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PARTIAL REGULARITY FOR OPTIMAL DESIGN PROBLEMS INVOLVING BOTH BULK AND SURFACE ENERGIES**

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Abstract

This paper studies a class of variational problems which involving both bulk and surface energies. The bulk energy is of Dirichlet type though it can be in very general forms allowing unknowns to be scalar or vectors.The surface energy is an arbitrary elliptic parametric integral which is defined on a free interface. One also allows other constraints such as volumes of partitioning sets. One establishes the existence and regularity theory, in particular, the regularity of the free interface of such problems.

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§0. Introduction

In this article we study a large class of nonlinear variational problems involving both bulk and interface energies. Such problems arise in many applied sciences, such as nonlinear elasticity, material sciences and image segmentations in the computer version theory (see for example [1, 5, 15, 18]). The regularity of solutions to these problems is often a rather subtle issue. In [17], the second author first established the regularity theory for a simple model problem. The present paper is a natural extension of that work. It has substantially improved [17] in several aspects. For example, it allows much more general bulk and interfacial energies and it also allows to have other constraints such as volume of partitioning sets. Therefore, it makes the theory more applicable. The preprint of the present paper has been circulated since early 1993. Both authors believed the other has submitted the paper for its publication until recently. They found out it has not been submitted anywhere. Though more than five years have passed, we find the work remains to be of interest. Indeed, it is still the only work treat this sort of problem in such great generality. In the last couple years, there are some rather nice regularity results proved for image segmentation problems (free discontinuity problems) (see for examples [2, 3, 10]). However, the problems treated here are somewhat different from these works and, in many aspects, it is more general. As in [17], all results remain valid for the vector valued case. For the details we refer to section 1 below.

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§1. Preliminaries and the Statement of Main Results

(a) The Variational Problems

Let Ω be a bounded, Lipschitz domain in $\mathbb{R}^n$, and let $\phi : \Omega \to R$ be an $H'(\Omega)$ function. For a given constant $\beta \in (0,1)$, we want to find a measurable subset A of Ω and a function u in $H'(\Omega)$ such that the pair (A, u) minimizes

$$\int_\Omega [F(x, u, \nabla u(x)) + \chi_A(x)\, G(x, u\nabla u(x))]\, dx + \Psi(\partial\Omega) \tag{1.1}$$

subject to the following constraints

$$u = \phi \text{ on } \partial\Omega, \text{ and } \mathrm{vol}\,(A) = \beta\ \mathrm{vol}\,(\Omega). \tag{1.2}$$

Here vol (A) denotes the Lebesgue measure of A, χ_A is the characteristic function of A, and Ψ is an elliptic parametric integral (cf. [8, §5.1]) which will be specified below. Here we also require the integrands F, G to be in class $\mathcal{H}$ where we say a function $H(x, u, p) = \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is in the class $\mathcal{H}$ if H satisfies:

$$\text{(smoothness) } H \in \mathbb{C}^{\ell,\eta}\left(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n\right),\ \ell \geq 2,\ 0 < \eta < 1, \tag{1.3}$$

i.e., H has derivatives up to order ℓ which are Hölder continuous with Hölder exponent η;

$$\text{(Legendre condition) } \lambda|\xi|^2 \leq H_{p_i p_j}(x, u, p)\, \xi_i \xi_j \leq \lambda^{-1} |\xi|^2 \tag{1.4}$$

for some constant $\lambda \in (0,1)$, and for all $\xi \in \mathbb{R}^n$, $(x, u, p) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n$;

(controlled growth)

$$|H_{pu}| + |H_{px}| \leq M(1 + |p|), \quad |H_{uu}| + |H_{ux}| + |H_{xx}| \leq M(1 + |p|^2) \tag{1.5}$$

for some constant $M > 0$.

The typical example of integrands in (1.1) is given by

$$\begin{cases} F(x, u, p) = \sum a_{ij}(x, u)\, p_i p_j + a_k(x, u)\, p_i + a(x, u), \\ G(x, u, p) = \sum b_{ij}(x, u)\, p_i p_j + b_i(x, u)\, p_i + b(x, u), \end{cases} \tag{1.6}$$

where a_{ij}, b_{ij}, a_i, b_i, a, b belong to $C^{\ell,r}$ and (a_{ij}), (b_{ij}) are positive definite, $1 \leq i, j \leq n$.

Though we study only those variational problems (1.1) involving a scalar function in this paper, readers can easily see that all the results can be generalized to the case that u is a vector valued function. In particular, our main results valid for variational problems (optimal design) arise in linear or nonlinear elasticity[1].

(b) The surface energy $\Psi(\partial A)$

To explain the surface energy term $\Psi(\partial A)$ in (1.1) we need to introduce a few notations. For the basic definitions and properties of normal and integral currents, we refer to the classical paper of Federer and Fleming[9]; a complete account of this theory is, of course, in [8]. Our currents $\mathbf{T}$ will be so-called oriented boundaries (see [8, §4.4]). In particular, they will have dimensions $m \equiv n - 1$. We are interested in functionals (or interfacial energies) Ψ on the space of integral currents $\mathbf{I}_m(\mathbb{R}^n)$ with compact support in $\overline{\Omega}$.

A parametric integrand of degree m on an open set $\Omega \subseteq \mathbb{R}^n$ is a continuous real-valued function $\Psi(x, \vec{\alpha})$ defined for $x \in \Omega$ and $\vec{\alpha} \in \mathbb{R}^n$ which is homogeneous of degree 1 in $\vec{\alpha}$, hence $\Psi(x, t\vec{\alpha}) = t\Psi(x, \vec{\alpha})$ for $t > 0$. Thus Ψ is determined by its restriction on $\Omega \times \mathbb{S}^m$. Moreover, for $\mathbf{T} \in \mathbf{I}_m(\mathbb{R}^n)$, we can define $\Psi(\mathbf{T}) = \int \psi(x, \vec{T}(x))\, d\|\mathbf{T}\|(x)$. Ψ thus defined is called a parametric integral.

We shall say Ψ is of class $\mathbb{C}^\ell$, if this is so, for the restriction of the function $\Psi(x,\vec{\alpha})$ to $\Omega \times \mathbb{S}^m$. In this paper we always assume $\ell \geq 2$. Thus there is a nonnegative monotone increasing function $v(t)$, with $v(t) \to 0$ as $t \to 0$, with the following properties:

For $x_1,\, x_2 \in \overline{\Omega}$ and $\vec{\alpha} \in \mathbb{S}^m$ we have

$$|\Psi\,(x_1,\vec{\alpha}) - \Psi\,(x_2,\vec{\alpha})| \leq \Lambda\,|x_2 - x_1| \tag{1.7}$$

for some constant $\Lambda > 0$.

For every $(x_0,\vec{\alpha}_0) \in \overline{\Omega}\times\mathbb{S}^m$ there is a $\xi_0 \in \mathbb{R}$ with $|\xi_0| \leq \Lambda$ such that for all $(x,\vec{\alpha}) \in \overline{\Omega}\times\mathbb{S}^n$ we have

$$|\Psi(x,\vec{\alpha}) - \Psi(x_0,\vec{\alpha}_0) - \langle \xi_0,\, \vec{\alpha}-\vec{\alpha}_0\rangle| \leq \Lambda\left(|x-x_0| + |\vec{\alpha}-\vec{\alpha}_0|^2\right). \tag{1.8}$$

For every $x_0 \in \overline{\Omega}$ and every reference frame $\{e_1,\, e_2, \cdots,\, e_n\}$ in $\mathbb{R}^n$ centered at x_0, the associated nonparametric integrand $\Psi^*(\mathbf{x},y,p)$ (see [8, §5.1]) has the following bounds:

(i) for all $(\mathbf{x},y) \in \overline{\Omega}$ and all $|p| \leq 1$, Λ is a common bound for Ψ^*, $\frac{\partial \Psi^*}{\partial\, p_i}$, $i = 1, \cdots, n-1$, and therein first derivatives.

(ii) for all $(x,y) \in \overline{\Omega}$ and all $|p| < 1$, and for every $|p^0| \leq 1$, one has

$$\begin{aligned} &\Big|\frac{\partial \Psi^*}{\partial p_i}(x,y,p) - \frac{\partial \Psi^*}{\partial p_i}(0,0,p^0) - \sum_{j=1}^{n-1} \frac{\partial^2 \Psi^*}{\partial p_i\, \partial p_j}(0,0,p^0)(p_j - p_i^0)\Big| \\ &\leq \Lambda(|x| + |y|) + v(|x| + |y| + |p-p^0|)\cdot|p-p^0|. \end{aligned} \tag{1.9}$$

We shall also assume Ψ is Λ-elliptic in Ω. That is, for every $m = n-1$ dimensional disk D and every integer k and $a \in \Omega$, we have

$$\Lambda^{-1}\,[M(kD) + X - M(kD)] \leq \Psi_a(kD+X) - \Psi_a(kD) \tag{1.10}$$

for all rectifiable X with compact support such that $\partial X = 0$. Here Ψ_a be the integrand $\Psi_a(x,\vec{\alpha}) = \Psi(a,\vec{\alpha})$, for $a \in \overline{\Omega}$, $x \in \mathbb{R}^n$ and here $M(T)$ denotes the mass of the current $\mathbf{T}$ (cf. [8]). Note that a special and important case of parametric integrands is the m-dimensional area.

(c) Main results

Our first main result concerning the existence and preliminary regularity properties of solutions to the problem (1.1). The hypothesis on F, G and Ψ in the following theorem can be further weakened (for the precise statements see section 2 below).

Theorem 1.1. *Suppose F and G are in the class $\mathcal{H}$, i.e., satisfy* (1.3)–(1.5)*, and suppose that Ψ satisfies* (1.7)–(1.10). *Then the minimization problem* (1.1)–(1.2) *has a solution (A,u). Moreover, $u \in H'(\Omega) \cap \mathbb{C}^{\frac{1}{2}}(\Omega)$ and ∂A is $(n-1)$ countable rectifiable.*

It is clear that if (A,u) is a minimal solution of (1.1), then u has to minimize the functional

$$\int_\Omega [F(x,u,\nabla u) + \chi_A(x)\, G(x,u,\nabla u)]\, dx. \tag{1.11}$$

In particular, u satisfies the Euler-Lagrange equation

$$\begin{aligned} &\sum_{i=1}^n \frac{\partial}{\partial x_i}\left[F_{u_{x_i}}(x,u,\nabla u) + \chi_A(x)\, G_{u_{x_i}}(x,u,\nabla u)\right] \\ &= F_u(x,u,\nabla u) + \chi_A(x)\, G_u(x,u,\nabla u). \end{aligned} \tag{1.12}$$

For the variational problem (1.11) an extensive theory has been developed and it is known that all minimals bounded in a ball B, which does not intersect with ∂A, belong to $C^{\ell,\eta}(B)$.

This is the consequence of the regularity theory for such problems. This theory is well presented in the books [14] and [19], and it is built on the basic works by DeGiorgi and Moser. The Hölder regularity of minimals of (1.11) can also be derived from the work of Giaquinta and Giusti (see e.g. [12]), who established that quasi-minimals, a generalization of the concept of minimals, belong to the so-called DeGiorgi class for which DiBenedetti and Trudinger[6] proved their Hanack inequality. It will be also clear from our proof that $u \in C^{\frac{1}{2}}(\overline{\Omega})$ provided that ϕ is Lipschitz on $\partial\Omega$ and $\partial\Omega$ is $\mathbb{C}^1$.

