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DARBOUX EQUATIONS AND ISOMETRIC EMBEDDING OF RIEMANNIAN MANIFOLDS WITH NONNEGATIVE CURVATURE IN R^3**

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Abstract

The present paper is concerned with the existence of golbal smooth solutions for the homogeneous Dirichlet boundary value problem of the Darboux equation and the case degenerate on the boundary is contained. As some applications the smooth isometric embeddings of positively and nonnegatively curved disks into R^3 are constructed.

Keywords Darboux equation, Isometric embedding, Riemannian manifold, Nonnegative curvature
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§0. Introduction

As Darboux pointed out, the isometric embedding of two dimensional Riemannian manifolds in R^3 leads to solve a nonlinear partial differential equation of Monge Ampere type

$$\det(\nabla^2 z) = k\det(g_{ij})(1 - g^{ij}z_i z_j) \quad \text{in } \Omega \subset R^2, \tag{0.1}$$

where $\nabla^2 z = (z_{ij} - \Gamma_{ij}^k z_k)$ denotes the Hessian of z with respect to the given smooth metric $g = g_{ij}du^i du^j$ defined on $\overline{\Omega}$, g^{ij} the inverse of the metric tensor and k the curvature of the metric g. Indeed, from the Gauss equations of the required isometric embedding $\vec{r} = (x, y, z)$,

$$\vec{r}_{ij} = \Gamma_{ij}^l \vec{r}_l + \Omega_{ij}\vec{n}, \qquad i, j = 1, 2,$$

where Ω_{ij} are the coefficients of its second fundamental form and $\vec{n}$ is its normal, computing the inner products of the last expressions and the unit vector $\vec{k}$ of the z axis we have

$$\det(\nabla^2 z) = \det(\Omega_{ij})(\vec{n}, \vec{k})^2.$$

Notice that $\det(\Omega_{ij}) = k\det(g_{ij})$ and

$$(\vec{n}, \vec{k})^2 = 1 - \Big(\frac{\vec{r}_1 X \vec{r}_2}{|\vec{r}_1 X \vec{r}_2|} X\vec{k}\Big)^2 = 1 - g^{ij}z_i z_j.$$

It turns out that z satisfies (0.1). The equation (0.1) is called the Darboux equation. Ones are very interested in the existence of the smooth convex caps, i.e., a convex surface whose

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boundary is on some plane and the projection of this surface on this plane is a one to one mapping. This leads to solve (0.1) and

$$z = 0 \text{ on } \partial\Omega \tag{0.2}$$

with a subsidiary condition

$$|\nabla z|^2 = g^{ij} z_i z_j < 1 \text{ on } \overline{\Omega}. \tag{0.3}$$

If z is a global smooth solution to the problem (0.1) with (0.3), it follows that $g - dz^2$ is a flat metric defined on $\overline{\Omega}$ (for details, refer to [2]) and in some conditions we can find two smooth functions x and y in $C^\infty(\overline{\Omega})$ (Lemma 1.5 of the present paper) such that $dx^2 + dy^2 = g - dz^2$, i.e., $\vec{r} = (x, y, z)$ is the required convex cap. There has been a counter example in [7, Appendix 3] to show that (0.1) with (0.2) and (0.3) is not always solvable even if ($\overline{\Omega}$,g) is an analytic geodesic disk with strictly positive Gauss curvature. It should be emphasized that the boundary of Ω in the above counter example has negative geodesic curvature. Therefore, naturally, we assume that

$$\text{the curvature } k > 0 \text{ on } \overline{\Omega}, \tag{0.4}$$

$$\partial\Omega \text{ has positive geodesic curvature } k_g. \tag{0.5}$$

Under the assumptions (0.4) and (0.5), a local smooth solution (in $C^\infty(\Omega) \cap C(\overline{\Omega})$) to the problem (0.1) and (0.2) with $|\nabla z| < 1$ in Ω was obtained in [18, p.104, Theorem 4]. In order to study the isometric embedding of metric with curvature changing its sign, for example, the metric defined on a neighbourhood of Ω with curvature positive inside and negative outside, the existence of a global smooth solution to (0.1) with (0.2) and (0.3) is very important and necessary to smoothly extend the solution to the outside Ω. If there exists a smooth geodesically convex subsolution to the problem (0.1)–(0.3), a global smooth solution is constructed in [4] for (0.4) and also in [12] for the degenerate case of the Gauss curvature respectively. But up to now it is unknown whether (0.4) and (0.5) are enough or not for getting a global smooth solution of (0.1)–(0.3). This is the motivation of the present paper. The problem is that the factor $(1 - |\nabla z|^2)$ in the right hand side of (0.1) may destroy its ellipticity. The main difficulty in getting the global smooth solution is how to obtain the above bound over $\overline{\Omega}$ of $|\nabla z|$ strictly less than 1. Lemma 2.1 in the present paper gives a stable intrinsic estimation for the above bound of $|\nabla z|$. Particularly, it is worth pointing out that this estimation obtained here only involves the lower bound of the curvature k in a certain interior closed domain of Ω and does not depend on the lower bound of the curvature k on $\overline{\Omega}$. So we can take this advantage to some degenerate case. Meanwhile, the existence of global smooth solution to (0.1)–(0.3) for smooth metric is also proved.

In the sequel, unless otherwise stated, by a smooth solution we always mean a global smooth solution, namely, a solution smooth up to the boundary.

Theorem A. *If* (0.4), (0.5) *are fulfilled, then* $(\overline{\Omega}, g)$ *always has a smooth isometric embedding in* R^3 *which is a convex cap.*

Theorem B. *If* (0.5) *is fulfilled and the Gauss curvature satisfies*

$$k > 0 \textit{ in } \Omega \textit{ and } k = 0, \quad dk \neq 0 \textit{ on } \partial\Omega, \tag{0.6}$$

then the conclusion in Theorem A continues to be true.

