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ON TOPOLOGICAL LINEAR CONTRACTIONS ON NORMED SPACES AND APPLICATION****

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Abstract

Simultaneous contractifications, simultaneous proper contractifications and semigroup (countable family or finite family) of commuting operators and of non-commuting operators are first given. Characterizations are given for a single bounded linear operator being a topological proper contraction. By using complexification of a real Banach space and by applying a fixed point theorem of Edelstein, it is shown that every compact topological strict contraction on a Banach space is a topological proper contraction. Finally, results on simultaneous proper contractification are applied to study the stability of a common fixed point of maps which are Fréchet differentiable at that point.

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§1. Introduction and Preliminaries

Throughout this paper $(X, \|\cdot\|)$ denotes a normed space and $\mathcal{N}(\|\cdot\|)$ denotes the collection of all norms on X which are equivalent to $\|\cdot\|$. For each $\|\cdot\|' \in \mathcal{N}(\|\cdot\|)$ and each $r > 0$, let $X_r(\|\cdot\|') = \{x \in X : \|x\|' \le r\}$ and $X_r^0(\|\cdot\|') = \{x \in X : \|x\|' < r\}$. If $A : X \to X$ is linear and is $\|\cdot\|$-continuous, then $\|A\|_p = \sup\{\|Ax\| : \|x\| \le 1\}$ is finite, and is called the $\|\cdot\|$-operator norm of A. Let $\mathcal{B}(X)$ be the unital normed algebra of all $\|\cdot\|$-continuous linear operators from X into itself, then $\|\cdot\|_p$ is a norm on $\mathcal{B}(X)$ such that $\|I\|_p = 1$ (where I denotes the identity operator on X) and $\|AB\|_p \le \|A\|_p\|B\|_p$ for all $A, B \in B(X)$; i.e. $\|\cdot\|_p$ is a unital algebra norm on $\mathcal{B}(X)$. Let $\mathcal{M}(\|\cdot\|_p)$ denote the collection of all unital algebra norms on $\mathcal{B}(X)$ which are equivalent to $\|\cdot\|_p$.

Remark 1.1. It is clear that if $\|\cdot\|' \in \mathcal{N}(\|\cdot\|)$, then $\|\cdot\|_p' \in \mathcal{M}(\|\cdot\|_p)$. However, if $\|\cdot\|_* \in \mathcal{M}(\|\cdot\|_p)$, $\| \ \|_*$ may not be an operator norm, i.e. there may not exist a $\|\cdot\|' \in \mathcal{N}(\|\cdot\|)$ such that $\|\cdot\|_p' = \|\cdot\|_*$ (see [10]). For characterizations of $\|\cdot\|_* \in \mathcal{M}(\|\cdot\|_p)$ being an operator norm on an infinite dimensional Banach space $(X, \|\cdot\|)$, we refer to [3].

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For each $A \in \mathcal{B}(X)$, the limit $r_\sigma(A) = \lim_{n\to\infty} \|A^n\|_p^{1/n}$ exists and is called the spectral radius of A. It is easy to see that $r_\sigma(A) = \inf\{\|A^n\|_p{}^{1/n} : n = 1, 2, \cdots\} = \inf\{\|A^n\|_p'{}^{1/n} : n = 1, 2, \cdots\}$ for any $\|\cdot\|' \in \mathcal{N}(\|\cdot\|)$. I. M. Gel'fand proved that, if $(X, \|\cdot\|)$ is a complex Banach space, $r_\sigma(A) = \sup\{|\lambda| : \lambda \in \sigma(A)\}$, where $\sigma(A)$ is the spectrum of A.

Definition 1.1. *Let $A \in \mathcal{B}(X)$. Then A is* (a) *a $\|\cdot\|$-proper contraction*[7,p.76] *if $\|A\|_p < 1$;* (b) *a topological proper contraction if there exists a $\|\cdot\|' \in \mathcal{N}(\|\cdot\|)$ such that A is a $\| \ \|'$-proper contraction;* (c) *a $\|\cdot\|$-strict contraction if $\|Ax\| < \|x\|$ whenever $x \neq 0$;* (d) *a topological strict contraction if there exists a $\|\cdot\|' \in \mathcal{N}(\|\cdot\|)$ such that A is $\|\cdot\|'$-strict contraction;* (e) *a $\|\cdot\|$-contraction if $\|A\|_p \le 1$;* (f) *a topological contraction if there exists a $\|\cdot\|' \in \mathcal{N}(\|\cdot\|)$ such that A is a $\|\cdot\|'$-contraction;* (g) *$\|\cdot\|$-power bounded if $\sup\{\|A^n\|_p : n = 1, 2, \cdots\} < \infty$.*

Remark 1.2. Let $A \in \mathcal{B}(X)$. (a) If A is $\|\cdot\|$-power bounded, define $\|x\|' := \sup\{\|A^n x\| : n \ge 0\}$ for all $x \in X$, then $\|\cdot\|' \in \mathcal{N}(\|\cdot\|)$, $\|A\|_p' \le 1$ and $\|B\|_p' \le \|B\|_p$ for all $B \in \mathcal{B}(X)$ with $AB = BA$; in particular, A is a topological contraction. (b) If A is $\|\cdot\|$-power bounded, then $r_\sigma(A) \le 1$. (c) If $r_\sigma(A) < 1$, then A is power bounded. (d) If $r_\sigma(A) > 1$, then A is not power bounded. (e) If $r_\sigma(A) = 1$, then A may or may not be power bounded.

Definition 1.2. *Let $A \in \mathcal{B}(X)$. Then $\{0\}$ is an attractor for $\|\cdot\|$-compact sets* (*respectively, $\|\cdot\|$-bounded sets, $\|\cdot\|$-bounded closed sets, $\|\cdot\|$-bounded open sets*) *under A if given any $r > 0$ and any $\|\cdot\|$-compact set* (*respectively, $\|\cdot\|$-bounded set, $\|\cdot\|$-bounded closed set, $\|\cdot\|$-bounded open set*) *K in X, there exists a positive integer N such that $A^n(K) \subseteq X_r^0(\|\cdot\|)$ for all $n \ge N$.*

Definition 1.3. *Let $\mathcal{F} \subseteq \mathcal{B}(X)$. Then $\mathcal{F}$ is $\|\cdot\|$-equicontinuous if given any $\epsilon > 0$, there exists a $\delta > 0$ such that $\|Ax\| \le \epsilon$ for all $\|x\| \le \delta$ and all $A \in \mathcal{F}$.*

In this paper, simultaneous contractifications, simultaneous proper contractifications and semigroup (countable family or finite family) of commuting operators and of non-commuting operators are first given. Characterizations are given for a single bounded linear operator being a topological proper contraction. By using complexification of a real Banach space and by applying a fixed point theorem of Edelstein, it is shown that every compact topological strict contraction on a Banach space is a topological proper contraction. Finally, results on simultaneous proper contractification are applied to studying the stability of a common fixed point of maps which are Fréchet differentiable at that point.