The Hölder exponent $\frac{1}{2}$ is critical exponent in the sense that the two terms in the energy functional (1.1) locally have the same dimension $(n-1)$ under appropriate scalings. In fact, if one could show that $u \in \mathbb{C}^{\frac{1}{2}+\eta}(\Omega)$, for some $\eta > 0$, then one would easily conclude that ∂A is a $\mathbb{C}^{1,\alpha}$ (for some $\alpha > 0$) hypersurface away from a relatively closed subset of Ω of Hausdorff $m = n-1$ dimensional measure zero. The latter follows from the general regularity theory for minimizing or almost minimizing currents developed by DeGiorgi, F. Almgren, and many others (see e.g [4, 21]). We should also point out that the singular set of an almost area-minimizing $(n-1)$ current in $\mathbb{R}^n$ has the Hausdorff dimension $\leq n-8$, and which is also optimal.

Our second main result is the almost everywhere regularity of free interfaces.

Theorem 1.2. *Let* (A, u) *be a minimal solution of* (1.1), (1.2). *Then with the same hypothesis of Theorem* 1.1 *and, in addition, that F and G are of form* (1.6)*, we have ∂A is a $\mathbb{C}^{1,\alpha}$-hypersurface in Ω (for some $\alpha > 0$) away from a relatively closed subset Σ of the Hausdorff $(n-1)$-dimensional measure zero.*

The proof of the above theorem, though involves various arguments from the general regularity theory for minimal surfaces aforementioned, contains several important new estimates. The first one is to establish the so-called mass ratio lower bound for the current $\mathbf{T} = \partial A$, from which the Height Bound Lemma (cf. [21]) can be applied. The second key point is to establish an energy-comparison lemma. Such estimate is necessary for our blow up arguments. The third one is certain first variation estimates. It is here we noticed that the blow-up equations (for free interfaces) does not involve any minimizing properties of the current $\mathbf{T} = \partial A$ but to show the convergence of blow-up sequences to the limiting functions which satisfy some linear elliptic equations involving some necessary energy variation estimates.

We should also point out that in Theorem 1.2 we have made an additional hypothesis that F and G are of form (1.6) due to some technical reasons (see §3 for the details). We conjecture that the same statements remain true even without this additional assumption. It is clearly the case if the principle terms F and G are close to some constant multiples of $|\nabla u|^2$ (cf. [17]). Also this hypothesis is only needed in proving the Mass ratio lower bound.

§2. Proof of Theorem 1.1

(a) Existence of Minimizers

From the form of the functional (1.1), we naturally seek a function $u \in H'(\Omega)$ and a set of finite perimeter $A \subset \Omega$ so that the pair (A, u) will minimize (1.1). Here we say $A \subset \Omega$ is of finite perimeter in Ω if χ_A, the characteristic function of A, belongs to the space of functions of bounded variations, $BV(\Omega)$, in Ω (cf. [8. §4.4]).

If $\chi_A \in BV(\Omega)$, then ∂A is an $(n-1)$-dimension integral current. Moreover, the term $\Psi(\partial A)$ is given by

$$\Psi(\partial A) \equiv \int_\Omega \Psi(x, \vec{v}_A(x))\, d\|\partial A\|, \tag{2.1}$$

where $\vec{v}_A$ is the orientation unit normal vector field of ∂A, and $\|\partial A\|$ denotes the total variation measure of ∂A which by a theorem of DeGiorgi and Federer (cf. [8, Chapter 4]) is given by $\mathcal{H}^{n-1}\lfloor\partial A$, i.e., the restriction of the $(n-1)$-Hausdorff measure to the rectifiable set ∂A in Ω.

To prove the existence of a minimizer of (1.1) subject to the constraints (1.2), we note first that (1.3)–(1.5) imply that

(i) Both F and G are continuous in u, p and measurable in x; moreover, they satisfy the inequalities

$$\lambda_0|p|^2 - C_0 \le F(x,u,p),\ G(x,u,p) \le \lambda_0^{-1}\,|p|^2 + C_0 \tag{2.2}$$

for all x, u, p where $\lambda_0 \in (0,1)$, $C_0 > 0$ are constants which may depend on λ, M, etc., in (1.3)–(1.5).

(ii) Let $\{u_i\}$ be a sequence of $H'(\Omega)$ functions and such that $u_i \to u$ weakly in $H'(\Omega)$. Then

$$\int_\Omega F(x,u,\nabla u)\,dx \le \lim_{i\to\infty} \int_\Omega F(x,u,\nabla\, u_i)\,dx \tag{2.3}$$

(similarly for G).

(2.3) can be easily deduced from (2.2) and (1.4) (cf. [11]).

Next we assume that the parametric integrand Ψ satisfies

$$\Psi(a,\vec{\alpha}) \ge \Lambda_0^{-1} \text{ for some constant } \Lambda_0 \in (0,1) \quad \text{for all} \quad (a,\vec{\alpha}) \in \Omega\times\mathbb{S}^m, \quad m = n-1. \tag{2.4}$$

The condition (2.4) is implied by the Λ-ellipticity Ψ.

Theorem 2.1. *Let Ω, ϕ be as above, and suppose that F, G and Ψ satisfy the hypothesis* (2.2)–(2.4). *Then there is a pair (A,u) with $\chi_A \in BV(\Omega)$ and $u \in H'(\Omega)$ such that it minimizes the functional* (1.1) *subject to the constraints* (1.2) *among all such pairs. Moreover, we have $u \in W^{1,q}_{\mathrm{loc}}(\Omega)\cap C^\alpha(\Omega)$ for some constant $\alpha > 0$ and $q > 2$.*

Proof. We use the direct method in the calculus of variations. Let (A, u_i), $\chi_{A_i} \in BV(\Omega)$, $u_i \in H'(\Omega)$, be a minimizing sequence for the energy functional (1.1) subject to (1.2). We let u_i^* be a minimizer of (1.11) with χ_A replacing by χ_{A_i} such that $u_i^* = \phi$ on $\partial\Omega$. The existence of u_i^* is guaranteed via (2.2) and (2.3). Then (A_i, u_i^*) has clearly smaller energy than that of (A_i, u_i) and hence they form a new minimizing sequence of (1.1) which satisfies (1.2).

By (2.4) we conclude that χ_{A_i}'s remain uniformly bounded in $BV(\Omega)$. Thus we may assume, by compactness theorem for BV-functions (cf. [8, Chapter 4]), and by passing to a subsequence if necessary, that $\chi_{A_i} \to \chi_A$ in $L'(\Omega)$ and $\chi_{A_i} \to \chi_A$ weakly in $BV(\Omega)$. Here A is a subset of Ω with finite perimeter in Ω. We also note that the strong convergence of χ_{A_i} to χ_A in $L'(\Omega)$ implies automatically that $|A| = \beta\cdot|\Omega|$.

Next, since u_i^* minimizes (1.11) with $u_i^* = \phi$ on $\partial\Omega$, and since F and G satisfy (2.2), we have u_i^* remains uniformly bounded in $H'(\Omega)$. Moreover, via a theorem of Giaquinta and

Giusti[12] we have u_i^* remains locally uniformly bounded in $\mathbb{C}^\alpha(\Omega) \cap W^{1,q}_{\mathrm{loc}} B(\Omega)$, for some constants $\alpha > 0$ and $q > 2$. Therefore, we may assume (by passing to a subsequence if it is needed) that $u_i^* \to u$ weakly in $H'(\Omega)$. Finally since $\chi_{A_i} \to \chi_A$ in $L'(\Omega)$ and (2.3) we conclude that u minimizes (1.11) with $u = \phi$ on $\partial\Omega$. Therefore (A, u) solves the minimization problem (1.1)–(1.2).

The final conclusion of Theorem 2.1 follows again from [12, Chapter V, §3; Chapter VII, §2].

Remark 2.1. Since A is a set of finite perimeter in Ω, DeGiorgi's Theorem (cf. [8, Chapter 4]) says that $\partial A \lfloor \Omega$ is an $(n-1)$-dimensional countably rectifiable set. That is, $\partial A \lfloor \Omega$ is contained in a countable union of $\mathbb{C}^1$-hypersurfaces in Ω and a set of $\mathbb{H}^{n-1}$-measure zero.

(b) The $\mathbb{C}^{\frac{1}{2}}$-Estimate

Unless otherwise specified, various constants in this subsection and below will depend on n, β, ϕ, Ω, F, G and Ψ. As in [17], in order to show $u \in \mathbb{C}^{\frac{1}{2}}(\Omega)$, we need a series of lemmas.

Lemma 2.1. *Let (A, u) be a minimal solution of* (1.1) *and* (1.2). *These are positive constants C_1, R_1 with the following properties for any $x_0 \in \mathrm{spt}(\partial A) \cap \Omega$, there is a $y \in \Omega \setminus B_{R_1}(x_0)$ such that*

$$\sup_{0<r\le R_1} \frac{1}{r^n} \int_{B_r(y)} |\nabla u|^2 \, dx \le C_1 \tag{2.5}$$

and that

$$\inf_{0<r\le R_1} \frac{1}{r^n} \left|B_r(y) \cap A^C\right| \le \frac{1}{C_1}. \tag{2.6}$$

Proof. Since u minimizes (1.11), and since F, G satisfy (2.2), one has $\int_\Omega |\nabla u|^2 \, dx \le C$, for some C which depends on ϕ, λ_0 and C_0 in (2.2). Let us consider the function $f(x) = |\nabla u|^2(x)$, and let

$$M\, f(x) = \sup_{r>0} \frac{1}{r^n} \int_{B_r(x)\cap\Omega} |f(y)| \, dy, \quad x \in \Omega$$

be the Hardy-Littlewood maximal function of f. Then, for any $t > 0$, the weak L'-estimate holds

$$|\{x \in \Omega : M\, f(x) > t\}| \le \frac{C(n)}{t} \, \|f\|_{L'(\Omega)} \le \frac{C(n)}{t}\, C. \tag{2.7}$$

Now we choose t_0 so large that

$$\frac{C(n)\, C}{t_0} \le \frac{1}{4}\,(1-\beta)\,|\Omega|, \tag{2.8}$$

and hence

$$\left|\left\{x \in A^C : M\, f(x) \le t_0\right\}\right| \ge \frac{3}{4}\,(1-\beta)\,|\Omega|.$$

Next we choose R_0 so small that

$$2^n\, R_0^n \le \frac{1}{4}\,(1-\beta)\,|\Omega| \tag{2.9}$$

and that

$$\sum_{i\in I} |Q_i| \ge \left[1 - \frac{1}{8}\,(1-\beta)\right]|\Omega|. \tag{2.10}$$

Here $\{Q_i\}_{i=1}^{\infty}$ form a decomposition of $\mathbb{R}^n$ into closed cubes of side length R_0, and $I = \{i \in N : Q_i \subset \overline{\Omega}\}$.

We claim there is an $i_0 \in I$ so that

$$|Q_{i_0} \cap E| \geq \epsilon_0 \, |Q_{i_0}|, \quad \epsilon_0 = \frac{1}{8}(1-\beta), \tag{2.11}$$

where $E = \{x \in A^C : M f(x) \leq t\} \setminus Q_{R_0}(x_0)$, $Q_{R_0}(x_0)$ is the cube of side length $2R_0$ and centered at x_0.