If (0.6) is fulfilled and if we smoothly extend the metric g to outside $\overline{\Omega}$, the Gauss curvature will clearly change its sign. Combining Theorem B with the result in [13] on existence of smooth solution for degenerate hyperbolic equation of Monge-Ampere type, one can obtain a smooth semiglobal isometric embedding in R^3 for a Riemannian manifold with curvature clearly changing its sign on a curve. This is a generalization of [15]. Under the present case, the region where the curvature is positive, is global and the region where the Gauss curvature is negative, is local. This is the reason why we call “semiglobal”.

§1. Several Lemmas

This section is concerned with some geometric consideration which are useful to the later realization in R^3 of the Riemannian manifolds with positive or nonnegative curvature.

Lemma 1.1. *Let g be a smooth metric defined on a simply connected bounded closed domain $\overline{\Omega}$ whose boundary has positive geodesic curvature k_g. Then there exists a smooth noncompact complete extension $(M, \tilde{g})$ of $(\overline{\Omega}, g)$, which is also simply connected such that, outside a compact set, the metric is of the form*

$$\tilde{g} = dt^2 + \widetilde{G}^2 ds^2 \ \textit{whose curvature} = \frac{1}{4} \ \min k_g^2 e^{-2(\max k_g)t}$$

and

$$\frac{\widetilde{G}_t}{\widetilde{G}} \geq \frac{1}{2} \ \min k_g e^{-(\max k_g)t}. \tag{1.1}$$

Proof. Indeed it suffices to extend the Riemannian manifold $(\overline{\Omega}, g)$ to a smooth complete noncompact Riemannian manifold $(M, \tilde{g})$ for which outside a compact set (1.1) holds. Since $\partial\Omega$ is of positive geodesic curvature, under the geodesic coordinates with the base curve $\partial\Omega$, the metric g can be expressed as

$$g = dt^2 + G^2(s,t)ds^2, \quad -\delta \leq t \leq 0$$

$$\text{with } G(s,0) = 1 \text{ and } G_t(s,0) = \text{the geodesic curvature } k_g > 0 \tag{1.2}$$

for some positive constant δ. Next we try to extend g to $\tilde{g}$ by finding a suitable smooth extension $\tilde{k}$(s,t) of the Gauss curvature $k(s,t) = -G_{tt}/G$, $t \leq 0$ and by solving the Gauss equation

$$G_{tt} = -\tilde{k}G, \quad t \geq 0. \tag{1.3}$$

Denote by k_1 an arbitrary smooth extension of $k(s,t)$ in the region $t \geq 0$ and by $\alpha(\varsigma)$ the smooth cutoff function with $\alpha \equiv 1$ as $|\varsigma| \leq 1/2$ and $\alpha \equiv 0$ as $|\varsigma| \geq 1$. Suppose that the required function $\tilde{k}$ is of the form

$$\tilde{k} = \alpha\Big(\frac{t}{\epsilon}\Big)k_1(s,t) + \frac{1}{4}\Big(1 - \alpha\Big(\frac{t}{\epsilon}\Big)\Big) \min k_g^2 e^{-2t \max k_g}, \quad t \geq 0. \tag{1.4}$$

Choose ϵ so small that for the solution G to the problem (1.3) (1.2) we have

$$W = \frac{G_t(s,t)}{G(s,t)} \geq \frac{2}{3} \min k_g \text{ at } t = \epsilon. \tag{1.5}$$

Obviously ϵ depends only on $\max|k_1|$ and $\max k_g$. From (1.3) it is not difficult to derive

$$W_t = -\tilde{k} - W^2, \ t > \epsilon. \tag{1.6}$$

With $W^- = \frac{1}{2}\min k_g \exp(-t\max k_g)$ we have

$$W_t^- \le -\tilde{k} - (W^-)^2, \quad t > \epsilon.$$

In view of (1.5) an application of the comparison principle soon gives the assertion in (1.1). This proves the present lemma.

Lemma 1.2. *Let the assumption in Lemma* 1.1 *be fulfilled. Then there exists a geodesically convex function $\tilde{h}$ in $C^\infty(\overline{\Omega})$ satisfying*

$$(\nabla^2\tilde{h}) \ge \frac{1}{C}I > 0 \quad \text{and} \quad |\tilde{h}|_2 \le C, \tag{1.7}$$

where the constant C depends only on the reciprocal of $\min k_g$ *and the intrinsic diameter L of Ω and $|k|_0$. Here and later we always denote by $|k|_s$ the C^s-norm of k.*

Proof. Lemma 1.1 gives a smooth extension $(M, \tilde{g})$ of $(\overline{\Omega}, g)$ which is a complete simply connected noncompact Riemannian manifold with positive curvature outside a compact set. In [5] the authors proved that any noncompact complete Riemannian manifold has a strictly convex function provided that outside a compact set the sectional curvature is positive. Translating the argument in [5, Section 2] from line to line for $(M, \tilde{g})$, in view of (1.1), without difficulty we can find a Busemann like function which satisfies (1.7) in the sense of the second difference quotient. Meanwhile, by a smooth convolution approximate technique given in [6] we can get a smooth strictly convex function $\tilde{h}$ which also satisfies (1.7) for another C. The details for the construction of $\tilde{h}$ are omitted here.

Lemma 1.3. *Let the assumption in Lemma* 1.1 *be fulfilled. Then there exists a smooth geodesically convex function ψ vanishing on $\partial\Omega$ and satisfying $|\nabla\psi| \le \frac{1}{2}$ and* (1.7) *for another constant C depending only on the same quantities as before.*

Proof. Under the geodesic coordinates (s, t) as used in (1.2), near the boundary $\partial\Omega$, we define

$$h_0 = e^t - 1 \text{ as } -\delta \le t \le 0.$$

Following [3] and [12, Lemma 2.1], setting a function $g \in C^\infty((-\infty, 0])$ subject to

$$g(0) = 0, \quad g(\theta) = -1 \text{ if } \theta \le h_0\Big(-\frac{3\delta}{4}\Big); \quad g'(\theta) > 0 \text{ if } \theta \ge h_0\Big(-\frac{\delta}{2}\Big)$$

$$\text{and} \quad g'' \ge 0 \text{ in } (-\infty, 0],$$

after constructing a smooth geodesically convex function

$$\psi = g(h_0) + \lambda\zeta\tilde{h}, \text{ where } \lambda \text{ is small enough and } \zeta \text{ a cutoff function},$$

we can soon find the required smooth geodesically convex function by scaling ψ. Hence this completes the proof of the present lemma.

