to see that

$$\begin{cases}|P_j(t)|\le \dfrac{C}{(1+|t|)^{2\mu}},\\|Q_l(t)|\le \dfrac{C}{(1+|t|)^{2\mu}},\\|R_s(t)|\le \dfrac{C}{(1+|t|)^{2\nu}},\end{cases} \tag{2.6}$$

hence integration by parts are legitimate. Then, we need only to consider integrals of the form

$$I = \iiiint\!\!\int e^{i((x-y)\xi+y\zeta-x\eta)}\widehat{u}(\zeta)\overline{\widehat{v}(\eta)}b(x,y,\xi)\cdot P(\xi-\eta)Q(\zeta-\xi)R(x-y)dydxd\xi d\eta d\zeta, \tag{2.7}$$

where $b(x,y,\xi)$ is bounded, P,Q and R satisfy (2.6). Denote

$$A(y,\xi) = \int \widehat{u}(\zeta)Q(\zeta-\xi)e^{iy\zeta}d\zeta, \qquad B(x,\xi) = \int \overline{\widehat{v}(\eta)}P(\xi-\eta)e^{-ix\eta}d\eta.$$

It is easy to see, A and $B \in L^2(R_{2n})$ when $4\mu > n$ and

$$\|A\|_{L^2(R_{2n})} = \|u\|_{L^2(R_n)}\|Q\|_{L^2(R_n)}, \quad \|B\|_{L^2(R_{2n})} = \|v\|_{L^2(R_n)}\|P\|_{L^2(R_n)}. \tag{2.8}$$

For I, we have for $4\mu > n, 2\nu > n$,

$$\begin{aligned} |I| &\le C\iint |R(x-y)|dxdy \int |A(y,\xi)B(x,\xi)|d\xi \\ &\le C\|R\|_{L^1(R_n)}\|P\|_{L^2(R_n)}\|Q\|_{L^2(R_n)}\|u\|_{L^2(R_n)}\|v\|_{L^2(R_n)} \\ &\le C\|u\|_{L^2(R_n)}\|v\|_{L^2(R_n)}. \end{aligned} \tag{2.9}$$

Hence, finally we have

$$|(Au,v)| \le C\|u\|_{L^2(R_n)}\|v\|_{L^2(R_n)}. \tag{2.10}$$

From (2.10) we have that A can be extended to a linear operator $L^2 \to L^2$ with norm $\|A\| \le C$, where C depends on $\partial_x^\alpha\partial_y^\beta\partial_\xi^\gamma a(x,y,\xi)$ with orders up to $|\alpha|, |\beta| \le 2\mu, |\gamma| \le 2\nu$ and $\mu = [\frac{n}{4}]+1, \nu = [\frac{n}{2}]+1$.

§3. L_p^Ω and Fundamental Propositions

Now consider our problem in Besov spaces and other related spaces. Their definitions are based on that of L_p^Ω.

Definition 3.1. *L_p^Ω is the set of $f \in \mathcal{S}'(R_n)$ which belongs itself to L^p with spectrum concentrated in a compact set $\Omega \subset R_n$:*

$$L_p^\Omega = \{f;\quad f \in \mathcal{S}'(R_n)\bigcap L^p(R_n), \mathrm{supp}\hat{f}(\xi) \subset \Omega \subset\subset R_n\}. \tag{3.1}$$

Proposition 3.1. *When $a(x,\xi) \in S^0_{\rho,\delta}, 0 \le \rho,\delta \le 1$ and*

$$a(x,\xi) = 0 \qquad \text{for } |\xi| \ge kR, \tag{3.2}$$

the Weyl quantization $A = a^w(x,D)$ of (1.1) is a bounded linear operator $L^p \to L^p (1 \le p \le \infty)$ with norm

$$\|A\| \le CR^n. \tag{3.3}$$

C is independent of R but depends on $\partial_\xi^\alpha a(x,\xi)$ up to order $|\alpha| \le 2\lambda$ with $\lambda = [\frac{n}{2}]+1$.

Proof. Using integration by parts and the Young's or Schwarz's inequality, one can write the proof easily.

Now consider the case when the symbol vanishes in a ball centered at $\xi = 0$. We have

Proposition 3.2. *Let $a(x,\xi) \in S^0_{\rho,\delta}, 0 \le \rho,\delta \le 1, \delta < 1$ and*

$$a(x,\xi) = 0 \qquad \text{for } |\xi| \le hR. \tag{3.4}$$

Then $A = a^w(x,D) : L_p^\Omega \to L^p$ *is bounded and*

$$\|A\| \le CR^{n+2\mu\delta}. \tag{3.5}$$

Here $1 \le p \le \infty$ *and* $\Omega = \{\xi; |\xi| \le R\}, C$ *is independent of* R *but depends on* $\partial_x^\beta \partial_\xi^\alpha a(x,\xi)$ *with* $|\alpha| \le 2\lambda, \lambda = [\frac{n}{2}] + 1; |\beta| \le 2\mu, \mu = [\frac{n}{2(1-\delta)}] + 1.$

Proof. It is well-known that $\mathcal{S}(R_n) \bigcap L_p^\Omega$ is dense in L_p^Ω (see [4, p.24]), where Ω is required to have a 'segment property' which is true when Ω is a ball. Hence we may assume for the moment that $u \in \mathcal{S}(R_n) \bigcap L_p^\Omega$ and a standard density argument would lead to the final result. Integrating by parts in y gives

$$Au(x) = (2\pi)^{-n} \iint e^{i(x-y)\xi} (-|\xi|^2)^{-\mu} (\Delta_y)^\mu \Big(a\Big(\frac{x+y}{2}, \xi\Big) u(y)\Big) dy d\xi.$$

This is legitimate since $u(y) \in \mathcal{S}(R_n)$ causing all terms outside the integral in y vanishing and the fact that $a(x,\xi) = 0$ for $|\xi| \le hR$ makes the factor $|\xi|^{-2\mu}$ harmless.