In fact, if (2.11) were false for all $i \in I$, then one would have

$$|E| \leq \epsilon_0 \sum_{i\in I} |Q_i| + \Big|\Omega \setminus \bigcup_{i \in I} Q_i\Big| \leq \epsilon_0 |\Omega| + \frac{1}{8}(1-\beta)\,|\Omega| \leq \frac{1}{4}(1-\beta)\,|\Omega|.$$

On the other hand, $|E| \geq \frac{3}{4}(1-\beta)\,|\Omega| - \frac{1}{4}(1-\beta)\,|\Omega| = \frac{1}{2}(1-\beta)\,|\Omega|$, by (2.9) and the definition of the set E, this is a contradiction and thus (2.11) is valid.

Now we would like to show (2.11) implies that there is a point $y \in Q_{i_0} \cap E$ such that

$$\inf_{0<r\leq R_0} \frac{1}{r^n}\,|B_r(y) \cap E| \geq C(n)\,\epsilon_0 \tag{2.12}$$

for some constant $C(n)$. The conclusion of Lemma 2.1 will follow by simply choosing $R_1 = R_0$ and $C_1 = \max\left\{t_0, \frac{1}{C(n)\,\epsilon_0}\right\}$.

To show (2.12), we simply decompose Q_{i_0} diadically into 2^n equal to smaller cubes $\left\{Q_1', Q_1^2, \cdots Q_1^{2n}\right\}$. By (2.11) there exists at least one Q_1^j, $1 \leq j \leq 2^n$, so that $|Q_1^j \cap E| \geq \epsilon_0\,|Q_1^j|$. We choose one of such Q_1^j and repeat the above process. In this way we obtain a sequence of cubes $\{Q_i^*\}_{i=1}^{\infty}$ such that

(i) $Q_1^* = Q_{i_0}$,

(ii) Q_{i+1}^* is one of the 2^n smaller cubes in the diadic decomposition of Q_i^*,

(iii) $|Q_i^* \cap E| \geq \epsilon_0\,|Q_i^*|$, for $i = 1, 2, \cdots$.

Let now $y = \bigcap\limits_{i=1}^{\infty}, Q_i^*$, we want to show y satisfies (2.12). In fact, if $2^{-i} R_0 \leq r \leq 2^{-i+1} R$ then $|B_r(y) \cap E| \geq |Q_{i+k_0}^* \cap E| \geq \epsilon_0\,|Q_{i+k_0}^*| \geq C(n)\,\epsilon_0\,r^n$. Here k_0 is a constant depending only on n so that $2^{-k_0}\sqrt{n} \geq 1$. This completes the proof of Lemma 2.1.

Lemma 2.2. *For $0 < \rho \leq \frac{1}{2+C_1} R_1 \leq 1$, and $x_0 \in \operatorname{spt}(\partial A) \cap \Omega$, one has*

$$\int_{B_\rho(x_0)} \chi_A |\nabla u|^2\,dy + |\partial A \lfloor B_\rho(x_0)| \leq C_2\,\rho^{n-1} \tag{2.13}$$

for a constant C_2 where $|\cdot|$ denotes Hausdorff measure of approximate dimensions.

Proof. For $x_0 \in \operatorname{spt}(\partial A) \cap \Omega$, we choose a $y \in \Omega \setminus B_{R_1}(x_0)$ as in Lemma 2.1. Then there is a $\widetilde{\rho} \leq n\sqrt{2+C_1}\,\rho$ so that $|B_{\widetilde{\rho}}(y) \cap A^C| = |B_\rho(x_0) \cap A|$. We then define a new set $\widetilde{A} = (A \sim B_\rho(x_0)) \cup (B_{\widetilde{\rho}}(y) \cap \Omega) \subseteq \Omega$ so that $\left|\widetilde{A}\right| = |A| = \beta|\Omega|$.

By minimality of the pair (A, u), one obtains

$$\begin{aligned}
&\int_{B_\rho(x_0)} [F(x,u,\nabla u) + \chi_A\,G(x,u,\nabla u)]\,dx + \Psi(\partial A \lfloor B_\rho(x_0)) \\
\leq &\int_{B_\rho(x_0)} F(x,u,\nabla u)\,dx + \Psi(\partial B_\rho(x_0) + \Psi(\partial B_{\widetilde{\rho}}(y)) + \int_{B_{\widetilde{\rho}}(y)} G(x,y,\nabla u)\,dx.
\end{aligned} \tag{2.14}$$

Here, by (2.2), (2.4) one has

$$\int_{B_\rho(x_0)} \left(\lambda_0|\nabla u|^2 - C_0\right)\chi_A\,dx + \Lambda^{-1}|\partial A\cap B_\rho(x_0)| \tag{2.15}$$
$$\le \Lambda\left(|\partial B_\rho(x_0)| + |\partial B_{\widetilde\rho}(0)|\right) + \int_{B_{\widetilde\rho}(y)} \left(\lambda_0^{-1}|\nabla u|^2 + C_0\right)dx.$$

The conclusion of Lemma 2.2 follows easily from (2.15) by choosing $C_2 = C_2(\lambda_0, C_0, n, \Lambda, B_1, R_1)$ suitably.

Lemma 2.3. *There are two positive constants M and $\theta\in(0,\frac12)$ such that, for $0<\rho<R_2$, one has either*

$$\int_{B_{\theta\,\rho}(x_0)} |\nabla u|^2\,dx \le \theta^{n-\frac12}\int_{B_\rho(x_0)}|\nabla u|^2\,dx \tag{2.16}$$

or

$$\int_{B_{\theta\,\rho(x_0)}} |\nabla u|^2 \le M\,C_2(\theta\,\rho)^{n-1}, \tag{2.17}$$

where $x_0\in \operatorname{spt}(\partial A)\cap\Omega$ and $B_{2\rho}(x_0)\subseteq K\subset\subset\Omega$. Here M may also depend on K.

Proof. We first note that, as u is a minimizer of (1.11),

$$\|u\|_{H'(\Omega)} + \|u\|_{\mathbb{C}^\alpha(K)} \le C_K, \tag{2.18}$$

for some constants C_K and $\alpha>0$. We also have, from Lemma 2.2, that

$$\int_{B_{\rho\,(x_0)}} \chi_A|\nabla u|^2\,dx + |\partial A\lfloor B_\rho(x_0)| \le C_2\,\rho^{n-1}, \tag{2.19}$$

for $0\le\rho\le R_2$.

In above inequalities (2.16), (2.17), θ is a constant which depends only on n and λ in (1.4), and which we shall choose later.

Suppose that both (2.16) and (2.17) were not true, there would be sequences of $\rho_i\in(0,R_2)$, $x_i\in\operatorname{spt}(\partial A_i)\cap\Omega$, and a sequence of minimal solutions, (A_i,u_i) of (1.1) so that

$$\int_{B_{\theta\rho_i}(x_i)} |\nabla u|^2\,dx \ge i\,C_2\,\rho_i^{n-1}, \quad \text{for } i=1,2,\cdots, B_{2_{\rho_i}}(x_i)\subset K, \tag{2.20}$$

and that

$$\int_{B_{\theta\rho_i}(x_i)} |\nabla u_i|^2\,dx \ge \theta^{n-\frac12}\int_{B_{\rho_i}(x_i)}|\nabla u_i|^2\,dx. \tag{2.21}$$

It is obvious, via (2.18), that $\rho_i\to0$ as $i\to\infty$ we set

$$v_i(y) = \left(u_i(x_0+\rho_i\,y) - \overline{u}_i\right)/\delta_i,$$
$$\delta_i = \|\nabla_y\,u_i\,(x_0+\rho_i\,y)\|_{L^2(B_1)}, \quad \overline{u}_i = ⨍_{B_{\rho_i}(x_i)} u_i\,dx$$
$$A_i^* = \left\{y\in B_1 : y = \rho_i^{-1}(x-x_i),\ \text{for some } x\in A_i\right\}.$$

Then v_i and $\chi_{A_i^*}$ are uniformly bounded in $H'(B_1)$ and $BV(B_1)$, respectively (cf. (2.19)). Thus we may assume, by passing to subsequences, if necessary, that

(i) $\chi_{A_i^*}\to\chi_{A_\infty}$ weakly in $BV(B_1)$ and strongly in $L'(B_1)$ for a measurable subset A_∞ of B_1;

(ii) $v_i\to v$ weakly in $H'(B_1)$ and strongly in $L^2(B_1)$;

(iii) $x_i \to x^* \in K$, as $i \to \infty$.

We claim that v is a minimizer of the functional of the form $\int_{B_1} F^*(\nabla u)\,dx$ with F^* satisfying (1.4) and that $v_i \to v$ in $H'_{\rm loc}(B_1)$.

We now can derive a contradiction from the above claim and (2.20), (2.21). In fact, since $v \in C^{1,\eta}$ for some $\eta > 0$ (cf. [12]), and

$$\int_{B_\theta} |\nabla v|^2 \le \frac{1}{2}\,\theta^{n-\frac{1}{2}} \int_{B_1} |\nabla v|^2\,dy \le \frac{1}{2}\,\theta^{n-1} 22$$

for some $\theta = \theta(n,\lambda) \in (0,1)$, we get, via $v_i \to v$ in $H'(B_\theta)$, that $\int_{B_\theta} |\nabla v_i|^2\,dy \le \theta^{n-\frac{1}{2}}$, and this contradicts to (2.21).

To show the claim, we first note $u_i(x)$ is a minimizer of

$$\int_{B_{\rho_i}(x_i)} [F(x,u_i,\nabla u_i) + \chi_{A_i}\,G(x,u_i,\nabla u_i)]\,dx. \tag{2.23}$$

If we write

$$F(x,u,p) \equiv F(x,u,0) + F_p(x,u,0)\,\nabla u + a_{k\ell}\,u_{x\ell}\,u_{xk},$$

where $a_{k\ell} = \int_0^1 F_{pk\,p\ell}(x,u,t\nabla u)\,dt$, and hence $\lambda I \le (a_{k\ell}) \le \lambda^{-1}\,I$, we divide (2.23) by

$$\int_{B_{\rho_i}(x_i)} |\nabla u_i|^2\,dx = \rho_i^{n-2}\,\delta_i^2 \ge i\rho_i^{n-1}\,c_2$$

to obtain v_i is a minimizer of the functional

$$\left(\frac{\rho_i}{\delta_i}\right)^2 \int_{B_1(0)} Q\left(x_i+\rho_i y, u_i(x_i+\rho_i y),\, \frac{\delta_i}{\rho_i}\,\nabla v_i\right) dy + 0_i(1). \tag{2.24}$$

Here the $0_i(1)$ term is given by

$$\begin{aligned}&\left(\frac{\rho_i}{\delta_i}\right)^2 \int_{B_1} F(x_i+\rho_i y, u_i(x_i+\rho_i y),\,0)\,dy\\ &+\frac{\rho_i}{\delta_i}\int_{B_1} F_p(x_i+\rho_i y, u_i(x_i+\rho_i y),\,0)\,\nabla v_i(y)\,dy\\ &+\frac{\rho_i^2}{\delta_i^2}\int_{B_1} \chi_{A_i^*}\,G(x_i+\rho_i y),\,u_i(x_i+\rho_i y)\,\frac{\epsilon_i}{\rho_i}\,\nabla v_i\,dy\end{aligned}$$

which converges to zero as $i \to \infty$ via (2.18), (2.19) and (2.20). Here the first term of (2.24) is given by

$$\int_{B_1(0)} \int_0^{\prime} F_{pk\,p\ell}\left(x_i+\rho_i y, u(x_i+\rho_i y),\, t\,\frac{\epsilon_i}{\rho_i}\,\nabla v_i\right) \frac{\partial v_i}{\partial y_k}\,\frac{\partial v_i}{\partial y_\ell}. \tag{2.25}$$

Moreover, by the Caccippoli-type estimate of Giaquinta-Giusti (cf. [12, p. 161]) for v_i, we obtain that v_i converges to v in $H'_{\rm loc}(B_1)$. Finally, since the integrand of (2.25) satisfies (1.4) uniformly (independent of i), and since $x_i \to x^*$, $u_i(x_i+\rho_i y) \to u^*$ for some constant u^* (by (2.18)), we obtain the limiting v minimizes a functional of type $\int_{B_1} F^*(\nabla v)\,dy$ with F^* satisfying (1.4)

We point out that the above claim follows from a general fact in Γ-convergence theory (see, for example, [7]). This completes the proof of the lemma.