Next we shall further discuss some geometric properties of Riemannian manifold with boundary of positive geodesic curvature.

Lemma 1.4. *Let the assumption in Lemma* 1.1 *be fulfilled. Then $\overline{\Omega}$ is geodesically convex.*

Proof. Without loss of generality, we may assume that the metric g is defined in a simply connected neighbourhood Ω' of Ω. First of all we claim:

for each point $p \in \partial\Omega$ there is a geodesic

$$\text{disk } B_R(p) \text{ such that } \overline{\Omega} \cap B_R(p) \text{ is geodesically convex.} \tag{1.8}$$

Near $\partial\Omega$ we take geodesic coordinates with the base curve $\partial\Omega$, under which the metric g is of the form

$$g = dt^2 + G^2(s,t)ds^2 \text{ in } N(\partial\Omega) = \{-t_0 \le t \le t_0\}, \tag{1.9}$$

where $\partial\Omega$ is defined by $t = 0$ and parameterized by the arclength. By the Liouville theorem the geodesic curvature of $\partial\Omega = G_t(s,0) > 0$. Therefore, without loss of generality, we assume $G_t > 0$ in the region considered. For each $p \in \partial\Omega$ there is a positive constant R such that the geodesic disk $B_R(p)$ is convex and contained in $N(\partial\Omega)$. Then for arbitrary two points $p_i \in B_R(p) \cap \overline{\Omega}$, $i = 1, 2$ with the coordinates (s_i, t_i) there is a unique geodesic γ connecting p_1 and p_2, lying in B_R(p). Next we illustrate this geodesic completely lying in $B_R(p) \cap \overline{\Omega}$. Suppose that this geodesic $\gamma(\sigma)$ is parameterized by its arclength. Then the geodesic equation provides

$$t_{\sigma\sigma} = -\Gamma_{11}^2 s_\sigma^2 = GG_t s_\sigma^2 \ge 0. \tag{1.10}$$

This means that $t(\gamma(\sigma))$ is convex and hence, attains its maximum at the endpoints, namely, $t(\sigma) \le \max\, t_i \le 0$. This implies the assertion in (1.8).

Now we turn to the following arbitrary case: $p_i \in \Omega$, $i = 1, 2$. In order to illustrate that any minimizing curve γ connecting p_1 and p_2 completely lies in Ω we must show that γ has no point on $\partial\Omega$. If it was false and $p \in \gamma \cap \partial\Omega$, we would find two points q_1 and q_2 respectively on the coming ray and outgoing ray, such that q_1 and q_2 are all contained in the convex geodesic disk of p mentioned in (1.8). Thus it follows that there is a unique geodesic connecting q_1 and q_2, lying in Ω. This contradicts the minimizing of γ and completes the proof of the present lemma.

Lemma 1.5. *Let the assumption in Lemma* 1.1 *be fulfilled and let the metric g be of nonpositive curvature. Then for each point $p \in \Omega$ there is a simply connected domain ω in T_p such that* $\exp_p$ *is globally diffeomorphic from $\overline{\omega}$ onto $\overline{\Omega}$.*

Proof. As mentioned before we can extend this metric to a neighbourhood Ω' of Ω where $\partial\Omega'$ has positive geodesic curvature either. If $p \in \Omega$, we define

$$\omega' = \left\{(\rho,\phi)|\rho < \sup\{s|\exp_p sv \subset \Omega'\},\ v \in T_p,\ \phi \text{ is the polar angle of } v\right\},$$

where v is a unit vector. So the exponential map $\exp_p$ is a surjective smooth map from ω' onto Ω'. Besides

$$\langle (d\exp)_{\rho v} v,\ (d\exp)_{\rho v} v\rangle = 1$$

and if the Gauss curvature of the metric is nonpositive, an application of Jacobian equation tells us

$$\langle (d\exp)_{\rho v} w, (d\exp)_{\rho v} w\rangle\ \ge 1$$

for all $v, w \in S^1$ and $w \perp v$. These two expressions imply the exponential map $\exp_p$ locally diffeomorphic. Lemma 1.4 tells us that the exponential map $\exp_p$ is surjective from $\omega' \longrightarrow \Omega'$. It remains to illustrate $\exp_p$ to be injective. Indeed, suppose there are two unit vectors $v_i \in T_p$, and real numbers ρ_i such that

$$q = \exp_p \rho_1 v_1 = \exp_p \rho_2 v_2 \quad \text{and} \quad \exp_p \rho v_1 \ne \exp_p \rho' v_2$$

for $0 < \rho < \rho_1$ and $0 < \rho' < \rho_2$. On the region Q enclosed by

$$\gamma_1 = \{\exp_p \rho v_1 | \, 0 \le \rho \le \rho_1\} \quad \text{and} \quad \gamma_2 = \{\exp_p \rho v_2 | \, 0 \le \rho \le \rho_2\},$$

using the Gauss-Bonnet formula we have

$$2\pi = \int_Q k + \int_{\partial Q} k_g + \theta_1 + \theta_2 \le \theta_1 + \theta_2, \quad -\pi \le \theta_1, \theta_2 \le \pi,$$

where θ_1 and θ_2 are the angles between γ_1 and γ_2 at the points p and q respectively. It turns out that $\theta_1 = \theta_2 = \pi$. This implies that they coincide with each other. This is a contradiction.

Thus we have proved that $\exp_p$ is globally diffeomorphic from ω' onto Ω', so is the map from $\omega = (\exp_p)^{-1}\overline{\Omega}$ to $\overline{\Omega}$. This completes the proof of Lemma 1.5.

Remark. When the metric is flat, Lemma 1.5 is just the consequence of [4, Theorem 1].