§2. Simultaneous Contractification

Theorem 2.1 *Let $\mathcal{S}$ be a subsemigroup* (*under composition*) *of $\mathcal{B}(X)$. Then the following conditions are equivalent:*

(i) *$\mathcal{S}$ is $\|\cdot\|_p$-bounded;*

(ii) *$\mathcal{S}$ is $\|\cdot\|$-equicontinuous;*

(iii) *there exists a $\|\cdot\|' \in \mathcal{N}(\|\cdot\|)$ such that* (a) *$\|S\|_p' \le 1$ for all $S \in \mathcal{S}$ and* (b) *for each $T \in \mathcal{B}(X)$ which commutes with all $S \in \mathcal{S}$, $\|T\|_p' \le \|T\|_p$.*

Proof. Since (i) $\Longleftrightarrow$ (ii) and (iii) (a) $\Rightarrow$ (i) are trivial, we only need to show (i) $\Rightarrow$ (iii). Without loss of generality we may assume that $I \in \mathcal{S}$. Let $M = \sup\{\|S\|_p : S \in \mathcal{S}\}$; then $M < \infty$. Define $\|x\|' = \sup\{\|Sx\| : S \in \mathcal{S}\}$ for all $x \in X$, then $\|x\| \le \|x\|' \le M\|x\|$ for all $x \in X$ and $\|\cdot\|' \in \mathcal{N}(\|\cdot\|)$. If $A \in \mathcal{S}$, then for each $x \in X$, $\|Ax\|' = \sup\{\|SAx\| : S \in \mathcal{S}\} \le \sup\{\|Sx\| : S \in \mathcal{S}\} = \|x\|'$. It follows that $\|A\|_p' \le 1$ for all $A \in \mathcal{S}$. This proves (iii) (a). (iii)(b) is immediate.

We note that by Remark 1.1, while Theorem 2.1 is an analogue of Theorem 1 in [2, p.18] (see also Lemma 7 in [1, p.21]), it is not its direct consequence. Similarly the following Theorem 2.2 is an analogue of Lemma 8 in [1, p.21] but not its direct consequence.

Theorem 2.2. *If $A_1, \cdots, A_N \in \mathcal{B}(X)$ are commuting, then corresponding to each $\epsilon > 0$, there exists a $\|\cdot\|' \in \mathcal{N}(\|\cdot\|)$ such that* (i) *$\|A_i\|'_p \le r_\sigma(A_i) + \epsilon$, for all $i = 1, 2, \cdots, N$,* (ii) *$\|A\|'_p \le \|A\|_p$ for each $A \in B(X)$ which commutes with $A_1, \cdots, A_N$.*

Proof. Let $\epsilon > 0$. For each $i = 1, \cdots, N$, let $S_i := A^i/(r_\sigma(A_i) + \epsilon)$; then $r_\sigma(S_i) = r_\sigma(A_i)/(r_\sigma(A_i)+\epsilon) < 1$. Let $\mathcal{S}$ be the subsemigroup (under composition) of $\mathcal{B}(X)$ generated by $\{I, S_1, \cdots, S_N\}$. Since $A_1, \cdots, A_N$ are commuting, $I, S_1, \cdots, S_N$ are also commuting. Thus $\mathcal{S} = \{S_1^{n_1} S_2^{n_2} \cdots S_N^{n_N} : n_i = 0, 1, 2, \cdots, i = 1, 2, \cdots, N\}$. Since for each $i = 1, \cdots, N$, $r_\sigma(S_i) = \lim_{n\to\infty} \|S_i^n\|_p{}^{1/n} < 1$, there exists a positive integer $m(i)$ such that $n \ge m(i)$ implies $\|S_i^n\|_p{}^{1/n} < 1$. Hence $\sup\{\|S\|_p : S \in \mathcal{S}\} \le \max\{\|S_1^{n_1}\|_p \|S_2^{n_2}\|_p \cdots \|S_N^{n_N}\|_p : n_i = 0, 1, \cdots, m(i) - 1, i = 1, \cdots, N\} < \infty$. Therefore $\mathcal{S}$ is $\|\cdot\|$-bounded so that by Theorem 2.1, there exists a $\|\cdot\|' \in \mathcal{N}(\|\cdot\|)$ such that $\|S\|'_p \le 1$ for all $S \in \mathcal{S}$ and $\|A\|'_p \le \|A\|_p$ for all $A \in B(X)$ commuting with $A_1, \cdots, A_N$. The desired conclusion follows.

The following extension of a result of Holmes[8] is obvious by Theorem 2.2.

Corollary 2.1. *Let $A \in \mathcal{B}(X)$, then $r_\sigma(A) = \inf \|A\|'_p$, where the infimum is taken over all norms $\|\cdot\|' \in \mathcal{N}(\|\cdot\|)$. Thus A is a topological proper contraction if and only if $r_\sigma(A) < 1$.*

Corollary 2.2. *Let $A_1, \cdots, A_N \in \mathcal{B}(X)$ be commuting. If for each $i = 1, \cdots, N$, there exists $\|\cdot\|^{(i)} \in \mathcal{N}(\|\cdot\|)$ such that $\|A_i\|_p^{(i)} < 1$, then there exists $\|\cdot\|' \in \mathcal{N}(\|\cdot\|)$ such that $\|A_i\|'_p < 1$ for all $i = 1, \cdots, N$.*

Proof. By Corollary 2.1, $r_\sigma(A_i) < 1$ for each $i = 1, \cdots, N$. Let $\alpha := \max\{r_\sigma(A_i) : i = 1, \cdots, N\}$, then $0 \le \alpha < 1$. Let $\epsilon > 0$ be such that $\alpha + \epsilon < 1$. The desired conclusion follows from Theorem 2.2.

It is easy to show by examples of matrices that Corollary 2.2 cannot be extended to a countably infinite family, and that commutativity is essential in Corollary 2.2.