The proof of $\mathbb{C}^{\frac{1}{2}}$-estimate of u in Ω follows from the identical reasoning as in [17].

Remark 2.2. If $\partial\Omega$ is of class $\mathbb{C}^1$ and if $\phi : \partial\Omega \to \mathbb{R}$ is Lipschitz continuous, then the statements of Lemma 2.3 can be formulated as follows for a point $x_0 \in \mathrm{spt}(\partial A) \cap \partial\Omega$:

There are two positive constants M and $\theta \in (0,1)$ such that, for $0 \le \rho \le R_2$, one has either

$$\int_{B_{\theta\rho}(x_0)\cap\Omega} |\nabla u|^2\, dx \le \theta^{n-\frac{1}{2}} \int_{B_\rho(x_0)\cap\Omega} |\nabla u|^2\, dx \tag{2.26}$$

or

$$\int_{B_{\theta\rho}(x_0)\cap\Omega} |\nabla u|^2\, dx \le M\, C_2(\theta\rho)^{n-1}. \tag{2.27}$$

(Here the constant will depend, in addition, on Lipschitz norm of ϕ and $\mathbb{C}^1$-character of $\partial\Omega$).

The proofs of these statements are similar to (2.16) and (2.17). One does need an additional fact that if v minimizes a functional of type $\int F^*(\nabla v)\, dx$ in $B^+ = \{x \in B_1 : x_n \ge 0\}$ with $v \equiv 0$ on $\{x_n = 0\}$ and F^* satisfying (1.4), then $v \in C^{1,\eta}(B^+_{\frac{1}{2}})$, for some $\eta > 0$ (cf. [17]).

It follows from (2.26), (2.27) that $u \in C^{\frac{1}{2}}(\overline{\Omega})$. We leave the details to readers.

§3. Mass Ratio Lower Bound

Let $m = n-1$, and $\mathbf{T} = \partial A\lfloor\Omega$. We will always use $B_r(x_0)$ to denote the ball centered at x_0 and of radius r in either $\mathbb{R}^m$ or $\mathbb{R}^n$ (but it will be clear in each context).

We assume the origin $\underline{o} \in \operatorname{spt}(T) \cap \Omega$, and let $\{e_1, \cdots e_{m+1}\}$ be an orthonormal frame in $\mathbb{R}^{n+1}$. With respect to such a frame, we can write any point $z \in \mathbb{R}^{n+1}$ as $z = (x,y) \in \mathbb{R}^n \times \mathbb{R}$. Let $\mathbb{P}$ be the orthonormal projection from $\mathbb{R}^{n+1}$ onto $\mathbb{R}^n \times \{o\}$, and let $\mathbb{C}_r = B_r(0) \times \mathbb{R} \subset \mathbb{R}^{n+1}$, $0 < r < \infty$. Then the cylindrical excess of $\mathbf{T}$ is given by

$$E(T,r) \equiv r^{-m}\left[M(T\lfloor\mathbb{C}_r) - M(\mathbf{P}_{\#}T\lfloor\mathbb{C}_r)\right]. \tag{3.1}$$

As in [17], we also define the Dirichlet Integral access of a function u by

$$D(u,r) \equiv r^{-m}\int_{B_r(0)} [F(x,u,\nabla u) + \chi_A\, G(x,u,\nabla u)]\, dx,\ B_r(0) \subset \mathbb{R}^{n+1},\ 0 < r < \infty. \tag{3.2}$$

The starting point of the proof of the regularity of interfaces in [17] is the following.

Mass Ratio Lower Bound

There is a positive constant c_0 depending only on Ω, ϕ, $K \subset\subset \Omega$ and n such that for all $x \in K \cap \operatorname{spt}(\mathbf{T})$, $0 < r < \operatorname{dist}(x,\partial\Omega)$,

$$M(T\lfloor B_r(x)) \ge c_0\, r^{n-1} \tag{3.3}$$

Here $\mathbf{T} = \partial A\lfloor\Omega$ and (A,u) is a minimal solution of (1.1) with $F \equiv G = |\nabla u|^2$, $u = \phi$ on $\partial\Omega$ (no volume constraint on A) and with $\Psi(\mathbf{T}) = M(\mathbf{T})$.

Here we want to prove the similar estimate as (3.3) for a minimal solution of (1.1), (1.2) with F and G being given by (1.6). To simplify the presentation, we proceed with the proof in three cases with increasing generalities.

Case I. We assume, in this case, that

$$F(x,u,\nabla u) \equiv G(x,u,\nabla u) = |\nabla u|^2 \text{ and that } \quad \Psi(\mathbf{T}) \equiv M(\mathbf{T}). \tag{3.4}$$

We should point out the following proof works also in the case that both F and G are given by constant multipliers of $|\nabla u|^2$ and, that $\Psi(\mathbf{T}) = CM(\mathbf{T})$ for some constant $C > 0$.

Let (A, u) be a minimal solution of (1.1), (1.2), and suppose that $B_{R_0}0) \subset \Omega$. We let

$$A^r = \mu_{r\,\#}\, A,\ u^r(z) = (u(rz) - \overline{u}_r)\,/r^{\frac{1}{2}}$$

for $0 < r \le R_0$. Here μ_r is the map $\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ defined by $\mu_r(x) = R^{-1} \times x$, and $\overline{u}_r = ⨍_{\partial B_r} u$. Then (A^r, u^r) is again a minimal solution of (1.1), (1.2) in $B_1(0)$ subject to suitable boundary conditions on u^r and A^r, and some suitable volume constraint on A^r. Thus one may assume that $B_1(0) \subset \Omega$ via a proper scaling and $\underline{o} \in K \cap \operatorname{spt}(T)$.

Next, by a theorem of DeGiorgi (cf. [11]), we have that for H^{n-1}-a.e., $x^* \in \operatorname{spt}(T) \cap \Omega$, there is a hyperplane Π passing x^* with respect to this hyperplane $E(T, R) \to o^+$ as $R \to o^+$. Moreover, if one denotes $H_\pm$, the two half spaces in $\mathbb{R}^{n+1}$ separated by Π, then

$$\|\chi_{H_+} - \chi_A\|_{L'(B_r(x^*))}\, r^{-n} \to o \text{ as } r \to o^+.$$

Similarly

$$\|\chi_{H_-} - \chi_A^c\|_{L'(B_r(x^*))}\, r^{-n} \to o \text{ as } r \to o^+.$$

For a given $x^* \in \operatorname{spt}(T) \cap K$, we let $y^* \in \operatorname{spt}(T) \cap K$ be so chosen that $x^* \in B_1(0) \subset B_2(0) \subset \Omega$ (after a suitable scaling) $y^* \in (B_2(0) \setminus B_1(0)) \cap \operatorname{spt}(T)$. Moreover, we need all three quantities $\|\chi_{H_+} - \chi_A\|_{L'(B_2(x^*))}$, $\|\chi_{H_-} - \chi_A\|_{L'(B_2(x^*))}$ and $E(T, 2)$ at x^* with respect to Π to be much smaller than 1 say, $\epsilon(n)$, $\epsilon(n) = 10^{-n}$.

By choosing a suitable coordinate in $\mathbb{R}^{n+1}$, we may assume that $\Pi = \mathbb{R}^n \times [0]$, and $H_+ = \{z \in \mathbb{R}^{n+1} : z = (x, y) \in \mathbb{R}^n \times \mathbb{R}_+\}$.

Let $\xi \in C_0^\infty(\mathbb{R}^n)$ be such that $0 \le \xi(x) \le 1$, $\xi \equiv 1$ on $B_{\frac{1}{4}}(\underline{o})$ and $\xi \equiv 0$ outside $B_{\frac{1}{2}}(\underline{o})$. Moreover, $\|\xi\|_{C^2(B_1(\underline{o}))} \le C_0$ and $\int_{B_1(0)} \xi(x)\, dx \ge \frac{1}{C_0}$ for some $C_0 > 0$.

For any such ξ, we introduce a diffeomorphism f_ϵ of $B_1(\underline{o}) \subset \mathbb{R}^n$ as follows

$$f_\epsilon(x, y) = (x, y + \epsilon\,\xi(x)\,\xi(y))$$

for $(x, y) \in B_1(\underline{o}) \subset \mathbb{R}^{n+1}$, and small $\epsilon \in (-1, 1)$. Let $T_\epsilon = f_{\epsilon\#}\,(T\lfloor B_1(\underline{o}))$, $A_\epsilon = f_{\epsilon\#}\, A$ and $u_\epsilon(x, y) = u \circ f_\epsilon(x, y)$, then it is easy to verify that

$$M(T_\epsilon) \le (1 + c\epsilon)\, M(T\lfloor B_1(\underline{o})), \tag{3.5}$$

$$\int_{B_1(\underline{o})} (1 + \chi_{A_\epsilon})\, |\nabla u_\epsilon|^2\,(z) dz \le (1 + c\epsilon) \int_{B_1(\underline{o})} (1 + \chi_A) |\nabla u|^2, \tag{3.6}$$

$$|A_\epsilon| \le (1 - \epsilon/c)\ |A|, \tag{3.7}$$

where $c = c(n) > 0$. In fact, the first one is trivial (cf. [8, §4.1]), the second one follows from a change variable formula and, the third one follows from the fact that

$$\|\chi_{H_+} - \chi_A\|_{L^1(B_2)} + \|\chi_{H_-} - \chi_{A^c}\|_{L^1(B_2)} + E(T, 2) \ll 1,$$

and the nonparametric approximation theory of F. Almgreen (see nonparametric approximation lemma below).

The above diffeomorphisms and estimate (3.5)–(3.7) are needed in our comparison estimates. Our construction of a comparison pair $(\widehat{A}, \widehat{u})$ will satisfy right energy estimates and preserves the volume constraints. This additional construction is not needed in [17] (see proof of Lemma 3.1 of [17]).

Lemma 3.1. *Let* (A, u) *be a minimal solution of problem* (1.1) *and* (1.2) *in the unit ball of* $\mathbb{R}^{n+1}$. *Suppose* (3.4) *valids and*

$$M(T\lfloor B_1(0)) + D(u, 1) \le \epsilon \tag{3.8}$$

then the function

$$\frac{E(r)}{r^{n+1}}\left(1-CE(r)^{\frac{1}{n-1}}\right)^{n-1} \tag{3.9}$$

is monotone increasing on $(0,1]$ *for some constant* C *whenever* $\epsilon \le \epsilon_0 \in (0,\frac{1}{2})$*, for a positive constant. Here*

$$E(r) = M(T\lfloor B_r(0) + \int_{B_r(0)} (1+\chi_A)\,|\nabla u|^2\,d.$$

Proof. We observe that $E(r)$ is a monotone increasing function of $r \in (0,1]$. Moreover, one has then

$$\lim_{r\to r_0,\, r>r_0} M(T\lfloor B_r(0)) = M(T\lfloor B_{r_0}(0)),$$
$$\lim_{r\to r_0,\, r<r_0} M(T\lfloor B_r(0)) \ge M(T\lfloor B_{r_0}(0)),$$

where $B_r(0)$ is a closed ball of radius r centered at 0 and that $f_{B_r(0)}(1+\chi_A)\,|\nabla u|^2$ is an absolute continuous function of $r \in (0,1)$.