Lemma 1.6. *Let $z \in C^\infty(\overline{\Omega})$ be a geodesically convex solution of* (0.1) *with* (0.3) *and let $\partial z/\partial n \ge 0$ on $\partial\Omega$ for the unit vector n pointing to outside Ω and perpendicular to $\partial\Omega$ in the given metric. Then if $\partial\Omega$ has positive geodesic curvature, so does the boundary $\partial\Omega$ with respect to the metric $\tilde{g} = g - dz^2$.*

Proof. Take the geodesic coordinates with the base curve $\partial\Omega$, as done in (1.2). Then by the hypotheses in this lemma we have, on $\partial\Omega$,

$$|\nabla z|^2 = z_s^2 + z_t^2 < 1. \tag{1.11}$$

It is easy to see that the metric $\tilde{g}$ is of the form, near $\partial\Omega$,

$$\tilde{g} = g - dz^2 = (G^2 - z_s^2)ds^2 - 2z_s z_t ds dt + (1 - z_t^2)dt^2 \tag{1.12}$$

and

$$\begin{aligned} e_1 &= \frac{1}{\sqrt{G^2 - z_s^2}} \partial_s, \\ e_2 &= -\frac{1}{\sqrt{G^2(1 - z_t^2) - z_s^2}} \Big(\frac{z_s z_t}{\sqrt{G^2 - z_s^2}} \partial_s + \sqrt{G^2 - z_s^2}\; \partial_t \Big) \end{aligned} \tag{1.13}$$

form a smooth unit orthonormal frame. Furthermore, e_1 is tangent to $\partial\Omega$ and running in the anticlock direction and e_2 is the interior normal. It turns out that the geodesic curvature of $\partial\Omega$ with respect to $\tilde{g}$

$$\begin{aligned} \tilde{k}_g &= \langle e_2, \nabla_{e_1} e_1 \rangle \\ &= \frac{k_g(1 - |\nabla z|^2) + z_t(z_{ss} - \Gamma_{11}^1 z_s - \Gamma_{11}^2 z_t)G^{-2}}{\sqrt{(1 - |\nabla z|^2)}(1 - z_s^2)^{3/2}} \\ &\ge \frac{k_g \sqrt{(1 - |\nabla z|^2)}}{(1 - z_s^2)^{3/2}}, \quad \text{at } t = 0, \end{aligned} \tag{1.14}$$

since $z_t \ge 0$ and z is geodesically convex. This proves the present lemma.

Lemma 1.7. *Let $g \in C^2(\overline{\Omega})$ be a positive curvature metric and let $\partial\Omega$ be of nonnegative geodesic curvature. Suppose that a C^2-convex cap $\Sigma : \overline{\Omega} \longrightarrow R^3$ is an isometric embedding in R^3 of $(g, \overline{\Omega})$, then the total mean curvature of Σ,*

$$\int_\Sigma H \le C\Big(L, |k|_0, \frac{1}{\int_\Omega k d\sigma}, \int_\Omega d\sigma\Big), \tag{1.15}$$

where L is the intrinsic arclength of $\partial\Omega$.

Proof. Suppose that Σ: $\vec{r} = (x, y, z)$ is convex with respect to the axis z and $\partial\Sigma$ lies on the plane $z = 0$. Denote by Π the projection on the plane $z = 0$ of Σ. First of all we show that Π contains a disk with radius r_0 of positive lower bound depending only on $\int_\Sigma k$, L and $\max k$. Indeed Lemma 1.6 tells us the geodesic curvature $\tilde{k}_g$ (indeed, curvature) of $\partial\Pi$ is nonnegative and hence Π is convex. And we can directly compute the area of Π,

$$\begin{aligned}|\Pi| &= \int_\Sigma \cos(\vec{n}, \vec{k}) ds = \int_{G(\Sigma)} \cos(\vec{n}, \vec{k}) \frac{d\omega}{k} \\ &\geq \frac{1}{4\pi \max k} \Big(\int_\Sigma k \Big)^2,\end{aligned}$$

where G is the spherical map: $\Sigma \longrightarrow S^2$. On the other hand, the arclength of $\partial\Pi$ = the arclength L of $\partial\Sigma$. Then the width of the convex domain Π, $d \geq |\Pi|/L$ and it is easy to find a triangle in Π for which one side $\geq d$ and the height to this side $\geq d/2$. It turns out that there is a disk in this triangle with the radius $r_0 \geq d^2/8L \geq |\Pi|^2/8L^3$. Thus we have proved the previous assertion.

Lift the center of this disk along the z axis to the plane $z = r_0$ and take this point as the new origin. If no confusion occurs, we still denote the position vector of Σ by $\vec{r}$. Set $w = -|\vec{r}|^2/2$. The Minkowsky function of Σ,

$$p = \langle -\vec{r}, \vec{n}_\Sigma \rangle = \sqrt{-2w - |\nabla w|^2},$$

where $\vec{n}_\Sigma$ is the interior normal of Σ. By the geometric meaning of p and the special choice of the origin, we know $p \geq r_0$ and p, $|\nabla w| \leq |\vec{r}|$ everywhere on Σ. Using the Gauss equation we have

$$\nabla_{ij} w + g_{ij} = \Omega_{ij} p, \quad 1 \leq i, j \leq 2,$$

which implies $H = (\Delta_g w + 2)/2p$. It turns out that, after the integration by parts,

$$2 \int_\Sigma H = \int_\Sigma \frac{\Delta_g w + 2}{\sqrt{-2w - |\nabla w|^2}} \leq r_0^{-1} \Big(\int_{\partial\Sigma} |\nabla w| + 2|\Sigma| \Big) \leq C$$

for some constant C depending only on the quantities mentioned in (1.15).