Now, we can rewrite $Au(x)$ as

$$Au(x) = (2\pi)^{-n} \iint e^{i(x-y)\xi} |\xi|^{-2\mu} \sum_{|\beta_1+\beta_2|=2\mu} C_{\beta_1,\beta_2} D_y^{\beta_1} a\Big(\frac{x+y}{2}, \xi\Big) D_y^{\beta_2} u(y) dy d\xi.$$

Since $\delta < 1$, when $|\beta| > 0$ we may apply to each term of $Au(x)$ the fact

$$(I - \Delta_\xi)^\lambda e^{i(x-y)\xi} = (1 + |x-y|^2)^\lambda e^{i(x-y)\xi}$$

and integrate by parts in ξ to reduce each term of $Au(x)$ to the form

$$\begin{aligned} J_{\beta_1}(x) &= (2\pi)^{-n} C_{\beta_1,\beta_2} \iint (I - \Delta_\xi)^\lambda \Big\{ |\xi|^{-2\mu} D_y^{\beta_1} a\Big(\frac{x+y}{2}, \xi\Big)\Big\} D_y^{\beta_2} u(y) \\ &\quad (1 + |x-y|^2)^{-\lambda} e^{i(x-y)\xi} dy d\xi \\ &= (2\pi)^{-n} \iint e^{i(x-y)\xi} a_{\lambda\beta_1}\Big(\frac{x+y}{2}, \xi\Big) (1 + |x-y|^2)^{-\lambda} D_y^{\beta_2} u(y) dy d\xi. \end{aligned}$$

Since $a(x,\xi) \in S^0_{\rho,\delta}$, we have

$$\Big| a_{\lambda\beta_1}\Big(\frac{x+y}{2}, \xi\Big)\Big| \le C|\xi|^{-2\mu(1-\delta)},$$

where C depends on $\partial_x^\beta \partial_\xi^\alpha a(x,\xi)$ with orders as high as $|\alpha| \le 2\lambda, \lambda = [\frac{n}{2}] + 1; |\beta| \le 2\mu, \mu = [\frac{n}{2(1-\delta)}] + 1$. Estimating the integral in ξ we will obtain

$$|J_{\beta_1}| \le CR^{n-2\mu(1-\delta)} \int |D_y^{\beta_2} u(y)| (1 + |x-y|^2)^{-\lambda} dy. \tag{3.6}$$

Integrate $|J_{\beta_1}|^p$ in x, by Young's inequality

$$\|J_{\beta_1}\|_{L^p} \le CR^{n-2\mu(1-\delta)} \|D_y^{\beta_2} u(y)\|_{L^p}.$$

Using the Bernstein's inequality(see [4, p.17]) and $u \in L_p^\Omega$, we obtain

$$\|Au\|_{L^p} \le CR^{n+2\mu\delta} \|u\|_{L^p}.$$

Combining Proposition 3.1 and Proposition 3.2, we obtain the following

Theorem 3.1. *Let* $a(x,\xi) \in S^0_{\rho,\delta}, 0 \le \rho, \delta \le 1, \delta < 1$. *Then the Weyl quantization* $A = a^w(x,D)$ *is a bounded linear operator* $L_p^\Omega \to L^p$ *with norm*

$$\|A\| \le CR^{n+2\mu\delta}, \tag{3.7}$$

where Ω *is the ball* $\Omega = \{\xi; |\xi| \le R\}, C$ *depends on* $\partial_x^\beta \partial_\xi^\alpha a(x,\xi)$ *with orders* $|\alpha| \le 2\lambda, \lambda = [\frac{n}{2}] + 1; |\beta| \le 2\mu, \mu = [\frac{n}{2(1-\delta)}] + 1$ *and is independent of* R.

Proof. Construct a function $\varphi(\xi) \in C_0^\infty(R_n)$ such that $0 \le \varphi(\xi) \le 1$, and

$$\varphi(\xi) = \begin{cases} 1, & |\xi| \le hR, \\ 0, & |\xi| \ge kR, \quad h < k \end{cases} \tag{3.8}$$

and denote

$$a_1(x,\xi) = \varphi(\xi)a(x,\xi), \quad a_2(x,\xi) = (1-\varphi(\xi))a(x,\xi).$$

Apply Proposition 3.1 to $A_1 = a_1^w(x, D)$ and Proposition 3.2 to $A_2 = a_2^w(x, D)$, then (3.7) is proved immediately.

Corollary 3.1. *If $a(x,\xi) \in S^m_{\rho,\delta}$ with all the assumptions on $a(x,\xi)$ as before, then $A : L_p^\Omega \to L^p$ is bounded with*

$$\|A\| \le CR^{m+n+2\mu\delta},$$

C depends on $\partial_x^\beta \partial_\xi^\alpha a(x,\xi)$ and $|\alpha| \le 2\lambda, \lambda = [\frac{n}{2}] + 1; |\beta| \le 2\mu, \mu = [\frac{m+n}{2(1-\delta)}] + 1$.