To show (3.9), it suffices to show

$$E(r) \le \frac{r}{n-1}\,E'(r) + C\,E(r)^{\frac{n}{n-1}} \tag{3.10}$$

for a.e. $r \in (0,1)$. In fact, we note that $E(r) \le C_0\, r^{n-1}$ via Lemmas 2.1 and 2.3. If (3.10) valids, then, after a simple integration, one has

$$\frac{E(r)}{\left(1-C\,E(r)^{\frac{1}{n-1}}\right)^{n-1}} \le \frac{E(\rho)}{\left(1-C\,E(\rho)^{\frac{1}{n-1}}\right)^{n-1}}\,\frac{r^{n-1}}{\rho^{n-1}}$$

prove that $1-c\,E(\rho)^{\frac{1}{n-1}} \ge \frac{1}{2}$ and $0, r \le \rho \le 1$. In particular, we obtain

$$\frac{E(r)}{r^{n-1}} \le \frac{\epsilon}{\left(1-c\,\epsilon^{\frac{1}{n-1}}\right)^{n-1}} \tag{3.11}$$

for $0 < r < 1$.

To show (3.10) we follow the cone-comparison arguments in [17]. That is, for a.e. $r \in (0,1)$, we replace $A\lfloor B_r(0)$ by $\widetilde{A}\lfloor B_r(0)$, and replace u by $\widetilde{u}$. Here $\widetilde{A}$ is the cone over the slice of T by the function $f(x)=|x|$ at the level $f=r$ and $\widetilde{u}$ is the minimizer of $\int_{B_r(0)}(1+\chi_{\widetilde{A}})\,|\nabla\widetilde{u}|^2$ subject to $\widetilde{u}=u$ on $\partial B_r(0)$. We show in [17] that $E(r) \le \frac{r}{n-1}: E'(r)$ if there is no volume constraint on $A\lfloor B_r(0)$. Considering the volume constraint in (1.2), we have to deform A somewhere to preserve this constraint.

Since, when $A\lfloor B_r(0)$ replaced by $\widetilde{A}\lfloor B_r(0)$ the volume change is at most

$$c(n)\,M(T\lfloor B_r(0))^{\frac{n}{n-1}},$$

the latter follows from the relative isoperimetric inequality as in [17]. Now we apply the defro-mentioned f_δ inside $B_1(x^*)$ and apply the estimates (3.5)–(3.7) where $|\delta| \simeq c(n)\,M(T\lfloor B_r(0))^{\frac{n}{n-1}}$. In this way we obtain (3.10) with the error term $c\,E(r)^{\frac{1}{n-1}}$ deduced from the estimates (3.5)–(3.7).

Following the exact same argument as in [17, Lemma 3.3] we can derive the following Mass Ratio Lower Bound.

Theorem 3.1. *Let* (A,u) *be a minimal solution of* (1.1), (1.2)*, and* $K \subset\subset \Omega$*. There is a constant* λ_0 *depending only on* Ω*,* ϕ*,* K *and* n *such that for all* $x \in K \cap \mathrm{spt}(T)$*,*

$0 < r < \operatorname{dist}(x, \partial\Omega)$.

$$M\,(T\lfloor B_r(x)) \geq \lambda_0\, r^{n-1} \tag{3.12}$$

provided that the assumption (3.4) *verified.*

Case II. In this case, we assume that

$$F \text{ and } G \text{ are of the form given in (1.6) and } \Psi(T) = M(T). \tag{3.13}$$

Since F and G are of form (1.6), and since $u \in C^{\frac{1}{2}}(\Omega)$, we obtain that for any $\mathbf{o} \in K \subset\subset \Omega$, and $\delta_0 > 0$, there is an $r_0(K, \delta_0) > 0$ such that

$$|a_{ij}(x, u(x)) - a_{ij}(\mathbf{o}, u(\mathbf{o}))| + |b_{ij}(x, u(x)) - b_{ij}(\mathbf{o}, u(\mathbf{o}))| < \delta_0 \tag{3.14}$$

whenever $|x - \mathbf{o}| \leq r_0$ (similarly for lower order terms).

After the scaling μ_{r_0} at $\mathbf{o}$ we may assume $r_0 = 1$ in (3.4). Suppose now that (3.8) is also true, then as in [17], we obtain either $|A| \leq c(n)\, \epsilon^{\frac{n}{n-1}}$ or $|A^c| \leq c(n)\, \epsilon^{\frac{n}{n-1}}$.

In the former case we introduce a suitable linear change of coordinates of $\mathbb{R}^n$ so that the resulting F, which is again of form (1.6), satisfies

$$a_{ij}(\mathbf{o}, u(\mathbf{o})) = \delta_{ij}. \tag{3.15}$$

In the latter case, we perform a linear change of coordinates to make

$$b_{ij}(\mathbf{o}, u(\mathbf{o})) = \delta_{ij}. \tag{3.16}$$

Also, we note that the assumption (3.8) implies in either of cases, that

$$M(T\lfloor B_{r_1}(\mathbf{o})) + D(u, r_1) \leq r_1^{n-1}\, \epsilon_1, \tag{3.17}$$

$\epsilon_1 = r_1^{-(n-1)}\, \epsilon \ll 1$ for a suitable constant r_1 (depending only on the coefficients of leading terms of F or G). Here $B_{r_1}(\mathbf{o})$ is the ball of radius r_1 centered at $\mathbf{o}$ in the new coordinates system obtained by one of these linear changes of coordinates.

Now, because of (3.14), one can easily verify as in [17] that

$$\int_{B_r(0)} (F + \chi_A\, G)\, dx \leq \frac{n-1}{r} \int_{\partial B_r(0)} (F + \chi_A\, G) \tag{3.18}$$

provided that $M(T\lfloor B_r(0)) \leq \epsilon_0\, r^{n-1}$, for some $\epsilon_0 > 0$.

To obtain in monotonicity estimate as for (3.9) we have to fix one of the cases in the above analysis.

For this purpose, we may assume that $|A| \leq c(n)\, \epsilon^{\frac{n}{n-1}}$. Then, after a linear transform we also assume that (3.15) is true.

Following the proofs of [17, Lemma 3.1] and Lemma 3.1 above, and by using (3.18), we obtain that

$$\frac{E(r)}{r^{n-1}} \left(1 - c\, E(r)^{\frac{1}{n-1}}\right)^{n-1}$$

is a monotone increasing function of r in interval $(r_\epsilon, r_0]$. Here $r_\epsilon \geq 0$ is chosen so that

$$|A\lfloor B_r(0)| < \frac{1}{2}\, |B_r|, \text{ for all } r \in (r_\epsilon, r_0].$$

We claim that r_ϵ can be chosen to zero. In fact, the monotonicity of the quantity

$$\frac{E(r)}{r^{n-1}} \left(1 - c\, E^{\frac{1}{n-1}}(r)\right)^{n-1}$$

implies, in particular, that

$$\frac{M(T\lfloor B_r(0))}{r^{n-1}} \le \epsilon_1 \left(1 - c\, r_1\, \epsilon_1^{\frac{1}{n-1}}\right) = \epsilon_2 \ll 1$$

for $r \in (r_\epsilon, r_0]$.

Thus, $|A\lfloor B_r(0)| < \frac{1}{2}\, |B_r(0))|$ is valid for all $r \in (0, r_0]$ via relative-isoperimetric inequality.

To summarize, we have proved the following

Lemma 3.2. *Suppose* (3.8), (3.13) *and* (3.14) *are true for a minimal solution* (A, u) *of* (1.1), (1.2), *then the function defined in* (3.9) (*see also* (3.2)) *is monotone increasing on* $(0, r_1]$. *Here the balls* $B_r(\mathbf{o})$ *in definition* (3.9) *must be those balls with respect to a new coordinate system obtained from a suitable linear change of the original coordinates system. As a consequence, we have the mass ratio lower bound* (3.12).

Case III. Here we assume that Ψ is a parametric elliptic integral satisfying (1.7)–(1.10), and that F, G are of form (1.6).

In dealing with general parametric integrals Ψ, the mass ratio lower bound cannot be obtained through an argument of the monotonicity of mass ratio (even for Ψ-minimizing currents). In other words, cone comparison will not work. Instead, we shall combine an isoperimetric-type inequality for Ψ-minimizing currents (cf. [21]) with our estimates on the Dirichlet integral growth. For this purpose and for simplicity, we shall consider only the case $F \equiv G = |\nabla u|^2$. When F and G are given by (1.6), the arguments in Case II and the proof given below apply. We leave this to the reader, however.

Let (A, u) be a minimal solution of (1.1) and (1.2) which satisfies (3.8). We replace $T = \partial A\lfloor B_1(o)$ by $\widetilde{T}$. Here $\widetilde{T}$ is a Ψ-minimizing current in $B_1(o)$ with $\partial\widetilde{T} = \partial T$. That is

$$\Psi(\widetilde{T}) \le \Psi(Q) \text{ for all } Q \in \mathbf{I}_{n-1}(\mathbb{R}^n) \text{ with } \partial\widetilde{T} = \partial Q = \partial T. \tag{3.19}$$

It is easy to verify that $\widetilde{T} = \partial\widetilde{A}\lfloor B_1(o)$ for some subset $\widetilde{A} \subset B_1(o)$.

Next, we let $\widetilde{u}$ be a minimizer of $\int_{B_1(o)} |\nabla\widetilde{u}|^2(1 + \chi_{\widetilde{A}})\, dx$ such that $\widetilde{u} = u$ on ∂B_1.

By (3.8), one has either $|A| + |\widetilde{A}| \le c(n)\, \epsilon^{\frac{n}{n-1}}$ or $|A^c| + |\widetilde{A}^c| \le c(n)\, \epsilon^{\frac{n}{n-1}}$. This is an easy consequence of the relative isoperimetric inequality (cf. [17, §3]). Then by the same argument as that in [17, §3], we may assume that

$$\begin{aligned}\int_{B_1(o)} (1 + \chi_{\widetilde{A}})|\nabla\widetilde{u}|^2\, dx &\le (1 + \delta(\epsilon)) \int_{B_1} (1 + \chi_A)|\nabla u|^2\, dx \\ &\le (1 + \delta(\epsilon))^2 \int_{B_1} (1 + \chi_{\widetilde{A}})\, |\nabla\widetilde{u}|^2\, dx,\end{aligned} \tag{3.20}$$

(here $\delta(\epsilon) \to 0$ as $\epsilon \to 0$) and

$$\begin{cases} \int_{B_1(o)} (1 + \chi_A)|\nabla u|^2\, dx \le \frac{1}{n-\frac{1}{2}} \int_{\partial B_1} (1 + \chi_A)|\nabla u|^2, \\ \int_{B_1(o)} (1 + \chi_{\widetilde{A}})|\nabla\widetilde{u}|^2\, dx \le \frac{1}{n-\frac{1}{2}} \int_{\partial B_1} (1 + \chi_A)|\nabla u|^2. \end{cases} \tag{3.21}$$

Both (3.20) and (3.21) follow from the fact that, as $\epsilon \to 0$, both u and $\widetilde{u}$ converge strongly in $H'(B_1)$ (after a suitable normalization, say $\int_{\partial B_1} (1 + \chi_A)|\nabla_T u|^2 = 1$) to a same harmonic function on $B_1(o)$ for which (3.21) is valid with $n - \frac{1}{2}$ replacing by n on the right hand of (3.21). We also note that for the general F and G of form (1.6), one can apply exactly the same argument as above by using the fact that u is uniformly Hölden continuous on $K \Subset \Omega$.