§2. The Existence of Global Smooth Convex Cap

In this section we shall use the continuity method to construct the required solution to the problem (0.1) with (0.2) and (0.3) and the isometric embedding in Theorem A and Theorem B. First of all we shall give a family of conformal metrics of the given metric g. Without loss of generality we may assume that Ω is just the unit disk D and moreover by the uniformization theorem, in global isothermal coordinates the metric g is of the form

$$g = E(du^2 + dv^2) \text{ in } \overline{D}$$

for some positive function $E \in C^\infty(\overline{D})$. Set

$$g_\lambda = E^\lambda \tilde{E}^{(1-\lambda)}(du^2 + dv^2), \ \lambda \in [0, 1], \quad \text{where } \widetilde{E} = (9 - r^2)^2. \tag{2.1}$$

By a direct computation it is easy to see the curvature of the metric g_0,

$$k_0 = 36/(9 - r^2)^4 \geq 1/200$$

and the geodesic curvature of ∂D, $k_{g_0} = 3/32$. Furthermore, for each g_λ its curvature

$$k_\lambda = \lambda k \left(\frac{E}{\widetilde{\widetilde{E}}}\right)^{(1-\lambda)} + (1-\lambda)k_0 \left(\frac{\widetilde{E}}{E}\right)^{\lambda}$$

and the geodesic curvature in g_λ of ∂D,

$$k_{g_\lambda} = \lambda k_g \left(\frac{E}{\widetilde{\widetilde{E}}}\right)^{(1-\lambda)/2} + (1-\lambda)k_{g_0} \left(\frac{\widetilde{E}}{E}\right)^{\lambda/2}.$$

Therefore there are two positive constants C and δ independent of λ such that for all $\lambda \in [0,1]$

$$|k_\lambda|_2 \le C \text{ and } k_\lambda \ge \frac{1}{C}\min\{k_0, k\} \text{ on } \overline{D},\ k_{g_\lambda} \ge \frac{1}{C} \text{ on } \partial D \tag{2.2}$$

and

$$g_\lambda = dt^2 + G_\lambda^2 ds^2,\ -3\delta \le t \le 0,\ \text{where } G_\lambda(s,0) = 1 \text{ and } \partial_t G_\lambda(s,0) = k_{g_\lambda}.$$

Set $S = \{\lambda \in [0,1] |$ for g_λ the problem (0.1) with (0.2) and (0.3) has smooth solution $z_\lambda\}$. Obviously S is not empty since both of g_0 and the domain D are all radius symmetric and it is easy to find the radius symmetric solution to the problem (0.1) with (0.2) and (0.3)

$$z_0 = -\int_r^1 \sqrt{1 - \exp\left[-\int_0^r \frac{2r(9-r^2)^3}{(9-3r^2)} k_0 dr\right]}\,(9-r^2)dr.$$

Lemma 2.1. *Let* (0.4),(0.5) *be fulfilled. Then for each $\lambda \in S$ the following inequality*

$$|\nabla z_\lambda| \le C\Big(L, |k|_1, \frac{1}{\int_D k d\sigma}, \frac{1}{\min_{\partial D} k_g}, \int_D d\sigma, \frac{1}{\min_{D_\delta} k}, \frac{1}{\delta}\Big) < 1 \tag{2.3}$$

holds, where δ is mentioned in (2.2) *and D_δ denotes the set of all the points of the distance $> \delta$ in g_λ from the boundary ∂D and L is the intrinsic arclength of ∂D.*

Proof. From Lemma 1.6, Lemma 1.5 and the definition of S, it follows that for each $\lambda \in S$ one can find x_λ and y_λ in $C^\infty(\overline{D})$ (i.e. the normal coordinates corresponding to the exponential map in Lemma 1.5) such that (x_λ, y_λ) is a diffeomorphism defined on $\overline{D}$ and the graph $\vec{r}_\lambda = (x_\lambda, y_\lambda, z_\lambda)$ is an isometric embedding of $(g_\lambda, \overline{D})$ in R^3. Under the geodesic coordinates, by (2.2) we have

$$\sup_{-3\delta \le t \le 0} \frac{2G_\lambda k_\lambda}{\partial_t G_\lambda} \le M(L, |k|_0, 1/\min k_g).$$

Set

$$\phi = -\int_t^0 \sqrt{1 - e^{-M(R+\tau)}}d\tau,\ -3\delta \le t \le 0. \tag{2.4}$$

A simple computation yields

$$\det(\nabla_\lambda^2 \phi) \ge k_\lambda G_\lambda^2 (1 - |\nabla_\lambda \phi|^2),\ -3\delta \le t \le 0 \tag{2.5}$$

for arbitrary $R > 3\delta$.

Next we choose R so large that ϕ is a subbarrier as $-3\delta \le t \le 0$ to (0.1) with (0.2). Since $\vec{r}_\lambda$ is a convex cap, Lemma 1.7 and (2.2) provide us with uniformly above bound for the total mean curvature $\int H_\lambda$ for all $\lambda \in S$. Thus Heinz's interior estimation [8, Satz 3] (also [18, p. 102]) provides us that

$$|D^2 \vec{r}_\lambda| \le C \quad \text{on } D_{2\delta}$$

for some constant C depending only on $1/\delta$, L, $|k|_1$, the total mean curvature and the reciprocal of the lower bound of k over D_δ. Thus with the aid of Lemma 1.7 it is easy to see this constant C controlled by the quantities mentioned in (2.3) and independent of λ. So their principal curvatures k^i_λ, $i = 1, 2$ satisfy

$$C \geq k^i_\lambda \geq \frac{1}{C} \quad \text{on } \vec{r}(\overline{D}_{2\delta})$$

for another constant C depending only on the same quantities in (2.3). Therefore the distance of the image of the spherical mapping $G(\vec{r}_\lambda(\partial D_{3\delta}))$ from the equator $\geq \delta/(2C)$ since $\vec{r}_\lambda$ is a convex cap and the spherical mapping for convex surfaces is homeomorphic. Consequently

$$|\nabla_\lambda z_\lambda|^2 = 1 - \langle \vec{n}_\lambda, \vec{k}\rangle^2 \leq 1 - \sin^2\Big(\frac{\delta}{2C}\Big) = q < 1 \quad \text{on } \partial D_{3\delta}. \tag{2.6}$$

Now let us fix the constant R so large that

$$\sqrt{1 - e^{-M(R-3\delta)}} \geq q, \tag{2.7}$$

which implies

$$\phi_t \geq \partial_t z_\lambda \quad \text{on } t = -3\delta. \tag{2.8}$$

With the aid of (2.5), (0.1), (2.8) and the fact that $\phi = z_\lambda = 0$ on ∂D, an application of the maximum principle to $\phi - z_\lambda$ on the region $-3\delta \leq t \leq 0$ provides at once

$$\phi \leq z, \text{ as } -3\delta \leq t \leq 0. \tag{2.9}$$

Thus, we have

$$1 > \phi_t(0) = \sqrt{1 - e^{-MR}} \geq \sup_{\partial D} \partial_t z_\lambda.$$

This completes the proof of the present lemma since $|\nabla z_\lambda|$ attends its maximum on ∂D.