§4. Boundedness in Scales of Function Spaces

Many function spaces which are important in mathematical physics, PDE and harmonic analysis can be organized into various scales, the most remarkable are $B^s_{p,q}$ and $F^s_{p,q}$ (for notation, see [4, 5]). These scales are defined through the Littlewood-Paley dyadic decomposition in frequency domain (see [4, 6, 7]): let Φ denote the set of vectors $\Phi(\xi) = \{\varphi_j(\xi)\}_{j=0}^\infty$ such that $\varphi_j(\xi) \in C_0^\infty$ and

$$\begin{cases} \mathrm{supp}\varphi_0(\xi) \in \{\xi; |\xi| \le 2\}, \\ \mathrm{supp}\varphi_j(\xi) \in \{\xi; 2^{j-1} \le |\xi| \le 2^{j+1}\}, & (j > 0), \\ \varphi_j(\xi) \ge 0, \quad \sum\limits_{j=0}^\infty \varphi_j(\xi) = 1, \end{cases} \tag{4.1}$$

and moreover

$$2^{j|\alpha|}|\partial^\alpha \varphi_j(\xi)| \le C_\alpha, \tag{4.2}$$

where C_α are independent of j. Usually, we always take $\varphi \in C_0^\infty$ such that $\{\varphi_j(\xi)\}_{j=1}^\infty = \{\varphi(2^{-j}\xi)\}_{j=1}^\infty$ satisfy (4.1). With the help of vectors Φ, we can associate to every $f \in \mathcal{S}'(R_n)$ a vector

$$\{f_j\} = \{\varphi_j(D)f\}_{j=0}^\infty \big(= \{\varphi_0(D)f, \varphi(2^{-j}D)f, j > 0\}\big), \tag{4.3}$$

and we obtain Littlewood-Paley decomposition in $\mathcal{S}'(R_n)$:

$$f(x) = \sum_{j=0}^\infty f_j(x). \tag{4.4}$$

Now we can define two scales of function spaces in L^p frame. Namely,

Definition 4.1. (1) *$B^s_{p,q}$ is the set of $f \in \mathcal{S}'(R_n)$ such that f_j in (4.3) are in L^p and*

$$\|f\|_{B^s_{p,q}} = \Big(\sum_{j=0}^\infty \|2^{sj} f_j\|^q_{L^p}\Big)^{1/q} = \Big(\sum_{j=0}^\infty \|2^{sj} F^{-1}(\varphi_j F f)\|^q_{L^p}\Big)^{1/q} < \infty, \tag{4.5}$$

or more directly, the weighted (by 2^{sj}) L-P decomposition of f belongs to $l_q(L^p(R_n))$.

(2) *$F^s_{p,q}$ is the set of $f \in \mathcal{S}'(R_n)$ such that $\big(\sum\limits_{j=0}^\infty |2^{sj} f_j|^q\big)^{1/q} \in L^p$ and*

$$\|f\|_{F^s_{p,q}} = \Big\|\Big(\sum_{j=0}^\infty |2^{sj} f_j|^q\Big)^{1/q}\Big\|_{L^p} < \infty. \tag{4.6}$$

By Paley-Wiener theorem, we see all the sums and integrals above make sense.

We assume now $-\infty < s < +\infty, 1 \le p < +\infty, 1 \le q < +\infty$. We can prove that both $B^s_{p,q}$ and $F^s_{p,q}$ are Banach spaces with $\|f\|_{B^s_{p,q}}$ in (4.5) and $\|f\|_{F^s_{p,q}}$ in (4.6) their norms respectively.

$B^s_{p,q}$ is called the Besov space.

Remark 4.1. We can prove that for various $\varphi \in C_0^\infty$, the norms in (4.5) (or (4.6)) are equivalent, hence the structure of the resulting Banach spaces are equivalent. This is why we omit a mark for φ such as $\|f\|^{\varphi}_{B^s_{p,q}}$ in our notations (4.5) and (4.6).

Remark 4.2. $B^s_{p,q}$ and $F^s_{p,q}$ actually can be defined even for $-\infty < s < +\infty, 0 \le p \le +\infty, 0 \le q \le +\infty$ (but for $F^s_{\infty,q}$, a slight modification is needed). Many function spaces are their particular cases, e.g., Zygmund spaces $\mathcal{Z}^s = B^s_{\infty,\infty}$ $(s > 0)$, Sobolev space $H^s_p = F^s_{p,2}$ $(-\infty < s < +\infty, 1 < p < +\infty)$, inhomogeneous Hardy space and inhomogeneous BMO space etc. Particularly $L^p(R_n)$ is equivalent to $F^0_{p,2}$ (this is just the classical Littlewood-Paley theorem for L^p), while $L^2(R_n)$ (i.e., H^0) is equivalent to $B^0_{2,2}$. Hence $L^p(R_n)$ is not a Besov space, which helps to explain, at least partially, why the L^p-boundedness arguments are different from that of L^2-boundedness. For more details see [4, 6, 7].

Now deal with the boundedness of A in $B^s_{p,q}$. For $u \in B^s_{p,q}$, consider

$$Au = \sum_{j=0}^{\infty} A\, u_j, \tag{4.7}$$

which will be shown later to be convergent in L^p. Our goal is to prove the following

Theorem 4.1. *Let $a(x,\xi) \in S^0_{\rho,\delta}, 0 \le \rho, \delta \le 1, \delta < 1$, then the Weyl quantization $A = a^w(x,D)$ is a bounded linear operator $A : B^s_{p,q} \to L^p$ with norm*

$$\|A\| \le C.$$

C depends on $\partial_x^\beta \partial_\xi^\alpha a(x,\xi)$ with $|\alpha| \le 2\lambda, \lambda = [\frac{n}{2}] + 1, |\beta| \le 2\mu, \mu = [\frac{n}{2(1-\delta)}] + 1$. Here we also assume

$$s > \sigma = n + 2\mu\delta. \tag{4.8}$$

Proof. We first prove the convergence of (4.7) in L^p. Actually, by Theorem 3.1, $Au_j \in L^p$ and

$$\|Au_j\|_{L^p} \le C 2^{j(n+2\mu\delta)} \|u_j\|_{L^p} = C 2^{j\sigma} \|u_j\|_{L^p}.$$