Corollary 2.3. *Let $\mathcal{S}$ be a semigroup* (*under composition*) *of commuting topological proper contractions in $\mathcal{B}(X)$. If there exists $\|\cdot\|' \in \mathcal{N}(\|\cdot\|)$ such that $\|S\|'_p < 1$ for all but a finite number of $S \in \mathcal{S}$, then there exists $\|\cdot\|^* \in \mathcal{N}(\|\cdot\|)$ such that $\|S\|^*_p < 1$ for all $S \in \mathcal{S}$.*

Theorem 2.3. *Let $\mathcal{S}$ be a subsemigroup* (*under composition*) *of $\mathcal{B}(X)$ such that*

(1) *$\lim_{n\to\infty} \|S_1 \cdots S_n\|_p = 0$ for any choice of $S_1, \cdots, S_n \in \mathcal{S}$ and the limit depends on n only, i.e., for each $\epsilon > 0$, there is a positive integer N such that $\|S_1 \cdots S_n\|_p < \epsilon$ for all $S_1, \cdots, S_n \in \mathcal{S}$ with $n \ge N$;*

(2) *$\mathcal{S}$ is $\|\cdot\|_p$-bounded.*

Then there exist $\lambda \in (0, 1)$ and $\|\cdot\|^* \in \mathcal{N}(\|\cdot\|)$ such that $\|S\|^*_p \le \lambda$ for all $S \in \mathcal{S}$.

Proof. By (2) and Theorem 2.1, there exists a $\|\cdot\|' \in \mathcal{N}(\|\cdot\|)$ such that $\|S\|'_p \le 1$ for all $S \in \mathcal{S}$. By (1), there exists a positive integer N such that $\|S_1 \cdots S_n\|'_p < 1/2$ for all $S_1, \cdots, S_n \in \mathcal{S}$ with $n \ge N$. Let $\lambda = 1/\sqrt[N]{2}$, then $\lambda \in (0, 1)$. Define

$$\|x\|^* = \max\Big\{\|x\|', \quad \sup\Big\{\frac{\|S_1 \cdots S_n x\|'}{\lambda^n} : S_1, \cdots, S_n \in \mathcal{S} \text{ and } n \ge 1\Big\}\Big\}$$

for all $x \in X$. Clearly $\|\cdot\|^*$ is a norm on X such that $\|x\|' \le \|x\|^*$ for all $x \in X$. Note that for any $x \in X$, $S_1, \cdots, S_n \in \mathcal{S}$ with $kN \le n < (k+1)N$ where $k = 0, 1, 2, \cdots$,

$$\frac{\|(S_1 \cdots S_n)x\|'}{\lambda^n} \le \Big(\frac{\|S_1 \cdots S_N\|'_p}{\lambda^N}\Big) \cdots \Big(\frac{\|S_{(k-1)N} \cdots S_{kN}\|'_p}{\lambda^N}\Big)\Big(\frac{\|S_{kN+1} \cdots S_n\|'_p}{\lambda^{n-kN}}\Big)\|x\|'$$
$$\le \frac{1}{\lambda^{n-kN}}\|x\|' \le \frac{1}{\lambda^N}|x\|' = 2\|x\|'.$$

It follows that $\|x\|^* \le 2\|x\|'$ for all $x \in X$. Thus $\|\cdot\|^* \in \mathcal{N}(\|\cdot\|)$. Now let $S \in \mathcal{S}$. Then for each $x \in X$,

$$\|Sx\|^* = \max\Big\{\|Sx\|',\ \sup\Big\{\frac{\|S_1 \cdots S_n Sx\|'}{\lambda^n} :\ S_1, \cdots, S_n \in \mathcal{S} \text{ and } n \ge 1\Big\}\Big\}$$
$$= \lambda \max\Big\{\frac{\|Sx\|'}{\lambda},\ \sup\Big\{\frac{\|S_1 \cdots S_n Sx\|'}{\lambda^{n+1}} : S_1, \cdots, S_n \in \mathcal{S} \text{ and } n \ge 1\Big\}\Big\} \le \lambda\|x\|^*.$$

Hence $\|S\|^* \le \lambda$ for all $S \in \mathcal{S}$.

We remark here that Corollary 2.2 can also be derived from Theorem 2.3.

Theorem 2.4. *Let $\mathcal{S}$ be a countable family of commuting operators in $\mathcal{B}(X)$. Suppose* (i) *the subsemigroup generated by $\mathcal{S}$ in $\mathcal{B}(X)$ is $\|\cdot\|_p$-bounded and* (ii) *for each $S \in \mathcal{S}$, $\lim\limits_{n\to\infty} S^n x = 0$ for all $x \in X$. Then there exists a $\|\cdot\|' \in \mathcal{N}(\|\cdot\|)$ such that S is a $\|\cdot\|'$-strict contraction for all $S \in \mathcal{S}$.*

Proof. We may write $\mathcal{S} = \{S_1, S_2, \cdots\}$ for $\mathcal{S}$ is countable. By (i) and Theorem 2.1, there exists a $\|\cdot\|^* \in \mathcal{N}(\|\cdot\|)$ such that $\|S_i\|_p^* \le 1$ for all $i = 1, 2, \cdots$. For each $i = 1, 2, \cdots$, define $\|x\|^{(i)} = \sum\limits_{n=0}^{\infty} \frac{1}{2^n}\|S_i^n x\|^*$ for all $x \in X$. We have (a) $\|\cdot\|^{(i)} \in \mathcal{N}(\|\cdot\|)$ and $\|\cdot\|^* \le \|\cdot\|^{(i)} \le 2\|\cdot\|^*$; (b) S_i is a $\|\cdot\|^{(i)}$-strict contraction; and (c) $\|S_n\|_p^{(i)} \le \|S_n\|_p^* \le 1$ for all $n = 1, 2, \cdots$. Define $\|x\|' := \sum\limits_{n=1}^{\infty} \frac{1}{2^n}\|x\|^{(n)}$ for all $x \in X$. It follows that $\frac{1}{2}\|x\|^* \le \frac{1}{2}\|x\|^{(1)} \le \|x\|' \le 2\|x\|^*$ for all $x \in X$ and $\|\cdot\|' \in \mathcal{N}(\|\cdot\|)$. Let i be any positive integer. Then for each $x \in X$, by (c) above, we have

$$\|S_i x\|' = \sum_{n=1}^{\infty} \frac{1}{2^n}\|S_i x\|^{(n)} \le \sum_{n=1}^{\infty} \frac{1}{2^n}\|S_i\|_p^{(n)}\|x\|^{(n)} \le \sum_{n=1}^{\infty} \frac{1}{2^n}\|x\|^{(n)} = \|x\|'.$$

Thus if $\|S_i x\|' = \|x\|'$, we must have $\|S_i x\|^{(n)} = \|x\|^{(n)}$ for all $n = 1, 2, \cdots$. In particular, for $n = i$, $\|S_i x\|^{(i)} = \|x\|^{(i)}$. As S_i is a $\|\cdot\|^{(i)}$-strict contraction, $x = 0$. Therefore S_i is a $\|\cdot\|'$-strict contraction for all $i = 1, 2, \cdots$.