The following a priori estimates are very useful and, indeed, do not depend on Lemma 2.1.

Lemma 2.2. *Let* (0.4),(0.5) *be fulfilled and let* $z \in C^4(\overline{\Omega})$ *be a geodesic convex solution to* (0.1) *with* (0.2) *and* $|\nabla z| < 1$ *in* Ω. *Then*

$$\max_{\overline{\Omega}} |D^2 z| \leq C\Big(\max_{\partial\Omega} |D^2 z|, L, |k|_2, \frac{1}{\min_{\partial\Omega} k_g}\Big). \tag{2.10}$$

It is worth pointing out that the bounds given in (2.10) are independent of the lower bound of the Gauss curvature. We can take this advantage to the nonnegative curvature case.

Proof of Lemma 2.2. Throughout the argument in proving the present lemma, we shall use the orthonormal frame e_i and its dual frame ω_i, $i = 1, 2$ at the point considered and all the derivatives are covariant derivatives. Denote by Δ the Laplace-Beltrami operator Δ_g. A differentiation of (0.1) yields, with $f = k(1 - |\nabla z|^2)$ and $F^{ij} = \partial \det(\nabla^2 z)/\partial z_{ij}$,

$$F^{ij} z_{ijk} = f_k, \; k = 1, 2. \tag{2.11}$$

A differentiation of (2.11) again gives

$$F^{ij} z_{ijkk} = f_{kk} - \frac{1}{f}(F^{ml}F^{ij} - F^{im}F^{lj}) z_{ijk} z_{mlk}. \tag{2.12}$$

By means of the Ricci identity and the chain rules we have

$$z_{ijk} - z_{ikj} = z_m R_{mijk}, \tag{2.13}$$

$$\begin{aligned} z_{ijkk} - z_{kkij} &= 2z_{mk}R_{mijk} + z_{mj}R_{mkik} + z_{mi}R_{mkjk} \\ &\quad + z_m R_{mijk,k} + z_m R_{mkik,j}, \end{aligned} \tag{2.14}$$

where R_{mijk} and $R_{mijk,k}$ are the components of Riemannian tensor and their covariant derivatives. Inserting (2.13) (2.14) into (2.12) one can obtain

$$\begin{aligned} F^{ij}z_{kkij} &= f_{kk} - \frac{1}{f}(F^{ml}F^{ij} - F^{im}F^{lj})z_{ijk}z_{mlk} - 2F^{ij}z_{mk}R_{mijk} \\ &\quad - 2fk - (z_m R_{mijk,k} + z_m R_{mkik,j})F^{ij}, \end{aligned} \tag{2.15}$$

and

$$\Sigma f_{ll} = -2kz_l(\Delta z)_l - 2k(\Delta z)^2 + 0(1)(\Delta z + 1).$$

Now we are in a position to estimate the above bounds over $\overline{\Omega}$ for $|D^2 z|$. Consider an auxiliary function

$$w = \ln \Delta_g z + \psi,$$

where ψ is the function constructed in Lemma 1.3. To prove (2.10) we only need to discuss the case where w attains its maximum at some point $p \in \Omega$. If at the point p,

$$|\Delta z| \le \max\{1, \sqrt{8f}\},$$

(2.10) is trivial. Therefore it remains to discuss the case $|\Delta z| \ge \max\{1, \sqrt{8f})$ at the point p. Under the orthonormal frame near this point, without loss of generality, we may also assume $z_{12} = 0$ at p. It turns out that

$$|z_{11} - z_{22}| = \sqrt{(\Delta z)^2 - 4f} \ge \frac{1}{2}\Delta z \ge \frac{1}{2}, \quad \text{at } p, \tag{2.16}$$

and

$$w_i = \frac{(\Delta z)_i}{\Delta z} + \psi_i = 0, \quad i = 1, 2, \quad \text{at } p. \tag{2.17}$$

On the other hand, solving the following system, $k = 1, 2$,

$$F^{ll}z_{llk} = f_k \text{ and } z_{llk} = (\Delta z)_k,$$

we have

$$z_{11k} = \frac{f_k - F^{22}(\Delta z)_k}{F^{11} - F^{22}}, \tag{2.18}$$

$$z_{22k} = \frac{f_k - F^{11}(\Delta z)_k}{F^{22} - F^{11}}. \tag{2.19}$$

Hence

$$\begin{aligned} &- \frac{1}{f}(F^{ml}F^{ij} - F^{im}F^{lj})z_{ijk}z_{mlk} \\ &= 2(z_{12k}z_{12k} - z_{11k}z_{22k}) \\ &= 2\frac{\Delta z F^{ii}(\Delta z)_i^2 - f_i \Delta z(\Delta z)_i - 2F^{ii}(\Delta z)_i f_i + 2f_i f_i}{(F^{11} - F^{22})^2} \\ &\quad + 4k\frac{F^{11}(\Delta z)_1 z_1 - F^{22}(\Delta z)_2 z_2 + z_2 f_2 - z_1 f_1}{(F^{11} - F^{22})^2} + 2k^2|\nabla z|^2 \end{aligned}$$

which, in view of (2.16) and (2.17), is bigger than or equals

$$2\frac{\Delta z F^{ii}(\Delta z)_i^2}{(F^{11}-F^{22})^2} - C(1+\Delta z) \tag{2.20}$$