Note that $u_j \in L^\Omega_p$ and $\Omega = \{\xi; |\xi| \le 2^{j+1}\}$, hence we may replace R by 2^j. Since $s > \sigma$, we have

$$\|Au_j\|_{L^p} \le C 2^{j(\sigma - s)} \cdot 2^{js} \|u_j\|_{L^p}$$

and Schwartz's inequality gives

$$\sum_{j=0}^{\infty} \|Au_j\|_{L^p} \le C \Big(\sum_{j=0}^{\infty} 2^{j(\sigma-s)q'} \Big)^{1/q'} \Big(\sum_{j=0}^{\infty} (2^{js} \|f_j\|_{L^p})^q \Big)^{1/q} \le C_1 \|u\|_{B^s_{p,q}},$$

where q' is the dual index of q: $\dfrac{1}{q} + \dfrac{1}{q'} = 1$. This prove the convergence of (4.7) and

$$\|Au_j\|_{L^p} \le C \|u\|_{B^s_{p,q}}. \tag{4.9}$$

Remark 4.3. This proof is also valid for $p = \infty$ and $q = \infty$, the modification needed is very easy.

For the boundedness of A in $F^s_{p,q}$ we have

Theorem 4.2. *Let $a(x,\xi) \in S^0_{\rho,\delta}, 0 \le \rho, \delta \le 1, \delta < 1$. Then the Weyl quantization $A = a^w(x,D)$ is a bounded linear operator $A : F^s_{p,q} \to L^p$, here $-\infty < s < +\infty, 1 \le p < +\infty, 1 \le q \le +\infty$ with norm*

$$\|A\| \le C. \tag{4.10}$$

C depends on $\partial^\beta_x \partial^\alpha_\xi a(x,\xi)$ with $|\alpha| \le 2\lambda, \lambda = [\frac{n}{2}] + 1, |\beta| \le 2\mu,\ \mu = \big[\frac{n}{2(1-\delta)}\big] + 1$, and we also assume $s > \sigma = n + 2\mu\delta$.

Proof. It is well-known that there are continuous embeddings (see [4, 4.46])

$$B^s_{p,\min(p,q)} \hookrightarrow F^s_{p,q} \hookrightarrow B^s_{p,\max(p,q)}. \tag{4.11}$$

$A = a^w(x,D)$ can also be defined on $B^s_{p,\max(p,q)}$. Thus for $u \in F^s_{p,q}$, $l(u) \in B^s_{p,\max(p,q)}$, Au can be written as

$$Au\Big|_{u \in F^s_{p,q}} = (A \circ l)u\Big|_{u \in F^s_{p,q}}.$$

By Theorem 4.1, the conclusion of Theorem 4.2 is immediate.

In Theorems 4.1 and 4.2 we note the decreasing of the upper index s. In fact, $L^p = F^0_{p,2}$. Since in a certain sense, s is a measure of differentiability, the action of A causes loss of differentiablity. Thus, in order to make the action A more precise, it is well that we introduce differentiability of function into the definition of function space. Thus we introduce

Definition 4.2. *Let $D^{s,m}_{p,q}$ denote the completion of the set of vectors $f = \{f_0, f_1, \cdots, f_j\}, f_j \in C^\infty(R_n)$ with respect to the norm*

$$\|f\|_{D^{s,m}_{p,q}} = \Big(\sum_{j=0}^{\infty} 2^{sjq}\Big(\sum_{|\alpha| \le m} \|D^\alpha f_j\|_{L^p}\Big)^q\Big)^{1/q}. \tag{4.12}$$

Since $\|\cdot\|_{D^{s,m}_{p,q}}$ is evidently a norm, we see that $D^{s,m}_{p,q}$ is a Banach space when $1 \le p \le \infty, 1 \le q \le \infty, 1 < s < +\infty$ and the proofs of the following facts are staightforward.

Proposition 4.1. (1) *$B^s_{p,q} \subset D^{s,0}_{p,q}$ in the sense that $f = \{f_0, f_1, \cdots, f_j, \cdots\}, f_j = \varphi(2^{-j}D)f$.*

(2) *When $m' < m$, $D^{s,m'}_{p,q} \subset D^{s,m}_{p,q}$.*

(3) *$D^s_{p,q} \triangleq \bigcap_m D^{s,m}_{p,q}$ is a Frechet space with (4.12) as semi-norms.*

(4) *$H^s = D^s_{2,2} (s > 0)$ and $C^s = D^{s,[s]+1}_{\infty,\infty}$.*

The last statement is just the Littlewood-Paley decomposition of $H^s(R_n)$ and $C^s(R_n)$ (see [8, Theorems 2.1.4 and 2.1.7]).

As an application of this function space, we consider the boundedness of PsDO of $S^m_{1,1}$ class. As is well-known, classical PsDO of $S^0_{1,1}$ class is not bounded in $H^0 = L^2$. E. M. Stein proved that it is bounded in H^s when $s > 0$, but the proof is not published. L. Hörmander[9] offered a proof of this fact (see [10, 11] for more details), and we want to offer another proof here.