By an isoperimetric inequality for Ψ-minimizing current (cf. [21]), one has

$$\Psi\left(\widetilde{T}\right) \leq c\,(M(\langle T, f, 1\rangle))^{\frac{n-1}{n-2}},$$

where $\langle T, f, 1\rangle$ is the slice of T by $f(x) = |x|$ at the level $f \equiv 1$ (as in [17, §3], we should assume this exists).

Now we have two possibilities:

(i) $M(T\lfloor B_1(o)) \leq \frac{1}{2(n-1)}\, D(u, 1)$, and (ii) $M(T\lfloor B_1(o)) > \frac{1}{2(n-1)}\, D(u, 1)$.

In the case (i), we have

$$\begin{aligned} M(T\lfloor B_1(0)) + D(u, 1) &\leq \left(1 + \frac{1}{2(n-1)}\right) D(u, 1) \\ &\leq \frac{2n-1}{2(n-1)} \cdot \frac{2}{2n-1} \int_{\partial B_1} (1 + \chi_A)|\nabla u|^2 \\ &\leq \frac{1}{n-1}\Big[M\left(\langle T, f, 1\rangle\right) + \int_{\partial B_1} \left(1 + \chi_A\right)|\nabla u|^2\Big]. \end{aligned} \tag{3.22}$$

In the case (ii), since (A, u) is a minimal solution, one has, as for (3.10) that

$$\begin{aligned} \Psi(T\lfloor B_1) + D(u, 1) &\leq \Psi\left(\widetilde{T}\lfloor B_1\right) + D(\widetilde{u}, 1) + c(n)\, M\left(T\lfloor B_1\right)^{\frac{n}{n-1}} \\ &\leq c\, M\left(\langle T, f, 1\rangle\right)^{\frac{n-1}{n-2}} + c\, M\left(T\lfloor B_1\right)^{\frac{n}{n-1}} + D\left(\widetilde{u}, 1\right). \end{aligned} \tag{3.23}$$

By (3.20), (3.8), and (ii) above, we obtain from (3.23) that

$$M\left(T\lfloor B_1(0)\right) \leq c\left(M\left(\langle T, f, 1\rangle\right)\right)^{\frac{n-1}{n-2}}, \tag{3.24}$$

$$M\left(T\lfloor B_1(o)\right) + D(u, 1) < c\Big[M\left(\langle T, f, 1\rangle\right) + \int_{\partial B_1} \left(1 + \chi_A\right)|\nabla u|^2\Big]^{\frac{n-1}{n-2}}. \tag{3.25}$$

From (3.22), (3.25) and a scaling we see that either

$$E(r) \leq \frac{r}{n-1}\, E'(r) \tag{3.26}$$

or

$$E(r) \leq c\left(E'(r)\right)^{\frac{n-1}{n-2}}, \tag{3.27}$$

for a.e. $r \in (0, 1)$, such that $E(r)/r^{n-1} \leq \epsilon_0$. In other words,

$$E'(r) \geq \min\left\{(n-1)\,\frac{E(r)}{r},\ c(n)\, E(r)^{\frac{n-2}{n-1}}\right\} \text{ for a.e. such } r \in (0, 1) \text{ that } \frac{E(r)}{r^{n-1}} \leq \epsilon_0. \tag{3.28}$$

We want to show when $\frac{E(r)}{r^{n-1}} \leq \epsilon_0(n)$, we have $E'(r) \geq (n-1)\,\frac{E(r)}{r}$ for a.e. such r. In fact, otherwise $c(n)\, E^{\frac{n-2}{n-1}}(r) \leq \frac{(n-1)}{r}\, E(r)$ by (3.28), and therefore $E(r) \geq \left(\frac{c(n)}{n-1}\right)^{n-1} r^{n-1}$. This will contradict to $\frac{E(r)}{r^{n-1}} \leq \epsilon_0(n)$ whenever $\epsilon_0 \leq \left(\frac{c(n)}{n-1}\right)^{n-1}$.

As a consequence, we have

Lemma 3.3. *Let* (A, u) *be a minimal solution of* (1.1), (1.2) *and satisfy* (3.8)*. Then* $E(r)/r^{n-1}$ *is monotone increasing function for* $r \in [0, 1]$ *provided that* $E(1) \leq \epsilon_0$.

Remark 3.1. The above proof for the Case III fails when $n = 2$. But in the case that $n = 2$, the main ratio lower bound is trivial. Therefore, we have established the mass ratio lower bound (3.3) for any minimizer (A, u) of (1.1) and (1.2).

§4. Proof of Theorem 1.2

(a) Intermediate Remarks

The following two lemmas are fundamental in the regularity theory for minimal surfaces. Both of them follow from the mass ratio lower bound on the current T. For this reason, we shall assume in the remaining parts of this paper that all currents T we refer to satisfy the following hypothesis:

$$M(T\lfloor B_r(x)) \geq \lambda_0\, r^{n-1} \quad \text{for some } \lambda_0 > 0 \tag{4.1}$$

and all $r \in (0,1), x \in \operatorname{spt}(T) \cap B_1(0)$.

If (A, u) is a solution of (1.1), (1.2) and $K \Subset \Omega$ is an open set with compact closure in Ω, then there is an $R = R(K)$ positive number (which may depend on Ω and ϕ) such that for any $x_0 \in K$, $\mu_{x_0,R}\,\#T$ satisfies (H) for some $\lambda_0 > 0$. Here $\mu_{x_0,R}(x) = R^{-1}(x - x_0)$.

Height Bound Lemma.[8,§5.3] *Let T be an $(n-1)$ dimensional integral current in R^n with* $\operatorname{spt}(\partial T) \subset R^n - \mathbb{C}_\rho$, *and satisfy* (H). *Suppose that $\Theta^{n-1}(\|T\|, x) \geq 1$ for $\|T\|$ almost all $x \in \mathbb{C}_\rho$, and that $P_{\#}(T\lfloor \mathbb{C}_\rho) = E^{n-1}\lfloor B_\rho(0)$. Then there is an $\epsilon_0 > 0$ so that*

$$\sup |x_n - x_n'| \leq c(n)\, \rho\, \epsilon^{1/2(n-1)}$$

whenever $x = (x_1, \cdots, x_{n-1}, x_n)$, and $x' = \left(x_1', \cdots, x_{n-1}', x_n'\right)$ belong to $\operatorname{spt}(T) \cap \mathbb{C}_{\rho/2}$ *and* $E = E(T, \rho) \leq \epsilon \leq \epsilon_0$.

Nonparametric Approximation Lemma.[8,§5.3] *Let T be as above, and let $0 \leq \rho < 1$, $0 < r < \infty$ be two numbers such that*

(i) $E < \frac{1}{2} E^{1/2} < \left|B_1^{n-1}(0)\right|/6 \equiv \alpha(n)/6$ *where* $E = E(1)$;

(ii) $E/\alpha(n) < \tau\, E^\rho < 1$;

(iii) $\tau\, E^\rho < \frac{1}{2}(\tau\, E^\rho)^{1/2} < \alpha(n)/6$.

Then there is a Lipschitz function $f : B_{7/8}^{n-1}(0) \to R$ together with a partition of $B_{7/8}^{n-1}(0)$ into L^{n-1}-measurable sets A and B with the following properties:

(1) *lip* $(f) \leq c(n)\, \tau^{1/2(n-1)}\, E^{\rho/2(n-1)}$;

(2) *for each* $x \in A$, $[[x]] \times f(x) = \sum \left\{\Theta^{n-1}(\|\mathbf{T}\|, a)\, [[a]] : a \in \operatorname{spt}(T) \cap \{x\} \times R\right\}$;

(3) $L^{n-1}(B) \leq c(n)\, \tau^{-1}\, E^{1-\rho}$;

(4) $\|T\|(B \times R) \leq E + c(n)\, \tau^{-1}\, E^{1-\rho}$;

(5) $(T\lfloor(A \times R)) = (1_A \times f)_{\#}\, (E^n\lfloor A)$;

(6) $M\left((I \times f)^{\#}\, (E^n\lfloor B)\right) \leq (1 + \ lip^2 f)^{(n-1)/2}\, L^{n-1}(B)$.

These two lemmas will be needed.

As was mentioned earlier, if we know $u \in C^{1/2+\eta}(\Omega)$ for some $\eta \in (0, \frac{1}{2})$, then it is easy to check that, for any open set K with compact closure in Ω,

$$\int_{B_r(x)} |\nabla u|^2\, dy \leq C\, r^{n-1+2\eta}, \quad 0 < r < \delta_K, \quad x \in K,$$

where $\delta_K =$ dist $(K, \partial\Omega)$, C is a constant depending on K, ϕ and $\partial\Omega$. Let $T = \partial A\lfloor\Omega$ as before, then (H) implies $M\,(T\lfloor B_r(x)) \geq \lambda_0\, r^{n-1}$, for $0 < r < \delta_K$, and for $x \in \operatorname{spt}(T) \cap K$.

We claim T is (Ψ, ω, δ_k)-minimizing in K (cf. [4]). In fact, for any $\widetilde{A} \subset \Omega$ with $\widetilde{A}\Delta A = (\widetilde{A} - A) \cup (A - \widetilde{A}) \subset B_r(x)$ and $x \in \operatorname{spt}(T) \cap K$, $0 < r < \delta_K$, we let $\widetilde{T} = \partial\widetilde{A}\lfloor\Omega$. Then

$$\Psi(T\lfloor B_r(x)) \leq \Psi(\widetilde{T}\lfloor B_r(x)) + \omega(r)\, M(T\lfloor B_r(x)),$$

whhere $\omega(r) = \frac{C}{\lambda_0} : r^{2\eta}$, for $0 \le r \le \delta_K$. The latter is valid because (A, u) is a minimal solution of problem (1.1), (1.2) and that

$$\int_{B_r(x)} \left|\chi_{\widetilde{A}} - \chi_A\right| |\nabla u|^2 \, dy \le c\, r^{n-1+2\eta} \le \omega(r)\, M\left(T \lfloor B_r(x)\right).$$

It can be shown as in [4] that if T is (Ψ, ω, δ_K)-minimizing in K, then $\mathrm{spt}(T)$ in K is a $C^{1,\alpha}$-hypersurface away from a relatively closed subset of $\mathrm{spt}(T)$ of Hausdorff dimension $(n-1)$ measure zero.