for some constant C depending only on bounds on $\overline{\Omega}$ of k, $|\nabla k|$ and $|\nabla \psi|$. On the other hand, combining (2.15) with (2.20) we have, at the point p,

$$\begin{aligned} 0 \geq \Delta z F^{ii} w_{ii} &\geq F^{ii}(\Delta z)_{ii} - \frac{F^{ii}(\Delta z)_i^2}{\Delta z} + \Delta z F^{ii}\psi_{ii} - C(1+\Delta z) \\ &\geq \frac{\Delta z + 4f}{(F^{11}-F^{22})^2} F^{ii}(\Delta z)_i^2 + \lambda_{min}(D^2\psi)(\Delta z)^2 - C'(1+\Delta z), \end{aligned} \tag{2.21}$$

where the constant C' also depends on, in addition to the quantities mentioned above, the bounds on $\overline{\Omega}$ of $|D^2 k|$. Thus we have proved (2.10) if we note that the bounds for $|\nabla \psi|$ and the reciprocal of the lower bound of $\lambda_{\min}(D^2\psi)$ are controlled by some constant depending on $L, |k|_1, 1/\min k_g$. This ends the proof.

Naturally one also expects to get an estimation for the above bound on $\partial\Omega$ of $|D^2 z|$, independent of the lower bound of the Gauss curvature. This is possible for the solutions to the problem (0.1) with (0.2) and $|\nabla z| < 1$ in Ω. Consider the following problem

$$\Delta_g \tilde{\psi} = 2\sqrt{k}, \text{ in } \Omega \text{ and with } \tilde{\psi} = 0 \text{ on } \partial\Omega.$$

Define

$$\psi^+ = \tilde{\psi}/\sqrt{1 + \max|D\tilde{\psi}|^2}.$$

From the Hopf lemma it is easy to see

$$\min_{\partial\Omega} \frac{\partial \psi^+}{\partial n} > 0 \tag{2.22}$$

as long as k is not identically zero. And a direct computation at once provides

$$\Delta_g \psi^+ \leq 2\sqrt{k(1 - |\nabla \psi^+|^2)}, \text{ in } \Omega \text{ with } \psi^+ = 0 \text{ on } \partial\Omega. \tag{2.23}$$

Lemma 2.3. *Let the assumption in Lemma* 2.2 *be fulfilled. Then the following inequality*

$$\max_{\partial\Omega} |D^2 z| \leq C\Big(L, |k|_2, \frac{1}{\min k_g}, \frac{1}{\min \frac{\partial \psi^+}{\partial n}}\Big) \tag{2.24}$$

is valid.

Proof. For a given point $p \in \partial\Omega$ we take the normal coordinates centered at p and the interior normal at p as the y axis. Sometimes denote (x, y) by (x_1, x_2). It is easy to see

$$z_{xx}(0) + z_y(0)k_g(0) = 0$$

and hence $|z_{xx}(0)|$ is controlled by the right hand side of (2.24). To estimate $|z_{xy}(0)|$ we take a smooth tangent vector field $X = b^j\partial_j$ and act it on (0.1). As a result, we can get, with $F^{ij} = \partial(\det(\nabla^2 z))/\partial z_{ij}$ and $g = \det(g_{ij})$,

$$\begin{aligned} L(Xz) &= g^{-1}F^{ij}\nabla_{ij}(Xz) \\ &= (Xk)(1 - |\nabla z|^2) + A_0 f + F^{ij}A_{ij} + g^{-1}F^{ij}b^k(\nabla_{ijk}z - \nabla_{kij}z). \end{aligned}$$

Here and later A_i, A_{ij} $i, j = 0, 1, \cdots$ always denote the constants under control and A_i, A_{ij} may be different from line to line. By the Ricci identity (2.21) we have

$$\begin{aligned} L(Xz) &= g^{-1}F^{ij}\nabla_{ij}(Xz) \\ &= (Xk)(1 - |\nabla z|^2) + A_0 f + F^{ij}A_{ij} = f_1 \end{aligned} \tag{2.25}$$

for other A_{ij}. Since X is a tangent vector, it turns out that $|Xk| \leq C\sqrt{k}$. By the geometric and arithmetic mean theorem it is easy to see

$$|f_1| \leq C(\sqrt{f} + F^{11} + F^{22}) \leq C'(F^{11} + F^{22}).$$

With the aid of the geodesically convex smooth function ψ obtained in Lemma 1.3, without difficulty, one can show that $(Xz)\pm\lambda\psi$ are the subbarrier and the superbarrier for sufficiently large λ which is also controlled by the right hand side of (2.24). Thus the bound of $|z_{xy}(0)|$ can be obtained by a simple application of the maximum principle to the above subbarrier and superbarrier. To estimate the bound of $|z_{yy}(0)|$, from (0.1) we first derive

$$\Delta_g z \geq 2\sqrt{k(1-|\nabla z|^2)} \text{ in } \Omega \text{ with } z = 0 \text{ on } \partial\Omega. \tag{2.26}$$

With the aid of (2.23) and (2.26) the following inequality

$$\Delta_g(\psi^+ - z) + 2\sqrt{k}\frac{g^{ij}\psi_i^+(\psi_j^+ - z_j) + g^{ij}z_j(\psi_i^+ - z_i)}{\sqrt{1-|\nabla z|^2} + \sqrt{1-|\nabla\psi^+|^2}} \leq 0 \tag{2.27}$$

is valid. Thus in view of (2.27), the maximum principle implies

$$\min_{\partial\Omega}\frac{\partial\psi^+}{\partial n} \leq \frac{\partial\psi^+}{\partial n} \leq \frac{\partial z}{\partial n} \quad \text{on } \partial\Omega. \tag{2.28}$$

Therefore, under the geodesic coordinates as used in (1.2),

$$\nabla_{11}z = z_{ss} + k_g z_t = k_g\frac{\partial z}{\partial t} \geq \min k_g \min\frac{\partial\psi^+}{\partial n} \quad \text{on } \partial\Omega$$

Combining the last inequality with (0.1) yields the above bound for $|z_{yy}(0)|$ which is also controlled by the right hand side of (2.24). This completes the proof of the present lemma.