First consider the PsDO

$$Au(x) = (2\pi)^{-2n} \iint e^{i(x-y)\xi} a(x,y,\xi)u(y)dyd\xi \tag{4.13}$$

with $S^m_{1,1}$ amplitude $a(x,y,\xi)$. Its Littlewood-Paley decomposition is

$$a(x,y,\xi) = \sum_{j=0}^{\infty} a_j(x,y,\xi) = \sum_{j=0}^{\infty} \varphi_j(\xi)\, a(x,y,\xi).$$

We first prove the

Lemma 4.1. *Let A_j be the* PsDO *with amplitude $a_j(x,y,\xi)$ replacing $a(x,y,\xi)$ in* (4.13). *Then $D^\alpha_x A_j$ is L^p-bounded with norm*

$$\|D^\alpha_x A_j u\|_{L^p} \le C_\alpha 2^{j(m+|\alpha|)} \|u\|_{L^p}, \quad u \in \mathcal{S}(R_n). \tag{4.14}$$

Proof. It is easy to see (integrate by parts) that

$$D^\alpha_x A_j u(x) = (2\pi)^{-2n} \iint e^{i(x-y)\xi} b_{j,\alpha}(x,y,\xi)u(y)dyd\xi,$$

where $b_{j,\alpha}(x,y,\xi) = (\xi + D_x)^\alpha a_j(x,y,\xi) \in S^{m+|\alpha|}_{1,1}$. Since

$$\text{supp}_\xi b_{j,\alpha}(x,y,\xi) \subset \{\xi; 2^{j-1} \le |\xi| \le 2^{j+1}\},$$

the inequality (4.14) is easily derived by a simple calculation.

Theorem 4.3. *Let $a(x,\xi) \in S^m_{1,1}$. Then*

$$a(x,D)u = (2\pi)^{-2n} \int e^{ix\xi} a(x,\xi)\widehat{u}(\xi)d\xi$$

is a bounded linear operator from $B^s_{p,q}$ to $D^{s-m}_{p,q}$ and

$$\|a(x,D)u\|_{D^{s-m}_{p,q}} \le C\|u\|_{B^s_{p,q}}. \tag{4.15}$$

C depends on $\sup_{x,\xi} |\partial^\alpha_\xi a(x,\xi)(1+|\xi|^2)^{-\frac{m}{2}}|, |\alpha| \le 2([\frac{n}{2}]+1)$.

Proof. Let $a(x,\xi) = \sum\limits_{j=0}^{\infty} a_j(x,\xi)$ as in Lemma 4.1, then for $u \in B^s_{p,q}, u = \sum\limits_{k=0}^{\infty} u_k$ with $\text{supp}_\xi \widehat{u}_k(\xi) \subset \{\xi; 2^{k-1} \le |\xi| \le 2^k\}$. We have

$$a(x,D)u = \sum_{k,j=0}^{\infty} a_j(x,D)u_k, \tag{4.16}$$

where $a_j(x,D)u_k = (2\pi)^{-n} \int e^{ix\xi} a_j(x,\xi)\widehat{u}_k(\xi)d\xi$. Since $a_j(x,\xi)\widehat{u}_k(\xi) = 0$ whenever $|k-j| \ge 2$, we have $a_j(x,D)u = \sum\limits_{k=j-2}^{j+2} a_j(x,D)u_k$. By Lemma 4.1, we have

$$\|D^\alpha_x a_j(x,D)u\|_{L^p} \le C_\alpha 2^{j(m+|\alpha|-s)} \sum_{k=j-2}^{j+2} 2^{js}\|u_j\|_{L^p},$$

i.e.,

$$\sum_{j=0}^{\infty} (2^{j(s-m-|\alpha|)} \|D^\alpha_x a_j(x,D)u\|_{L^p})^q \le C_\alpha \sum_{j=0}^{\infty} (2^{js}\|u_j\|_{L^p})^q.$$

That is identical with (4.15).

Remark 4.4. When $s > 0, p = q = 2, a(x, D) : H^s(R_n) \to H^{s-m}(R_n)$ $(m < s)$ is bounded, thus generalizing the result of E. M. Stein.

Remark 4.5. $B^s_{\infty,\infty}(R_n) = C^s(R_n)$ is the Hölder-Zygmund space, thus our theorem also gives a boundedness result of PsDO on $C^s(R_n) \to C^{s-m}(R_n)$ $(m < s)$.

§5. Almost Diagonalizable Operators

There is a problem left open until now, i.e., the problem if the Weyl quantization $a^w(x, D)$ will map L_p^Ω into $L_p^{\Omega'}$ where $\Omega \subset\subset \Omega'$. Actually, this is a question of deep relevance. Since $u \in L_p^\Omega$ has spectrum concentrated in a compact domain Ω, and after acted by $a^w(x, D)$, the spectrum would be smeared, except when the symbol $a^w(x, \xi)$ is of very special form, for instance when $a(x, \xi) = a(\xi)$ and the Weyl quatization becomes convolution. In this sense we should say that $a^w(x, D)$ is not localized. This is just the result of the appearance of x-variables together with ξ-variables in the symbol. Thus it is clear that this phenomenon is connected with a deep-lying fundamental law of the nature, the principle of uncertainty. Thus, we should give a more reasonable formulation of our question: Can we decompose $a^w(x, D)$ in two parts

$$a^w(x, D) = A(x, D) + R(x, D), \tag{5.1}$$

where $A(x, D) : L_p^\Omega \to L_p^{\Omega'}, \Omega = \{\xi; |\xi| \le R\}, \Omega' = \{\xi; |\xi| \le kR\}$ with $k > 1$ and $R(x, D)$ is in a certain sense negligible, for instance, $R \in S^{-\infty}$? If so, we say $a^w(x, D)$ is almost diagonalizable and the main goal of this section is just to provide an affirmative answer.