Next, we let (A, u) be a minimal solution of problem (1.1), (1.2) and let $T = \partial A \lfloor \Omega, K \subset\subset \Omega$ be as above. DeGiorgi's Theorem[11] implies that for H^{n-1}-a.e. $\underline{o} \in \mathrm{spt}(T) \cap K$, there is a hyperplane with respect to which $E(T, R) \ll \epsilon_0$, for some $R > 0$. By the mass ratio lower bound (H), one then concludes (as in [8, §5.3]) that $\partial\left(\widetilde{T} \lfloor \mathbb{C}_{\theta R}\right) \subseteq R^n \sim \mathbb{C}_{\theta R}$, where $\widetilde{T} = T \lfloor B_R(\underline{o})$, and $\theta = \theta(n) \in \left(0, \frac{1}{4}\right)$.

We also note for some $\theta = \theta(n) \in \left(0, \frac{1}{4}\right)$, $D(u, \theta R) \le \epsilon_{0/2}$ (by Lemma 4.1 below). Therefore, if we set $r = \theta R$, we have that

$$E(T, r) + D(u, r) \le \epsilon < \epsilon_0(n), \quad P_{\#}(T \lfloor \mathbb{C}_r) = E^m \lfloor B_r(0), \quad \partial T \lfloor \mathbb{C}_2 = 0$$

provided $E(T, R)$ is sufficiently small.

(b) Energy Comparison Estimates

Let (A, u) be a minimal solution of (1.1), (1.2), and let $T = \partial A \lfloor B_2$. Suppose that $\mathbf{T}$ satisfies the hypothesis of the Height Bound Lemma with $\rho = 1$. Then, we have the following

Lemma 4.1. *There are two positive constants θ, $\epsilon_* \in \left(0, \frac{1}{16}\right)$ such that*

$$\theta^{-m} \int_{B_\theta(0)} \left(F(x, u, \nabla u) + \chi_A\, G(x, u, \nabla u)\right) dx$$
$$\le \theta^{\frac{1}{2}} \int_{B_1(0)} \left[F(x, u\nabla u) + \chi_A\, G(x, u, \nabla u)\right] dx \tag{4.2}$$

whenever $E(T, 1) \le \epsilon_*$ and $0 \in \mathrm{spt}(T)$. Here, we have also assumed that

$$|F(x, u, p) - F(0, u(0), p)| + |G(x, u(x), p) - G(0, u(0), p)) \le \epsilon_*(1 + |p|^2) \tag{4.3}$$

for all $x \in B_1(0)$.

Remark 4.1. Before giving the proof of Lemma 4.1, we note that (4.3) follows from $\mathbb{C}^{\frac{1}{2}}$-Hölder regularity of u in x, (1.3)–(1.5), and a proper scaling as indicated in the previous section. For this reason and for the simplicity of presentation, we should assume below that $F \equiv G = |\nabla u|^2$. It will be clear from the arguments showing below that the general case follows.

Proof of Lemma 4.1. By the Height Bound Lemma, we have that $\chi_A - \chi_{H^+} \to 0$ in $L'(B_1(0))$ as $E(T, 1) \to 0$. Here $H^+ = \{(x, x^{m+1}) \in \mathbb{R}^m \times \mathbb{R} : x^{m+1} > 0\}$.

Next we note that a solution $v \in \mathbf{H}^1(B_1)$ of the equation

$$\mathrm{div}\ \left[(1 + \chi_{H^+})\, \nabla u\right] = 0 \text{ in } B_1 \tag{4.4}$$

is Lipschitz continuous in B_1.

To prove (4.2), one only needs to notice that if $\int_{B_1} |\nabla u|^2 \, dx = 1$, and if $\|\chi_A - \chi_{H^+}\|_{L^1(B_1)} \le \delta$, then there is a solution v of (4.4) with $\int_{B_1} |\nabla v|^2 \, dx \le 1$, and $\|v - u\|_{H^1(B_{\frac{1}{2}})} \le \eta(\delta)$. Here $\eta(\delta) \to 0$ as $\delta \to 0$. By the Lipschitz continuity of v, we see that there is a $\theta \in \left(0, \frac{1}{10}\right)$ so that

$$\theta^m \int_{B_\theta(0)} |\nabla v|^2 \, dx \leq \frac{1}{2} : \theta^{\frac{1}{2}} \int_{B_1} |\nabla v|^2 \, dx \leq \frac{1}{2} : \theta^{\frac{1}{2}}. \tag{4.5}$$

We fix such θ, then there is a $\delta_\theta > 0$ such that $\theta^{-m} \int_{B_\theta(0)} |\nabla v|^2 \, dx \leq \theta^{\frac{1}{2}}$ provided that $|\chi_A - \chi_{H^+}|_{L'(B_1)} \leq \delta_\theta$.

Remark 4.2. One can also easily check that any minimizer of

$$\int_{B_1(0)} [F^*(\nabla v) + \chi_{H^+} G^*(\nabla v)] \, dx \tag{4.6}$$

is Lipschitz continuous inside $B_1(0)$. Here

$$F^*(\nabla v) = F(0, u(0), \nabla u), \ G^*(\nabla v) = G(0, u(0), \nabla v).$$

Energy Comparison Lemma. *Let A, u, T be as above. There are positive constants θ and $\epsilon_* \in (0,1)$ such that*

$$E(T, \theta) + D(u, \theta) \leq \frac{1}{2} \left(E(T, 1) + D(u, 1)\right) + c(n) \, H(1),$$

whenever $\epsilon^2 = E(T,1) + D(u,1) \leq \epsilon_*^2$. Here $H(1) = f_{\mathbb{C}_1} |y - \overline{y}|^2 \, d\|t\|, \quad y = x^{m+1}$.

Proof. The proof of the Energy Comparison Lemma uses so-called "squashing-deformation" (see for example [16]). To do so, we will fix u in our comparison and deform A to $\widetilde{A}$ by a squashing deformation with the following properties:

(i) $A = \widetilde{A}$ outside $\mathbb{C}_{\frac{3}{4}}$;

(ii) $\widetilde{T} \, L \, \mathbb{C}_\theta = E^n \, L \, \mathbf{B}$, with $B = \{(x, \overline{y}) : |x| \leq \theta\}$.

Here in $H(1)$ and below, we assume $\overline{y} = 0$ (by translating the horizontal plane up or down). The value $\overline{y}$ is chosen so that $|A| = |\widetilde{A}|$. This is essentially equivalent to the requirement

$$\int_{\mathbb{C}_1} (1 - \mu(x)) \, y \, d\|T\| \approx 0 \tag{4.7}$$

with error which can be controlled by $[E(T,1) + D(u,1)]^{1+\delta}$, for some $\delta_0 > 0$. Here $\mu(x) = \mu\left(x^1, \cdots, x^m\right)$ is the function used in the squashing deformation of $\mathbf{T}$ to the horizontal plane $x^{m+1} = \overline{y}$ (see [4, 16]).

Let $\widetilde{T} = \partial \widetilde{A} \lfloor B_2$. Then, by the minimality of (A, u) we have

$$\Psi(T \lfloor B_1) + \int_{B_1} (1 + \chi_A) \, |\nabla u|^2 \, dx \leq \int_{B_1} \left(1 + \chi_{\widetilde{A}}\right) |\nabla u|^2 \, dx + \Psi\left(\widetilde{T} \lfloor B_1\right). \tag{4.8}$$

We should point out that under the hypothesis on $\mathbf{T}$ we have via the Height-Bound-Lemma, that

$$\sup_{(x,y) \in \operatorname{spt}(\mathbf{T}) \cap \mathbb{C}_{\frac{3}{4}}} |y - \overline{y}| \leq C(n) \, \epsilon_*^{\frac{1}{2(n-1)}} < \frac{1}{16} \tag{4.9}$$

(if ϵ_* is small enough). Also $\partial \left(T \lfloor B_1\right) = \partial(\widetilde{T} \lfloor B_1)$.

We therefore conclude from (4.8) that

$$\Psi\left(T \lfloor B_1\right) - \Psi(\widetilde{T} \lfloor B_1) \leq \int_{B_1} \left(\chi_{\widetilde{A}} - \chi_A\right) |\nabla u|^2. \tag{4.10}$$

If T is Ψ-minimizing, then the right-hand-side can be replaced by zero. So our analysis will be to bound the term on the right-hand-side of (4.10).

Now we follow the calculations in [4] and [16] to obtain that

$$E(T,\theta) \le \lambda E(T,1) + \frac{c(n)}{\lambda}\int_{\mathbb{C}_1} |y|^2\, d\|T\| + \int_{B_1} \left|\chi_A - \chi_{\widetilde{A}}\right| |\nabla u|^2 \tag{4.11}$$

for any $\lambda \in (0,1)$.

Also, by $A = \widetilde{A}$ outside $\mathbb{C}_{3/4}$ and (4.9), we have $A = \widetilde{A}$ outside $B_{13/16}$ and thus

$$\begin{aligned}\int_{B_1} \left|\chi_A - \chi_{\widetilde{A}}\right| |\nabla u|^2 &\le \int_{B_{13/16}} \left|\chi_A - \chi_{\widetilde{A}}\right| |\nabla u|^2 \\ &\le \left\|\chi_A - \chi_{\widetilde{A}}\right\|_{L^p(B_{13/16})} \cdot \left\||\nabla u|^2\right\|_{L^q(B_{13/16})}.\end{aligned} \tag{4.12}$$

Here $q > 1$ is chosen so that the reverse Hölder estimate[12]

$$\left\||\nabla u|^2\right\|_{L^q(B_{13/16})} \le c(n)\int_{B_1} |\nabla u|^2 \tag{4.13}$$

is valid, where $p = \frac{q}{q-1} \in (1,\infty)$.

By applying the Height Bound Lemma again, we obtain

$$\left\|\chi_A - \chi_{\widetilde{A}}\right\|_{L^p(B_{13/16})} \le C(n)\, \epsilon^{1/2\,(n-1)\,p}. \tag{4.14}$$

On the other hand, by Lemma 4.1, we have

$$D(u,\theta) \le \theta^{1/2}\, D(u,1) \le \frac{1}{4}\, D(u,1). \tag{4.15}$$

Therefore, one has

$$\begin{aligned}E(T,\theta) + D(u,\theta) \le{}& \lambda\, E(T,1) + \frac{c(n)}{\lambda}\, H(1) + \frac{1}{4}\, D(u,1) \\ &+ C(n)\, E^{\delta_0}\,(T,1)\, D(u,1)\left(\delta_0 = \frac{1}{2(n-1)\,p}\right).\end{aligned} \tag{4.16}$$

By choosing $\lambda = \frac{1}{2}$, and ϵ_* so small that $c(n)\,\epsilon_*^{\delta} < \frac{1}{4}$, we obtain the conclusion of the lemma.

(c) Regularity of Interfaces

The following lemma is the key ingredient in the proof of the regularity of free interfaces. The hypothesis in this lemma can be deduced from remarks at the end of the previous section.

Excess Improvement Lemma. *Let* (A,u) *be a minimal solution of problem* (1.1) *and* (1.2) *in* $B_2(\underline{o})$, $\underline{o} \in \mathrm{spt}(T)$, $= \partial A \lfloor B_2(0)$. *Suppose that* T *satisfies the hypothesis in the Height Bound Lemma with* $\rho = 1$. *Then there are two positive constants depending only on* n, θ_* *and* ϵ_* *such that*

$$E(T,\theta_1) + D(u,\theta_1) \le \frac{2}{3}\,(E(T,1) + D(u,1)) \tag{4.17}$$

provided that $E(T,1) + D(u,1) \le \epsilon_*^2$. *Here the cylindrical excess on the left-hand side of* (4.17) *may be with respect to a new hyperplane which is a rotation of* $\mathbb{R}^n \times \{0\}$ *by an angle* ω *not larger than* $C(n)\,\epsilon$, $\epsilon^2 = E(T,1) + D(u,1)$.