Proof of Theorem A. Obviously, the set S is open. It suffices to prove the closeness. Indeed, from Lemma 2.1 it follows that

$$|\nabla z_\lambda| \leq q < 1 \quad \text{on} \quad \overline{\Omega} \quad \text{for all } \lambda \in S$$

and hence, the right hand side in (0.1) has uniformly positive lower bound for all g_λ , $\lambda \in S$. And Lemma 2.2 and Lemma 2.3 with the concrete construction of g_λ guarantee that the linearized operator of (0.1) for each g_λ , $\lambda \in S$ is uniformly elliptic. Therefore so is (0.1). A standard $C^{2+\alpha}$ estimation for completely nonlinear uniformly elliptic problem soon gives the closeness of the set S. This proves that if (0.4) and (0.5) are satisfied, (0.1) with (0.2) and (0.3) always admits a global smooth solution. Combining Lemma 1.5 and Lemma 1.6 one can soon completes the proof of Theorem A and moreover, (2.3) holds for the solution z.

Proof of Theorem B. Let σ be a smooth solution of the equation

$$\Delta_g\sigma = -1 \quad \text{in} \quad \Omega \quad \text{and} \quad \sigma < 0.$$

For the conformal metric $g_\epsilon = e^{2\epsilon\sigma}g$, its Gauss curvature satisfies

$$k_\epsilon = e^{-2\epsilon\sigma}(k+\epsilon) > 0, \quad k_\epsilon \geq \frac{1}{2}k, \tag{2.29}$$

for sufficiently small ϵ. Similarly we can define ψ_ϵ^+ for g_ϵ as before and assume that

$$\frac{\partial\psi_\epsilon^+}{\partial n} \geq \frac{1}{2}\min\frac{\partial\psi^+}{\partial n} \quad \text{on } \partial\Omega \tag{2.30}$$

if ϵ is very small. Denote by Ω_δ the set of all the points of the distance in the metric g_ϵ from $\partial\Omega > \delta$. Obviously, we may also assume that there is a positive constant δ such that the geodesic coordinate system with the base curve $\partial\Omega$ for each metric g_ϵ covers $\overline{\Omega}\setminus\Omega_\delta$ if ϵ is very small. Let (0.5) and (0.6) be fulfilled. Then by the continuity we can also assume that

$$k_g^\epsilon \geq \frac{1}{C} \text{ and } |\nabla k_\epsilon| \geq \frac{1}{C} \text{ on } \partial\Omega \text{ if } \epsilon \in (0, \epsilon_0] \tag{2.31}$$

for some positive constants C and ϵ_0. Using Theorem A, we can find a smooth isometric embedding: $\overline{\Omega} \longrightarrow \vec{r}_\epsilon = (x_\epsilon, y_\epsilon, z_\epsilon)$ subject to

$$\det(\nabla_{ij}^\epsilon z_\epsilon) = k_\epsilon g_\epsilon(1 - |\nabla^\epsilon z_\epsilon|^2) \quad \text{in } \Omega \text{ with } z_\epsilon = 0 \quad \text{on } \partial\Omega, \tag{2.32}$$

and furthermore, Lemma 2.2, Lemma 2.3 and Lemma 2.1 provide

$$|D^2 z_\epsilon| \leq C, \quad 1 - |\nabla^e z_\epsilon|^2 \geq \frac{1}{C} \tag{2.33}$$

for another constant C since under present circumstance, the above bounds of $L_\epsilon, |k_\epsilon|_2$, and the reciprocals of the $\min k_g^\epsilon$ over $\partial\Omega$, $\min k_\epsilon$ over Ω_δ and the total curvature over Ω are bounded above by a constant independent of ϵ and meanwhile

$$\frac{\partial z_\epsilon}{\partial n} \geq \frac{\partial \psi_\epsilon^+}{\partial n} \geq \frac{1}{2} \min \frac{\partial \psi^+}{\partial n} \qquad \text{on } \partial\Omega.$$

The main difficulty we are faced with consists in the curvature k vanishing on $\partial\Omega$. It is impossible to find a uniformly positive lower bound for the eigenvalues of the Hessian $\nabla^2 z_\epsilon$. To bypass this difficulty we note that the present degeneracy on $\partial\Omega$ of k is just of order 1, i.e., $dk \neq 0$, and hence the assumptions in [11, p.417] and [12, Theorem B] are satisfied. Thus one can find a function $\omega(t)$: $\overline{R}_+^1 \longrightarrow \overline{R}_+^1$ with $\omega(0) = 0$ and $\omega(t) \longrightarrow 0$ as $t \longrightarrow 0$, such that

$$|D^\alpha z_\epsilon(p) - D^\alpha z_\epsilon(p')| \leq C\omega(|p - p'|), \quad p, p' \text{ on } \overline{\Omega} \text{ and } |\alpha| = 2 \tag{2.34}$$

for some constant C depending only on the following quantities

$$L, |k|_3, \frac{1}{\int_\Omega k d\sigma}, \frac{1}{\min k_g}, \frac{1}{\min \frac{\partial \psi+}{\partial n}}, \frac{1}{\min_{\partial\Omega} |\nabla k|} \text{ and } \frac{1}{\min_{\Omega_\delta} k}.$$

So far we have proved the set $\{z_\epsilon\}$, $\epsilon \in (0, \epsilon_0]$ equicontinuous in $C^2(\overline{\Omega})$, namely, after extracting a subsequence we have $z_\epsilon \longrightarrow$ a solution z in $C^2(\overline{\Omega})$ to the problem (0.1) with (0.2) and (0.3). Moreover

$$1 - |\nabla z|^2 \geq 1/C.$$

Now [14, Main Theorem] (also [10, Theorem 3.1 and Theorem 3.2]) tells us that this geodesically convex solution in $C^2(\overline{\Omega})$ is smooth up to the boundary. This ends the proof for Theorem B.

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