By Littlewood-Paley decomposition of u,

$$Au = \sum_{j=0}^{\infty} Au_j. \tag{5.2}$$

By Littlewood-Paley decomposition of Au again, we have

$$\begin{aligned} Au &= \sum_{k=0}^{\infty} \varphi_k(D) Au = \sum_{k,j=0}^{\infty} \varphi_k(D) A \varphi_j(D) u = \sum_{k,j=0}^{\infty} A_{kj} u, \\ A_{kj} &= \varphi_k(D)\, A(x, D) \varphi_j(D). \end{aligned} \tag{5.3}$$

It is clear that A_{kj} is the composition of $\varphi_k(D), A(x, D)$ and $\varphi_j(D)$. Hence we may turn to the asymptotic expansion of Weyl quantization, which is given by Hörmander (see [12, Chapter 18, 18.4–18.6]). If $a_i(x, \xi) \in S(\Lambda^{m_i}, G)$, where G is a slow-varying metric, Λ^{m_i} the weights, then $a_1^w(x, D) \circ a_2^w(x, D)$ will have a symbol $(a_1 \# a_2)(x, \xi)$ and

$$(a_1 \# a_2)(x, \xi) \sim \sum_k \sum_{|\alpha|+|\beta|=k} \frac{(-1)^\beta}{\alpha! \beta!} D_\xi^\alpha \partial_x^\beta a_1(x, \xi) D_\xi^\beta \partial_x^\alpha a_2(x, \xi). \tag{5.4}$$

In our case, let $\varphi_\alpha = \partial^\alpha \varphi$, then

$$\varphi_k(\xi) \#\, a(x, \xi) \#\, \varphi_j(\xi) \sim \sum_{|\alpha|+|\beta|=0}^{\infty} \frac{(-1)^\beta}{\alpha! \beta!} 2^{-k|\alpha| - j|\beta|} \varphi_\alpha(2^{-k}\xi) \partial_x^{\alpha+\beta} a(x, \xi) \varphi_\beta(2^{-j}\xi).$$

At present, we have, at least formally, an expansion for the symbol $a(x, \xi)$ of $A(x, D)$ as

$$a(x, \xi) \sim \sum_{k,j=0}^{\infty} \sum_{|\alpha|+|\beta|=0}^{\infty} \frac{(-1)^\beta}{\alpha! \beta!} \varphi_\alpha(2^{-k}\xi) \partial_x^{\alpha+\beta} a(x, \xi) \varphi_\beta(2^{-j}\xi) 2^{-k|\alpha| - j|\beta|}. \tag{5.5}$$

We should prove that, roughly speaking, in the expansion (5.5), the summation in α and β is asymptotic, while that in k and j is convergent. To this end, we should notice that $\varphi(\xi)$ being contained in $C_0^\infty(R_n)$ lies in $S_{1,0}^{-\infty}=\bigcap_m S_{1,0}^m$. We may consider $\varphi(\xi)$ belonging to the class $S_{1,0}^M$ for arbitrary M.

For definite α and β, let $\sup|\varphi_\alpha|\cdot\sup|\varphi_\beta|=M_{\alpha\beta}$, since $1+|\xi|\sim 2^k$ on $\text{supp}_\xi\varphi(2^{-k}\xi)$ and $1+|\xi|\sim 2^j$ on $\text{supp}_\xi\varphi(2^{-j}\xi)$. It is easy to see that

$$\begin{aligned}&\Big|\frac{(-1)^\beta}{\alpha!\beta!}2^{-k|\alpha|-j|\beta|}\varphi_\alpha(2^{-k}\xi)\partial_x^{\alpha+\beta}a(x,\xi)\varphi_\beta(2^{-j}\xi)\Big|\\&\le C_{\alpha\beta}M_{\alpha\beta}2^{-k\lambda|\alpha|-j\lambda|\beta|}(1+|\xi|)^{(\lambda+\delta-1)|\alpha+\beta|},\end{aligned}\tag{5.6}$$

where $C_{\alpha\beta}$ are the suitable constants depending only on α and β. Substituting into (5.5), we see that the summation in k,j is convergent. Denoting the sum as $a_{\alpha\beta}(x,\xi)$, we have

$$a(x,\xi)\sim\sum_{|\alpha+\beta|=0}^{\infty}a_{\alpha\beta}(x,\xi).\tag{5.7}$$

Again, using (5.6) we see for λ small enough such that $\lambda+\delta<1$,

$$|a_{\alpha\beta}(x,\xi)|\le A_{\alpha\beta}(1+|\xi|)^{(\lambda+\delta-1)|\alpha+\beta|}.\tag{5.8}$$

Thus, (5.7) is an asymptotic expansion of $a(x,\xi)\in S^0_{\rho,\delta}$, here $0<\delta_1=\delta+\lambda<1$. Hence for any definite integer $N>0$, we can write

$$\begin{aligned}(a(x,\xi)&\sim\sum_{|\alpha+\beta|=0}^{N-1}a_{\alpha\beta}(x,\xi)+\sum_{|\alpha+\beta|=N}^{\infty}a_{\alpha\beta}(x,\xi)\\&=A_N(x,\xi)+R_N(x,\xi),\quad R_N(x,\xi)\in S_{\rho,\delta_1}^{(\delta_1-1)N}.\end{aligned}\tag{5.9}$$