Corollary 2.4. *Let $(X, \|\cdot\|)$ be a Banach space and $A_1, \cdots, A_N \in \mathcal{B}(X)$ be commuting such that for each $i = 1, \cdots, N$, $\lim\limits_{n\to\infty} A_i^n x = 0$ for all $x \in X$. Then there exists $\|\cdot\|^* \in \mathcal{N}(\|\cdot\|)$ such that A_i is a $\|\cdot\|^*$-strict contraction for all $i = 1, \cdots, N$.*

Proof. For each $i = 1, \cdots, N$, since $\lim\limits_{n\to\infty} A_i^n x = 0$ for all $x \in X$, $\sup\limits_{n\ge1}\|A_i^n\|_p < \infty$ by Uniform Boundedness Principle. Let $M := \prod\limits_{i=1}^{N} \sup\{\|A_i^n\|_p : n \ge 1\}$ and $\mathcal{S}$ be the semigroup generated by $\{A_1, \cdots, A_N\}$. Then the conclusion follows from Theorem 2.4.

Note that a contraction A satisfying the property $\lim\limits_{n\to\infty} A^n x = 0$ for all x need not be a $\|\cdot\|$-strict contraction, e.g., the backward shift on ℓ^2 is such an operator.

Example 2.1. Define $A \in \mathcal{B}(\ell^2)$ by

$$Ax = \Big(0, \Big(1 - \frac{1}{2^2}\Big)x_1, \Big(1 - \frac{1}{3^2}\Big)\, x_2, \cdots, \Big(1 - \frac{1}{(n+1)^2}\Big)x_n, \cdots\Big)$$

for all $x = (x_1, x_2, \cdots) \in \ell^2$. Then A is a strict contraction such that $\lim_{n\to\infty} A^n x \neq 0$ for some x.

§3. Characterizations of Topological Proper Contractions

Proposition 3.1. *Let $A \in \mathcal{B}(X)$. If $\dim X < \infty$ and A is a $\|\cdot\|$-strict contraction, then A is a $\|\cdot\|$-proper contraction.*

Proof. Since $X_1(\|\cdot\|)$ is compact, there exists an $x_0 \in X$ such that $\|x_0\| = 1$ and $\|A\|_p = \|Ax_0\|$. But then $\|A\|_p = \|Ax_0\| < \|x_0\| = 1$ so that A is a $\|\cdot\|$-proper contraction.

Theorem 3.1. *Let $A \in \mathcal{B}(X)$. Consider the following conditions:*

(i) *A is a topological proper contraction.*

(ii) *A^N is a topological proper contraction for some positive integer N.*

(iii) *$r_\sigma(A) < 1$.*

(iv) *$\{0\}$ is an attractor for $\|\cdot\|$-bounded (respectively, $\|\cdot\|$-bounded open, $\|\cdot\|$-bounded closed) sets under A.*

(v) *$\{0\}$ is an attractor for $\|\cdot\|$-bounded (respectively, $\|\cdot\|$-bounded open, $\|\cdot\|$-bounded closed) sets under A^N for some positive integer N.*

(vi) *A is a topological strict contraction and for each number λ with $|\lambda| = 1$ and for each $z \in X$, the affine operator $U_{\lambda,z}$, defined by $U_{\lambda,z}(x) = \lambda Ax + z$ for all $x \in X$, has the property:*

(E_0) *For each $x \in X$, the sequence $(U^n_{\lambda,z}(x))^\infty_{n=1}$ is Cauchy.*

(vii) *A is a topological strict contraction and for each number λ with $|\lambda| = 1$ and for each $z \in X$, the affine operator $U_{\lambda,z}$ as defined in (vi) has the property:*

(E_1) *For each $x \in X$, the sequence $(U^n_{\lambda,z}(x))^\infty_{n=1}$ is bounded.*

(viii) *A is a topological strict contraction and for each number λ with $|\lambda| = 1$ and for each $z \in X$, the affine operator $U_{\lambda,z}$ as defined in (vi) has the property:*

(E_2) *For each $x \in X$, the mean ergodic sequence $\big(\frac{1}{N}\sum_{n=1}^{N} U^n_{\lambda,z}(x)\big)^\infty_{N=1}$ converges to a (unique) fixed point of $U_{\lambda,z}$ in X.*

(ix) *A is a topological strict contraction and for each number β with $|\beta| = 1$, the operator $\beta I - A$ is surjective, where I is the identity operator on X.*

(x) *A is a topological strict contraction;*

(xi) *A^N is a topological strict contraction for some positive integer N.*

(xii) *$\lim_{n\to\infty} A^n x = 0$ for all $x \in X$.*

We have

(a) (i) *to* (v) *are all equivalent.*

(b) (i) $\Longrightarrow$ (vi), (vi) $\Longrightarrow$ (vii) *and* (viii) $\Longrightarrow$ (ix).

(c) *If $(X, \|\cdot\|)$ is a reflexive Banach space, then* (vii) $\Longrightarrow$ (viii).

(d) *If $(X, \|\cdot\|)$ is a complex Banach space, then* (ix) $\Longrightarrow$ (iii). *Thus if $(X, \|\cdot\|)$ is a reflexive complex Banach space,* (i) *to* (ix) *are all equivalent.*

(e) *If X is finite dimensional, then* (i) *is equivalent to* (x), (xi) *and* (xii).

Proof. (a) (i) $\Rightarrow$ (ii) and (iv) $\Rightarrow$ (v) are trivial; (ii) $\Rightarrow$ (iii) follows from the fact that $r_\sigma(A) = \inf\{\|A^n\|_*^{1/n} : n \geq 1\}$ whenever $\|\cdot\|_* \in \mathcal{N}(\|\cdot\|)$; (iii) $\Rightarrow$ (i) follows from Corollary 2.1.

(i) $\Rightarrow$ (iv). Let $\|\cdot\|' \in \mathcal{N}(\|\cdot\|)$ be such that $\|A\|'_p < 1$. Given any $\epsilon > 0$, there exists a positive integer N such that $\|A\|'^N_p < \epsilon$. Then it follows that $A^n(X_1(\|\cdot\|')) \subseteq X^0_\epsilon(\|\cdot\|')$, for all $n \geq N$.