Proof. The proof of this lemma follows from the standard blow-up arguments of [8, §5.3] (cf. also [16]). We, thus, shall only explain several new points in such a proof.

Step I. We let $\epsilon^2 = E(T,1) + D(u,1)$. If (4.17) were not true, there would be sequence of minimal solutions (A_i, u_i) in $B_2(\underline{o})$ with the following properties: $\underline{o} \in \mathrm{spt}(T_i)$,

$$T_i = \partial A_i \lfloor B_2(\underline{o}), \quad P_\#\,(T_i \lfloor \mathbb{C}_1) = E^m \lfloor B_1(\underline{o}), \quad \partial T_i = 0 \ \text{ in } \ \mathbb{C}_1,$$

and $\epsilon_i^2 = E(T_i, 1) + D(u_i, 1) \to 0^+$ as $i \to \infty$.

Let f_i be a Lipschitz function in the nonparametric approximation lemma for $\mathbf{T}_i$. Then

$$\overline{\lim_i} \int_{B_1(0)} |\nabla f_i|^2 \, \epsilon_i^2 \, dx \le c(n). \tag{4.18}$$

By a small translation in the vertical direction, we may also assume $\int_{\mathbb{C}_1} (1-\mu)\, y \, \|T\| = 0$. (Of course, under such a normalization, we may not have $\underline{o} \in \operatorname{spt}(T_i)$). But the conclusion of the Height Bound Lemma guarantees that there is $x_i \in \operatorname{spt}(T_i)$ and $x_i \to \underline{o}$).

Under the normalization, we have that $\frac{f_i}{\epsilon_i}$ are uniformly bounded in $H^1(B_1)$, and hence (by passing to a subsequence if necessary), that $\frac{f_i}{\epsilon_i} \to h$ in $L^2(B_1)$ and weakly in $H^1(B_1)$. Note that $\int_{B_1} (1-\mu)\, h \, dx = 0$ by (4.7), and that $B_1 \subset R^m$.

Step II. From the first variation estimates (see Lemma 4.2, Lemma 4.3 below) we have

$$L_0 \, h = 0 \text{ in } B_1(0) \subseteq \mathbb{R}^n, \tag{4.19}$$

where L_0 is a linear elliptic operator with constant coefficients, i.e., $L_0 = A_{ij} : \frac{\partial^2}{\partial x_i, \partial x_j}$. A_{ij}'s are constants depending only on Ψ.

Here we should remark that the function μ in the squashing-deformation may be chosen as follows. Let $x^* = Lx$ in a linear change of coordinates of $\mathbb{R}^n$ so that L_0 operators in x^*-coordinate become Δ. Then $\mu(x) = \mu(|x^*||)$ will be a good choice, as $0 = \int_{B_1(0)} (1-\mu)\, h dx$ implies $\int_{B_1(0)} (1 - \mu(x^*))\, h(x^*)\, dx^* = 0$ and hence $h(0) = 0$, by homoniticity of h in x^* variables.

Step III. We may proceed as in [16] to show that

$$r^{-m-2} \int_{\mathbb{C}_r} |y|^2 \, d\|r_{\omega_i} \# T_i\| \le \epsilon_i^2 \, r^{3/2} \tag{4.20}$$

for all $r \in \left(\frac{\theta_1}{2}, 2\theta_1\right)$ and for a suitable rotation γ_{ω_i}. In proving (4.17), both (4.20) and the energy comparison lemma play the important role. We refer the readers to [16] for details.

Corollary 4.1. *For $0 < r < 1$, one has*

$$E(T, r) + D(u, r) \le C(n)\, r^{\beta} \; (E(T, 1) + D(u, 1)) \tag{4.21}$$

for some $\beta = \beta(n) > 0$, whenever $E(T, 1) + D(u, 1) \le \epsilon_*^2$. Here $E(T, r)$ is the cylindrical excess of $\mathbf{T}$ with respect to a suitable hyperplane in R^n.

Proof. This follows easily from (4.17) by an iteration. We note that suppose π_j is a hyperplane in R^n which is obtained from π_{j-1} by a rotation r_{ω_j} with an angle $\omega_j \le c(n)\, \epsilon_{j-1}$, so that

$$E\left(T, \theta_1^j\right) + D\left(u, \theta_1^j\right) \le \frac{2}{3} \left(E\left(T, \theta_1^{j-1}\right) + D\left(u, \theta_1^{j-1}\right) \right)$$

for $j = 1, 2, \cdots$, where $\epsilon_j^2 = E\left(T, \theta_1^j\right) + D\left(u, \theta_1^j\right)$ for $j = 0, 1, 2, \cdots$, and $\pi_0 = R^m \times \{0\}$. Then, since

$$\sum_{j=1}^{\infty} \omega_j \le c(n) \sum_{j=1}^{\infty} \epsilon_{j-1} \le c(n)\, \epsilon_* \sum_{j=1}^{\infty} \left(\frac{2}{3}\right)^{j-1} = 3c(n)\, \epsilon_* < \frac{\pi}{3}$$

(if we take ϵ_* sufficiently small), we have $\lim_j = \pi_\infty$ exists. Moreover, we have $E(T, r) + D(u, r) \le c(n)\, r^{\beta} \cdot (E(T, 1) + D(u, 1))$. Here the excess on the left is taken with respect to the hyperplane π_∞.

Theorem 4.1. *Let (A,u) be a minimal solution of* (1.1) *and* (1.2). *Then $\partial A\lfloor\Omega \sim S$ is a $C^{1,\beta/2}$ hypersurface in Ω. Moreover, S is a relatively closed subset of* $\operatorname{spt}(\partial A\lfloor\Omega)$ *with $H^{n-1}(S)=0$.*

Proof. As we remarked at the beginning of this section, for H^{n-1} a.e $a\in\partial A\lfloor\Omega$, there is an $r=r_a\in(0,1)$ such that for $T=\partial A\lfloor\Omega$,

$$E(T,r)+D(u,r)\le\epsilon_*^2 \tag{4.22}$$

and that T satisfies the hypothesis of the Height Bound Lemma (with $\rho=r$). Then (4.21) and [8, §5.3] imply that $T\lfloor B_{\theta_r}(a)$ is represented by a $C^{1,\beta/2}$-graph. This completes the proof of Theorem 4.1.

We now return to the key first variation estimate which is needed in Step II of the proof of the Excess Improvement Lemma.

We consider deformations $h_t\#\mathbf{T}$ of $\mathbf{T}$ (and $h_t\# A$ of A, respectively), where

$$h_t(x,y)=(x,y+t\,\epsilon\,\eta(x))\in R^m\times R, \tag{4.23}$$

t is a parameter, $|t|\le1$, and $\eta(x)$ is a smooth function with compact support in $B_1(0)\subseteq R^m$ and $\epsilon^2=E(T,1)+D(u,1)$. We shall give an approximation for $\epsilon^{-2}\frac{d}{dt}\,\Psi(h_t\#T)$ in terms of η and f, the function in nonparametric approximation of $\mathbf{T}$. Here $\mathbf{T}$ satisfies all the hypotheses in the Excess Improvement Lemma.

Corresponding to Almgren's blow up map (cf. [8, §5.3]) $(x,y)\to(x,\epsilon y)\in R^m\times R$, we define $F(x)=\frac{1}{\epsilon}f(x)$.

Lemma 4.2. *If $\eta\in C_0^\infty(B_1)$ with $|\nabla\eta|\le1$, and if $\epsilon\le\epsilon_*$, $|t|\le1$, we have*

$$\Big|\epsilon^{-2}\frac{d}{dt}\Psi(h_{T\#}T)-\int_{B_1}A_{ij}F_{x_j}\cdot\eta_{xi}\,dx-t\int_{B_1}A_{ij}\,\eta_{xi}\,\eta_{xj}\,dx\Big|\le c(n)^{1/3}, \tag{4.24}$$

where $A_{ij}\frac{\partial}{\partial xi}\frac{\partial}{\partial xj}=L_0$ is an elliptic operator (*see* [4]).

Proof. See [4, p.114–115] or [8, §5.3]. We should point out that the proof of (4.24) uses only the Nonparametric Approximation Lemma. No minimality of $\mathbf{T}$ of any sort has been used in the arguments.

Lemma 4.3. *With the same hypothesis as Lemma* 4.2, *and in addition, $\eta\in C_0^\infty\left(B^m_{1-\delta}(0)\right)$, for some $\delta\in\left(0,\frac12\right)$, we have*

$$\Big|\int_{B_1(0)}A_{ij}\,F_{xi}\cdot\eta_{xj}\,dx\Big|\le C(n,\delta)(\epsilon^{\frac13}+\epsilon^{\frac{\delta_0}{2}}) \tag{4.25}$$

provided $\epsilon\le\epsilon_*(n,\delta)\in\left(0,\frac12\right)$, where $\delta_0=\frac{1}{2(n-1)\,p}$ (*see* (4.16)).

Proof. We use here the minimality of (A,u). If $|t|\le1$, by the Height Bound Lemma, one has that

$$\|\chi_A-\chi_{h_t\#A}\|_{L^1(B_1)}\le C(n)-\epsilon^{\frac{1}{2(n-1)}}$$

and also, that by minimality of (A,u),

$$\begin{aligned}\Psi(h_{t\#}T)-\Psi(T)&=\Psi(h_{t\#}TLB_1)-\Psi(TLB)\\&\ge-\int_{B_{1-\delta/2}}\left|\chi_A-\chi_{h_{t\#}A}\right||\nabla u|^2-C\,\epsilon^{\delta_0}\,\epsilon^2\end{aligned} \tag{4.26}$$

(cf. (4.14)). Here we have noted the fact that when $\epsilon\le\epsilon(n,\delta)$,

$$(A\sim h_{t\#}A)\cup(h_{t\#}A\sim A)\subset B_{1-\frac{\delta}{2}}(0).$$

Therefore, by reverse Hölder estimate for $|\nabla u|^2$,

$$\Psi\left(h_{t\#}T\right)-\Psi(t)\geq -C(n,\delta)\,\epsilon^{\delta_0}\,\epsilon^2. \tag{4.27}$$

On the other hand, Lemma 4.2 gives, after integration from 0 to t, the inequality

$$\left|\epsilon^{-2}\left(\Psi\left(h_{t\#}T\right)-\Psi(t)\right)-t\int_{B_1(0)}A_{ij}\,F_{x_j}\,\eta_{x_i}\,dx\right|\leq C(n)\left(\epsilon^{\frac{1}{3}}\,|t|+|t|^2\right). \tag{4.28}$$

By comparing (4.27) and (4.28) we obtain, after carefully choosing sgn(t), the bound

$$\left|\int_{B_1(0)}A_{ij}\,F_{x_j}\,\eta_{x_j}\,dx\right|\leq c(n)\,|t|+c(n)\,\epsilon^{\frac{1}{3}}+\frac{c(n,\delta)\,\epsilon}{|t|}. \tag{4.29}$$

Lemma 4.3 follows by taking $|t|=\epsilon^{\frac{\delta_0}{2}}$.

Finally, we note that as $\epsilon\to 0$ in the above lemma, we may take $\delta\to 0$. Thus, the blow-up function h in Step I of the Excess Improvement Lemma is a solution of $L_0\,h=0$ in $B_1(0)\subseteq R^m$.

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