By noticing that $\text{supp}\varphi(2^{-k}\xi)\cap\text{supp}\varphi(2^{-j}\xi)=\emptyset$, whenever $|k-j|>2$, (5.5) can be rewritten as

$$a(x,\xi)\sim\sum_{k=0}^{\infty}\sum_{j=k-2}^{k+2}\sum_{|\alpha|+|\beta|=0}^{\infty}\frac{(-1)^\beta}{\alpha!\beta!}\varphi_\alpha(2^{-k}\xi)\partial_x^{\alpha+\beta}a(x,\xi)\varphi_\beta(2^{-j}\xi)2^{-k|\alpha|-j|\beta|}.$$

Using the same procedure as above, we have an operator

$$\widetilde{A}(x,D)=\sum_{k=0}^{\infty}A_k(x,D),\tag{5.10}$$

$$A_k(x,D)=\sum_{j=k-2}^{k+2}\varphi_k(D)A(x,D)\varphi_j(D).\tag{5.11}$$

Convergence of (5.10) would be established later. At present we have asymptotic expansion of (5.11) as

$$\begin{aligned}A_k(x,D)&\sim\sum_{l=0}^{\infty}\sum_{j=k-2}^{k+2}\varphi_k(D)A_l(x,D)\varphi_j(D)\\&=\sum_{l=0}^{N-1}\sum_{j=k-2}^{k+2}\varphi_k(D)A_l(x,D)\varphi_j(D)+\sum_{l=N}^{\infty}\sum_{j=k-2}^{k+2}\varphi_k(D)A_l(x,D)\varphi_j(D)\\&=A_{kN}(x,D)+R_{kN}(x,D).\end{aligned}$$

Substituting into (5.10), we have $\widetilde{A}(x,D) = \widetilde{A}_N(x,D) + \widetilde{R}_N(x,D)$. Since $\widetilde{A}(x,D)$ and $A(x,D)$ have symbols with the same asymptotic expansion (5.5)

$$A(x,D) = \widetilde{A}(x,D) \bmod (S^{-\infty}),$$

we have finally

$$A(x,D) \sim \sum_{k=0}^{\infty} A_k(x,D) \mod (S^{-\infty}). \tag{5.12}$$

$A_k(x,D)$ is very remarkable, since we have $A_k : L_p^{\Omega_j} \to L_p^{\Omega_k}$ by (5.11), where $\Omega_l = \{\xi; |\xi| \le 2^{l+1}\}, l = j,k$. We give

Definition 5.1. *An operator $A : L_p^{\Omega} \to L_p^{\Omega'}, \Omega \subset \Omega'$ is called a diagonalizable (localizable) operator. An operator which can be expanded asymptotically as sum of diagonalizable operators*

$$A \sim \sum_{k=0}^{\infty} A_k \mod (S^{-\infty}), \tag{5.13}$$

is called an almost diagonalizable operator.

The discussion above can be summed as a

Theorem 5.1. *Let $a(x,\xi) \in S^0_{\rho,\delta}, 0 \le \rho, \delta \le 1, \delta < 1$, then its Weyl quantization $a^w(x,D)$ is almost diagonalizable.*

Definition 5.2. *If $A(x,D) = a^w(x,D)$ can be expanded asymptotically as $A(x,D) \sim \sum\limits_{k=0}^{\infty} A_k(x,D) \mod (S^{-\infty})$, while $A_k(x,D)$ are bounded linear operator $(\mathrm{mod}\, S^{-\infty}) : B^s_{p,q} \to B^{s'}_{p,q}, F^s_{p,q} \to F^{s'}_{p,q}$, we say $A(x,D)$ is bounded asymptotically.*

Theorem 5.2. *Let $a(x,\xi) \in S^0_{\rho,\delta}, 0 \le \rho, \delta \le 1, \delta < 1$, then its Weyl quantization $a^w(x,D)$ is an asymptotically bounded operator: $B^s_{p,q} \to B^{s'}_{p,q}$ with $s' = s - n$.*

Proof. By Theorem 5.1, we have $A(x,D) \sim \sum\limits_{k=0}^{\infty} A_k(x,D)$, and $A_k(x,D)$'s are defined by (5.11).

Now we can apply the same argument used in the proof of Theorem 5.1 to the operator $\varphi_k(D)A(x,D)$ and we have

$$\begin{aligned}\varphi_k(\xi)\#a(x,\xi) &\sim \sum_{|\alpha|=0}^{\infty} \frac{1}{\alpha!} 2^{-k|\alpha|} \varphi_\alpha(2^{-k}\xi)\partial_x^\alpha a(x,\xi) \\ &= \sum_{|\alpha|=0}^{N-1} \frac{1}{\alpha!} 2^{-k|\alpha|} \varphi_\alpha(2^{-k}\xi)\partial_x^\alpha a(x,\xi) + R_{Nk}(x,\xi),\end{aligned} \tag{5.14}$$

with $R_{Nk} \in S^{-N}$. Summing R_{Nk} in k, and denoting $R_N(x,\xi) = \sum\limits_{k=0}^{\infty} R_{Nk}(x,\xi)$, we can prove as before that $R_N(x,\xi) \in S^{-N}$. Here we make use of the fact that $2^k \sim 1 + |\xi|$ on $\mathrm{supp}_\xi \varphi(2^{-k}\xi)$.