(v) $\Rightarrow$ (ii). Suppose $\{0\}$ is an attractor for $\|\cdot\|$-bounded closed (respectively $\|\cdot\|$-bounded open) sets under A^{N_1} for some positive integer N_1. Then for $\epsilon = \frac{1}{2}$, there exists a positive integer N_2 such that $A^{N_1 n}(X_1(\|\cdot\|)) \subseteq X^0_{\frac{1}{2}}(\|\cdot\|)$ for all $n \geq N_2$ (respectively $A^{N_1 n}(X^0_1(\|\cdot\|)) \subseteq X^0_{\frac{1}{2}}(\|\cdot\|)$ for all $n \geq N_2$). Take $N = N_1 N_2$.

(b) (i) $\Rightarrow$ (vi). Let $\|\cdot\|' \in \mathcal{N}(\|\cdot\|)$ be such that $\|A\|'_p < 1$. Then for each $x, y \in X$, $\|U_{\lambda,z}(x) - U_{\lambda,z}(y)\|' = \|\lambda Ax - \lambda Ay\|' \leq \|A\|'_p \|x - y\|'$ so that $U_{\lambda,z}$ is a Banach contraction with Lipschitz constant $\|A\|'_p < 1$. It follows that for each $x \in X$, the sequence $(U^n_{\lambda,z}(x))^\infty_{n=1}$ is Cauchy.

(vi) $\Rightarrow$ (vii) is trivial.

(viii) $\Rightarrow$ (ix). Let β be a number with $|\beta| = 1$. Let $z_0 \in X$ be given. Set $\lambda = \frac{1}{\beta}$ and $z = \frac{1}{\beta} z_0$; then $|\lambda| = 1$. By (viii) the affine operator $U_{\lambda,z}$ defined by $U_{\lambda,z}(x) = \lambda Ax + z$ for all $x \in X$ has the property (E_2). Let $x \in X$; then the sequence $(\frac{1}{N}\sum\limits_{n=1}^{N} U^n_{\lambda,z}(x))^\infty_{N=1}$ converges to a (unique) fixed point $\hat{x}$ of $U_{\lambda,z}$ in X. Therefore $\hat{x} = \lambda A\hat{x} + z = \frac{1}{\beta} A\hat{x} + \frac{1}{\beta} z_0$ so that $(\beta I - A)(\hat{x}) = z_0$.

(c) Let $x \in X$; then by (E_1), the sequence $(U^n_{\lambda,z}(x))^\infty_{n=1}$ is bounded so that the closed convex hull C of $\{U^n_{\lambda,z}(x) : n = 1, 2, \cdots\}$ is also bounded and hence weakly compact as X is reflexive. But then $\big(\frac{1}{N}\sum\limits_{n=1}^{N} U^n_{\lambda,z}(x)\big)^\infty_{N=1}$ is a sequence in C so that by Eberlein-Smulian theorem[4,p.430], $\big(\frac{1}{N}\sum\limits_{n=1}^{N} U^n_{\lambda,z}(x)\big)^\infty_{N=1}$ contains a subsequence which converges weakly to some $\hat{x} \in X$. By Edelstein's Theorem[5,Theorem A], $\frac{1}{N}\sum\limits_{n=1}^{N} U^n_{\lambda,z}(x) \to \hat{x}$ as $N \to \infty$ and $U_{\lambda,z}(\hat{x}) = \hat{x}$. (Note that since A is a topological strict contraction, a fixed point of $U_{\lambda,z}$ is necessarily unique.)

(d) Since A is a topological strict contraction, let $\|\cdot\|' \in \mathcal{N}(\|\cdot\|)$ be such that $\|Ax\|' < \|x\|'$ for all $x \in X$ with $x \neq 0$. Since $(X, \|\cdot\|)$ is a complex Banach space, by Gel'fand's theorem, $\sigma(A)$ is a non-empty compact subset of the complex field $\mathbb{C}$ and $r_\sigma(A) = \sup\{|\lambda| : \lambda \in \sigma(A)\}$. Let $|\lambda| = 1$; $\lambda I - A$ is surjective by (ix). Suppose $(\lambda I - A)(x) = 0$, then $\|Ax\|' = \|x\|'$ so that $x = 0$. It follows that $\lambda I - A$ is also injective. Therefore by open mapping theorem, $\lambda I - A$ has a bounded inverse so that $\lambda \notin \sigma(A)$. By compactness of $\sigma(A)$, $r_\sigma(A) < 1$.

(e) Suppose $\dim X < \infty$. Then (i) $\Rightarrow$ (x) and (x) $\Rightarrow$ (xi) are trivial and (xi) $\Rightarrow$ (ii) follows from Proposition 3.1; (i) $\Rightarrow$ (xii) is immediate and (xii) $\Rightarrow$ (iii) follows from the fact that (xii) implies $\|A^n\|_p \to 0$ as $n \to \infty$ as $\dim X < \infty$.

Theorem 3.2. *Let $A \in \mathcal{B}(X)$ be of rank 1. Suppose that A is either a topological strict contraction or $\lim\limits_{n\to\infty} A^n x = 0$ for all $x \in X$. Then A is a topological proper contraction.*

Proof. Let $0 \neq A = \phi \otimes y$; i.e., $Ax = \phi(x)y$ for all $x \in X$, where ϕ is a $\|\cdot\|$-continuous linear functional on X.

Case 1. Suppose that A is a topological strict contraction. Let $\|\cdot\|' \in \mathcal{N}(\|\cdot\|)$ be such that A is a $\|\cdot\|'$-strict contraction. Replacing ϕ by $\|y\|'\phi$ and y by $\frac{y}{\|y\|'}$, we may assume that $\|y\|' = 1$. Note that for each $x \in X$ with $x \neq 0$, $|\phi(x)| = \|\phi(x)y\|' = \|Ax\|' < \|x\|'$. Since

$\|y\|' > \|Ay\|' = |\phi(y)\ |\ \|y\|'$, we have $|\phi(y)| < 1$. Moreover, for each $x \in X$, $A^2x = \phi(x)\phi(y)y$ so that $\|A^2\|'_p = \sup\{|\phi(x)|\,|\phi(y)|\,\|y\|' : \|x\|' \le 1\} \le |\phi(y)| < 1$. Therefore by Theorem 3.2, A is a topological proper contraction.

Case 2. Suppose $\lim\limits_{n\to\infty} A^n x = 0$ for all $x \in X$. As $A^n x = \phi(x)(\phi(y))^{n-1}y$ for $n \ge 1$, and as $A \ne 0$, it follows that $\lim\limits_{n\to\infty} |\phi(y)|^n = 0$, and $|\phi(y)| < 1$. As $r_\sigma(A) = \lim\limits_{n\to\infty} \|A^n\|_p^{1/n} \le |\phi(y)|$, it follows that $r_\sigma(A) < 1$ and A is a topological proper contraction by Theorem 3.1.