We shall prove now boundedness of $A_k(x,D) : B^s_{p,q} \to B^{s'}_{p,q}$ with $s' = s - n$. Actually

$$A_k(x,D) = \sum_{j=k-2}^{k+2} \varphi_k(D)Au_j.$$

Since $\varphi_k(D)A$ as a PsDO has a symbol supported on $\Omega_k = \{\xi; |\xi| \le 2^{k+1}\}$ and expandable asymptotically as (5.14), we can use the methods in section 3 to each term of (5.14) and note that $2^k \sim 1 + |\xi|$,

$$\Big|\iint e^{i(x-y)\xi}\varphi_\alpha(2^{-k}\xi)\partial_x^\alpha a\Big(\frac{x+y}{2},\xi\Big)u_j(y)dyd\xi\Big| \le C2^{-k|\alpha|}2^{kn}\|u_j\|_{L^p},$$

since $2^j \sim 2^k$ when $|k-j| \le 2$. Assuming $u_j = 0$ for $j < 0$, we see

$$\begin{aligned}\|A_k(x,D)u\|_{L^p} &\le C2^{kn}(\|u_{k-2}\|_{L^p} + \cdots + \|u_{k+2}\|_{L^p})\\ &= C2^{k(n-s)}(2^{(k-2)s}\|u_{k-2}\|_{L^p} + \cdots + 2^{(k+2)s}\|u_{k+2}\|_{L^p}).\end{aligned}$$

Hence for $A_k \mod (S^{-\infty})$, $2^{k(s-n)}\|A_k(x,D)u\|_{L^p} \le C\sum_{j=k-2}^{k+2} 2^{js}\|u_j\|_{L^p}$. Denote $w = Au$, then

$$\Big(2^{ks'}\|w_k\|_{L^p}\Big)^q \le \Big[C\sum_{j=k-2}^{k+2} 2^{js}\|u_j\|_{L^p}\Big]^q \le C\sum_{j=k-2}^{k+2}\big(2^{js}\|u_j\|_{L^p}\big)^q.$$

Summing up in k, we have $\|w\|_{B^{s'}_{p,q}} \le C\|u\|_{B^s_{p,q}}$, i.e., for A mod$(S^{-\infty})$, $\|Au\|_{B^{s'}_{p,q}} \le C\|u\|_{B^s_{p,q}}$, and the theorem is proved.

Next, we are to consider asymptotical boundedness of $A^w(x,D)$ in $F^s_{p,q}$, here $0 < p < +\infty$. We also use the embedding of the scales of function spaces as follows (see [4, §2.3.2, Proposition 2])

$$F^s_{p,q}(R_n) \hookrightarrow B^s_{p,\max(p,q)}(R_n),$$
$$B^s_{p,q_0}(R_n) \hookrightarrow B^{s-\varepsilon}_{p,q_1}(R_n),\ \varepsilon > 0,\quad B^{s-\varepsilon}_{p,\min(p,q)}(R_n) \hookrightarrow F^{s-\varepsilon}_{p,q}(R_n).$$

Theorem 5.3. *If $a(x,\xi) \in S^0_{\rho,\delta}, 0 \le \rho,\delta \le 1, \delta < 1$, then $a^w(x,D) = A(x,D)$ is an asymptotically bounded linear operator $F^s_{p,q} \to F^{s'-\varepsilon}_{p,q}$. Here $s' = s-n, \varepsilon > 0$ is arbitrary, $1 \le p < +\infty, 1 \le q \le +\infty$.*

Proof. The mapping $A : F^s_{p,q} \to F^{s'-\varepsilon}_{p,q}$ can be splitted into four steps:

$$F^s_{p,q} \xrightarrow{l_1} B^s_{p,q_0}\quad (q_0 = \max(p,q)),\qquad B^s_{p,q_0} \xrightarrow{A} B^{s'}_{p,q_0}\quad (s' = s-n),$$
$$B^{s'}_{p,q_0} \xrightarrow{l_2} B^{s'-\varepsilon}_{p,q_1}\quad (q_1 = \min(p,q)),\quad B^{s'-\varepsilon}_{p,q_1} \xrightarrow{l_3} F^{s'-\varepsilon}_{p,q}.$$

All mappings are continuous with norms depending only on $a(x,\xi)$. The theorem is proved.

References

[1] Wang, R. H. & Li, C. Z., On the L^p-boundedness of several classes of PsDO, *Chin. Ann. of Math.,* **5B:**2(1984), 193–214.

[2] Hwang, I. L., The L^p-boundedness of pseudo-dfferential operators, *Trans. Amer. Math. Soc.,* **102**(1987), 55–76.

[3] Lerner, N., Coherent states and evaluation equations, General Theory of PDE and Micro-local Analysis (Qi Minyou and L. Rodino, (eds)), Longman, 1996, 123–154.

[4] Triebel, H., Function spaces, Birkhanser, 1983.

[5] Schmeisser, H. J. & Triebel, H., Topics in Fourier analysis and function space, John Wiley, 1986.

[6] Chemin, J. Y., Fluides parfaits incompressibles, Astérisque, **230**, 1995.

[7] Chemin, J. Y., Incompressible Navier-Stokes systems, Lecture notes, Wuhan Univ. , 1995.

[8] Chi, M. Y., Xu Chaojiang & Wang Weike, An introduction to modern theory of partial differential equations, Wuhan Univ. Press, 1994 (in Chinese).

[9] Hörmander, L., Nonlinear hyperbolic differential equations, Lecture notes, Lund Univ, 1986-1987.

[10] Hörmander, L., Pseudo-differential operators of type 1.1, *Comm. PDE.,* **13**(1988), 1085–1112.

[11] Hörmander, L., Continuity of pseudo-differential operators of type 1.1, *Comm. PDE,* **14**(1989), 231–244.

[12] Hörmander, L., The analysis of linear partial differential operators, **3,** Springer, 1985.