Example 3.1. Let $X := \ell^1$ and $\|x\| := \sum\limits_{n=1}^{\infty} |x_n|$ for all $x = (x_n)_{n=1}^{\infty} \in X$. Define $\phi(x) := \sum\limits_{n=1}^{\infty} \frac{n}{n+1} x_n$, for all $x = (x_n)_{n=1}^{\infty} \in X$, and $A = \phi \otimes y$, i.e. $Ax = \phi(x)y$ for all $x \in X$. Then A is a rank 1 $\|\cdot\|$-strict contraction on X which is not a $\|\cdot\|$-proper contraction. This also shows that the conclusion of Proposition 3.1 may fail if X is infinite dimensional.

Example 3.2. Define $A : \ell^2 \to \ell^2$ by $Ax = \left(\frac{1}{2}x_1, \frac{2}{3}x_2, \cdots, \frac{n}{n+1}x_n, \cdots\right)$ for all $x = (x_1, x_2, \cdots) \in \ell^2$. Then A is a $\|\cdot\|$-strict contraction and $\lim\limits_{n\to\infty} A^n x = 0$ for all $x \in \ell^2$ but A is not a topological proper contraction.

§4. Topological Strict Contractions

Theorem 4.1 *Let* $(X, \|\cdot\|)$ *be a Banach space and* $A \in \mathcal{B}(X)$. *Then* (i) $\lim\limits_{n\to\infty} A^n x = 0$ *for all* $x \in X$ *if and only if* (ii) $\{0\}$ *is an attractor for compact sets under* A.

Proof. (i) $\Rightarrow$ (ii). By Corollary 2.4, let $\|\cdot\|^* \in \mathcal{N}(\|\cdot\|)$ be such that A is a $\|\cdot\|^*$-strict contraction. Let d be the metric on X induced by $\|\cdot\|^*$; i.e., $d(x, y) = \|x - y\|^*$ for all $x, y \in X$. Then the Theorem in [9, p.341] asserts that $\{0\}$ is an attractor for compact sets under A.

(ii) $\Rightarrow$ (i). Obvious.

Note that if A is a topological proper contraction, we must have $\lim\limits_{n\to\infty} A^n x = 0$ for all $x \in X$. In view of Corollary 2.4 and Theorem 4.1, when $(X, \|\cdot\|)$ is a Banach space, the phrase "A is a topological strict contraction" in (vi), (vii) and (viii) of Theorem 3.1 can be replaced by the phrase "$\lim\limits_{n\to\infty} A^n x = 0$ for all $x \in X$" or by the phrase "$\{0\}$ is an attractor for compact sets under A".

Theorem 4.2. *Let* $(X, \|\cdot\|)$ *be a Banach space. If* $A \in \mathcal{B}(X)$ *is a compact topological strict contraction, then* A *is a topological proper contraction.*

Proof. Let $\|\cdot\|' \in \mathcal{N}(\|\cdot\|)$ be such that $\|Ax\|' < \|x\|'$ for all $x \in X$ with $x \ne 0$.

Case 1. Suppose X is a complex Banach space. Let $\lambda \in \sigma(A) \setminus \{0\}$. Since A is compact, by Riesz-Schauder Theorem[18,p.284], there exists an $x \in X$ with $x \ne 0$ such that $Ax = \lambda x$. But then $|\lambda|\|x\|' = \|Ax\|' < \|x\|'$ so that $|\lambda| < 1$. By Gel'fand's theorem, $r_\sigma(A) = \sup\{|\lambda| : \lambda \in \sigma(A)\} < 1$ and hence A is a topological proper contraction by Theorem 3.1.

Case 2. Suppose X is a real Banach space. Let $\tilde{X}$ be the complexification of X and $\tilde{A}$ be the complexification of A (see [16]). Since A is compact, $\tilde{A}$ is compact. Let $x \in X$. Since $\|A\|' \le 1$, $(A^n x)_{n=1}^{\infty}$ is a bounded sequence in AX. Since A is compact, there exist a subsequence $(A^{n_i} x)_{i=1}^{\infty}$ of $(A^n x)_{n=1}^{\infty}$ and a $y \in X$ such that $\lim\limits_{i\to\infty} A^{n_i} x = y$. As A is a $\|\cdot\|'$-strict contraction, by a fixed point theorem of Edelstein[6,Theorem 1], we have $Ay = y$ and $\lim\limits_{n\to\infty} A^n x = y$. Then $y = 0$ and hence $\lim\limits_{n\to\infty} A^n x = 0$ for all $x \in X$. It follows that

$\lim\limits_{n\to\infty} \tilde{A}^n\tilde{x} = 0$ for all $\tilde{x} \in \tilde{X}$ so that $\tilde{A}$ is a topological strict contraction on $\tilde{X}$ by Corollary 2.4. Therefore, Case 1 and Theorem 3.1 imply that $r_\sigma(\tilde{A}) < 1$. But then $r_\sigma(A) < 1$, so that A is a topological proper contraction by Theorem 3.1.

§5. Application to Stability of a Common Fixed Point

Throughout this section, X is assumed to be a real Banach space and Ω denotes a non-empty subset of X. A map $f : \Omega \to X$ is said to be Fréchet differentiable at an interior point x^* of Ω if there is $Df(x^*) \in \mathcal{B}(X)$ such that

$$f(x^* + h) = f(x^*) + Df(x^*)h + w(x^*, h),$$
$$w(x^*, h) = \text{o}(|h|) \text{ as } h \to 0 \left(\text{i.e. } \frac{w(x^*, h)}{\|h\|} \to 0 \text{ as } h \to 0\right).$$

$Df(x^*)$ is necessarily unique and is called the Fréchet derivative of f at x^*.

In this section, as an application of Theorem 2.3, we shall generalize Kitchen's extension[11] of Ostrowski's theorem[14,2nd ed., pp.161−164] to a countable family of maps and study its relevance to the stability of perturbed linear difference equations. In particular, we shall show that Ostrowski's theorem is equivalent to O. Perron's stability theorem[15] for perturbed linear difference equations. Denote by I^+ the set of all non-negative integers. For the rest of this section, fix any map $f : I^+ \times \Omega \to X$ and let $f_n(\cdot) := f(n, \cdot)$ for each $n \in I^+$. If $x : I^+ \to \Omega$, consider the following non-stationary iterative process

$$x(n+1) = f_n(x(n)), \quad n \in I^+. \tag{NI}$$

Definition 5.1. *A solution $x : I^+ \to \Omega$ of* (NI) *is said to be stable if corresponding to each $\epsilon > 0$, there exists a $\delta > 0$ with the property that if $\hat{x} : I^+ \to \Omega$ is another solution of* (NI) *such that $\|\hat{x}(0) - x(0)\| \le \delta$, then $\|\hat{x}(n) - x(n)\| \le \epsilon$ for all $n \in I^+$. If, in addition, $\hat{x}(n) - x(n) \to 0$ as $n \to \infty$, x is said to be an asymptotically stable solution of* (NI).

Definition 5.2. *$\{f_n\}_{n\in I^+}$ is said to be uniformly Fréchet differentiable at an interior point ξ of Ω if corresponding to each $\epsilon > 0$, there exists a $\delta > 0$ such that $S(\xi;\delta) := \{y \in X : \|y - \xi\| \le \delta\} \subset \Omega$ and $\|f_n(y) - f_n(\xi) - Df_n(\xi)(y - \xi)\| \le \epsilon\|y - \xi\|$ for all $y \in S(\xi;\delta)$ and $n \in I^+$.*

Theorem 5.1. *Let ξ be an interior point of Ω such that ξ is a common fixed point of $\{f_n\}_{n\in I^+}$, i.e. $f_n(\xi) = \xi$ for each $n \in I^+$. Suppose that $\{f_n\}_{n\in I^+}$ is uniformly Fréchet differentiable at ξ and $\sup\limits_{n\in I^+} \|Df_n(\xi)\|_p < 1$. Then the constant map $x : I^+ \to \Omega$ defined by $x(n) := \xi$ for each $n \in I^+$ is an asymptotically stable solution of* (NI).

Proof. Since ξ is a common fixed point of $\{f_n\}_{n\in I^+}$, it is clear that the constant map defined by $x(n) := \xi$ for each $n \in I^+$ is a solution of (NI). Set $\alpha := \sup\limits_{n\in I^+} \|Df_n(\xi)\|_p$, then $\alpha < 1$. Let $\epsilon > 0$ be given such that $\beta := \epsilon + \alpha < 1$. Since $\{f_n\}_{n\in I^+}$ is uniformly Fréchet differentiable at ξ, there exists a $\delta \in (0, \epsilon)$ such that $S(\xi;\delta) \subset \Omega$ and $\|f_n(y) - f_n(\xi) - Df_n(\xi)(y - \xi)\| \le \epsilon\|y - \xi\|$ for all $y \in S(\xi;\delta)$ and $n \in I^+$. Now let $\hat{x} : I^+ \to \Omega$ be another solution of (NI) such that $\|\hat{x}(0) - x(0)\| \le \delta$, then $\hat{x}(0) \in S(\xi;\delta)$ so that

$$\begin{aligned}
\|\hat{x}(1) - x(1)\| &= \|f(0, \hat{x}(0)) - f(0, x(0))\| \\
&\le \|f_0(\hat{x}(0)) - f_0(\xi) - Df_0(\xi)(\hat{x}(0) - \xi)\| + \|Df_0(\xi)(\hat{x}(0) - \xi)\| \\
&\le \epsilon\|\hat{x}(0) - \xi\| + \|Df_0(\xi)\|\,\|\hat{x}(0) - \xi\| \\
&\le (\epsilon + \alpha)\|\hat{x}(0) - \xi\| = \beta\|\hat{x}(0) - \xi\| \le \delta < \epsilon.
\end{aligned}$$

We can now show by induction that $\|\hat{x}(n)-x(n)\| \leq \beta^n \|\hat{x}(0)-\xi\| < \delta < \epsilon$ for each $n \in I^+$. Since $\beta < 1$, we also have $\hat{x}(n) - x(n) \to 0$ as $n \to \infty$.

Theorem 5.2. *Let $\mathcal{S}$ be a subsemigroup of $\mathcal{B}(X)$ such that* (1) $\lim\limits_{n\to\infty} \|S_1 \cdots S_n\|_p = 0$ *for any choice of $S_1, \cdots, S_n \in \mathcal{S}$ and the limit depends on n only and* (2) *$\mathcal{S}$ is $\|\cdot\|_p$-bounded. Let ξ be an interior point of Ω which is a common fixed point of $\{f_n\}_{n\in I^+}$. Suppose that $\{f_n\}_{n\in I^+}$ is uniformly Fréchet differentiable at ξ such that for each $n \in I^+$, $Df_n(\xi) \in \mathcal{S}$. Then the constant map $x : I^+ \to \Omega$ defined by $x(n) := \xi$ for each $n \in I^+$ is an asymptotically stable solution of* (NI).

Proof. By Theorem 2.3, there exist $\lambda \in (0, 1)$ and $\|\cdot\|^* \in \mathcal{N}(\|\cdot\|)$ such that $\|S\|_p^* \leq \lambda$ for all $S \in \mathcal{S}$. The desired assertion follows from Theorem 5.1.

When $\mathcal{S}$ is generated by a single operator, Theorem 5.2 reduces to a result of M. H. Shih[17] which generalizes the finitely dimensional result of J.M.Ortega and W.C.Rheinboldt[13,p.349].

Let $g : \Omega \to X$. Now consider the following stationary iterative process

$$x(n + 1) = g(x(n)), \ \ n \in I^+. \tag{AI}$$

In Theorem 5.1, if $f_n = g$ for all $n \in I^+$, we have the following strengthened version of a result of J. W. Kitchen[11].

Corollary 5.1. *Let $g : \Omega \to X$ be a map and ξ be a fixed point of g in the interior of Ω. Suppose that g is Fréchet differentiable at ξ and $r_\sigma(Dg(\xi)) < 1$. Then the constant map $x : I^+ \to \Omega$ defined by $x(n) := \xi$ for each $n \in I^+$ is an asymptotically stable solution of* (AI).

Example 5.1. For each $n \in I^+$, let $A_{2n} := \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$ and $A_{2n+1} := \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$. Then $\{A_n\}_{n\in I^+}$ is uniformly Fréchet differentiable at $x^* = 0$ and $A_n x^* = x^*$ for all $n \in I^+$. As $r_\sigma(A_n) = 0$ for all $n \in I^+$, $\sup\limits_{n\in I} r_\sigma(A_n) = 0 < 1$. For any $\delta > 0$, let $x_0 := \begin{pmatrix} 0 \\ \delta \end{pmatrix}$. Define the sequence $\{x_n\}$ by $x_{n+1} = A_n x_n$ for $n = 0, 1, \cdots$. Then $\|x_n\| \to \infty$ as $n \to \infty$. Therefore the conclusion of Theorem 5.1 fails. Thus the condition $\sup\limits_{n\in I^+} \|Df_n(x^*)\|_p < 1$ in Theorem 5.1 cannot be replaced by $\sup\limits_{n\in I^+} r_\sigma(Df_n(x^*)) < 1$.

For the remaining of this section, Ω is assumed to be an open neighborhood of 0. Fix $p : I^+ \times \Omega \to X$, $x : I^+ \to \Omega$ and $A_n \in \mathcal{B}(X)$ for $n \in I^+$ and let $A(n, \cdot) := A_n(\cdot)$ for $n \in I^+$. Consider the following non-autonomous perturbed linear difference equation

$$x(n + 1) = A(n, x(n)) + p(n, x(n)), n \in I^+. \tag{NE}$$

Definition 5.3. *$p(n, x)$ is $o(\|x\|)$ uniformly with respect to $n \in I^+$ if given any $\epsilon > 0$, there exists a $\delta > 0$ such that $\|x\| \leq \delta$ implies $\|p(n, x)\| \leq \epsilon\|x\|$ for all $n \in I^+$.*

Note that the notion of $o(\|x\|)$ uniformly with respect to $n \in I^+$ is independent of equivalent norms.

Theorem 5.3. *Let $\mathcal{S}$ be a subsemigroup of $\mathcal{B}(X)$ such that* (1) $\lim\limits_{n\to\infty} \|S_1 \cdots S_n\|_p = 0$ *for any choice of $S_1, \cdots, S_n \in \mathcal{S}$ and the limit depends on n only and* (2) *$\mathcal{S}$ is $\|\cdot\|_p$-bounded. Suppose for each $n \in I^+$, $A_n \in \mathcal{S}$ and $p(n, x)$ is $o(\|x\|)$ uniformly with respect to $n \in I^+$. Then the zero map $x \equiv 0 : I^+ \to X$ is an asymptotically stable solution of* (NE).

Proof. Set $f(n, x) := A(n, x) + p(n, x)$ for $(n, x) \in I^+ \times \Omega$. Since $p(n, x)$ is $o(\|x\|)$ uniformly with respect to $n \in I^+$, we see that $\{f_n\}_{n\in I^+}$ is uniformly Fréchet differentiable at $\xi = 0$ such that $Df_n(\xi) = A_n$ for each $n \in I^+$. Note that $\xi = 0$ is also a common fixed point of $\{f_n\}_{n\in I^+}$. Thus the assertion follows from Theorem 5.2.

We have derived Theorem 5.3 from Theorem 5.2. In fact, these two results are equivalent. This can be seen from the following: If each f_n is Fréchet differentiable at ξ for which $f_n(\xi) = \xi$ for each $n = 0, 1, \cdots$, then (NI) may be written as $z(n+1) = A(n, z(n)) + p(n, z(n)), \quad n \in I^+$, where $A_n := Df_n(\xi), \quad z(n) := x(n) - \xi$, and $p(n, x) := f_n(x) - f_n(\xi) - Df_n(\xi)(x - \xi)$. Thus the conclusion of Theorem 5.2 follows from Theorem 5.3.

Now let the map $x' : I \to X$ be defined by $x'(n) = x(n+1)$ for each $n \in I^+$. Let $A \in B(X)$ and $q : \Omega \to X$. Consider the following autonomous perturbed linear difference equation

$$x' = A(x) + q(x). \tag{AE}$$

If the semigroup $\mathcal{S}$ is generated by A and $p(n, x) \equiv q(x)$ for all $n \in I^+$ in Theorem 5.3, we have

Corollary 5.2. *If $r_\sigma(A) < 1$ and $q(x)$ is $o(\|x\|)$, then the zero map $x \equiv 0 : I^+ \to X$ is an asymptotically stable solution of* (AE).

Corollary 5.2 generalizes O. Perron's theorem[15] from a finite dimensional space to an arbitrary Banach space. From the above remark, we readily see that Corollaries 5.1 and 5.2 are equivalent. Consequently, Ostrowski's theorem is equivalent to Perron's theorem. (For an elegant approach of Perron's theorem in $\mathbb{R}^n$ via Liapunov's direct method, we refer to J. P. LaSalle's book[12,p.18].)

References

[1] Bonsall, F. F. & Duncan, J., Numerical ranges of operators on normed spaces and of elements of normed algebras, Cambridge University Press, London, 1971.

[2] Bonsall, F. F. & Duncan, J., Complete normed algebras, Springer-Verlag, Berlin, 1973.

[3] Chuan, J. C. & Tan, K. K., Operator norms on $L(E)$, *J. Math. Anal. Appl.,* **97**(1983), 214–218.

[4] Dunford, N. & Schwartz, J. T., Linear operators, Part I: General theory, Interscience, New York, 1958.

[5] Edelstein, M., On some aspects of fixed point theory in Banach spaces, in The Geometry of Metric and Linear Spaces, Springer-Verlag, Berlin, Heidelberg, New York, **490** (1975), pp. 84–90.

[6] Edelstein, M., On fixed point and periodic points under contractive mappings, *J. London Math. Soc.,* **37**(1962), 74–79.

[7] Halmos, P. R., A Hilbert space problem book, Van Nostrand, Princeton, 1967.

[8] Holmes, R. B., A formula for spectral radius of an operator, *Amer. Math. Monthly,* **75**(1968), 163–166.

[9] Janos, L., Ko, H. M. & Tan, K. K., Edelstein contractivity and attractors, *Proc. Amer. Math. Soc.,* **76** (1979), 339–344.

[10] Ljubic, Ju. I., Operator norms of matrices, *Uspehi Mat. Nauk,* **18**(1963), 161–164 (in Russian).

[11] Kitchen, J. W., Concerning the convergence of iterates to fixed points, *Studia Math.,* **27**(1966), 247–249.

[12] LaSalle, J. P., The stability of dynamical systems, SIAM, Regional Conference Series in Applied Mathematics, **25**, 1976.

[13] Ortega, J. M. & Rheinboldt, W. C., Iterative solutions of nonlinear equations in several variables, Academic Press, New York, 1970.

[14] Ostrowski, A. M., Solution of equations and systems of equations, Academic Press, New York, 1960 (First ed.), 1966 (Second ed.).

[15] Perron, O., Über Sabilität und asymptotische Vechalten der Lösungen eines Systems endlicher Differenzengleichungen, *J. Reine Angew. Math.,* **161**(1929), 41–61.

[16] Rickart, C. E., General theory of Banach algebras, Van Nostrand, Princeton, 1960.

[17] Shih, M. H., A further generalization of the Ostrowski theorem in Banach spaces, *Proc. Japan Acad.,* **57**(1981), 168–170.

[18] Yosida, K., Functional analysis, Springer-Verlag, Berlin, Heidelberg, New York, 1968.