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EXISTENCE OF LIMIT CYCLES IN A MULTIPLY-CONNECTED REGION**

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Abstract

In this paper, a conjecture about the existence of limit cycles in a multiply-connected region proposed by Ye Yanqian is proved.

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Bendixson's annulus theorem in doubly connected region is well-known[1,2]. Recently, Ye Yanqian investigated the existence of limit cycles in multiply-connected region by using the annulus theorem, and proposed the following

Conjecture. Let G be an n multiply-connected region with outer boundary L_1. Assume the inner boundaries $L_2, L_3, \cdots, L_n$ are all sources of a given vector field $\overline{V}$, σ_1 and ν_1 are the numbers of inner and outer contacting points of the vector field $\overline{V}$ on L_1, respectively. Assume that

(a) $\sigma_1 = \nu_1 + 2(n-2)$, $n = 3, 4, \cdots$, $\nu_1 = 0, 1, 2, \cdots$;

(b) there is no critical point of $\overline{V}$ in G;

(c) all inner contacting trajectories of L_1 come from outside of L_1.

Then around each of L_i there exists a stable limit cycle for $i = 2, 3, \cdots, n$.

In order to prove this conjecture, we first introduce some concepts.

Definition 1. *If an inner contacting trajectory divides the inner boundaries of G into two groups, and the trajectory segment passing through this inner contacting point still remains in G (or goes out of G) with the increase of time, then we say this inner contacting trajectory is of type* I (*or type* II).

If an inner contacting trajectory does not separate the inner boundaries of G, and the trajectory segment passing through this inner contacting point still remains in G (or goes out of G) with the increase of time, then this inner contacting trajectory is called of type III (*or type* IV).

Remark 1. We know from Definition 1 that every inner contacting trajectory of type II, III or IV corresponds to at least one outer contacting point on L_1. Furthermore, the inner contacting trajectory of type IV has only two different topological phase portraits. As an example, Figure 1 shows the possible phase portrait of inner contacting trajectories of type IV in a triply connected region. Since every inner contacting trajectory of type

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IV corresponds to at least two outer contacting points on L_1, so when $\nu_1 = 1$, there is no trajectory of type IV in the corresponding phase portrait; but when $\nu_1 = 2$, there might be trajectory of type IV (see Fig.2).

(1) (2)

Fig.1

Fig.2 Fig.3

Proof of the Conjecture. We use mathematical induction in n to prove this conjecture. We distinguish several cases.

(A) Assume $n = 3$. G has two inner boundaries, denoted by L_2 and L_3.

(1) $\nu_1 = 0$. We know from the assumption (a) and Remark 1 that $\sigma_1 = 2$, there is no trajectory of type II, III and IV in G. So the two inner contacting trajectories l_1 and l_2 of type I must pass through G between L_2 and L_3, their topological structure can be shown in Fig.3. Otherwise, there is a critical point in G. This contradicts to the assumption (b). Thus, we can get two Bendixson annulus with l_1 and l_2 as parts of their boundaries, respectively. By the classical Bendixson annulus theorem in doubly connected region there are stable limit cycles around each of L_2 and L_3.

(2) When $\nu_1 = 1$, then $\sigma_1 = 3$. In a way similar to the analysis in (1), there are two trajectories of type I in G. If a third trajectory is also of type I, we can easily prove that G contains a critical point, this induces absurdity. The third trajectory must be of type II or type III, but not type IV. The phase portraits with different topological structure have only four cases (see Fig.4). Thus, we also can obtain two Bendixson's annulus with L_2 and L_3 as their boundaries, respectively. Therefore, there exist limit cycles in G surrounding L_2 and L_3, respectively.

Definition 2. *If two neighbouring inner contacting trajectories of type* I (*do not consider the trajectories of type* II *locating between them*) *divide the inner boundaries in G into the same number of two groups, and the trajectories after passing through inner contacting points lie in different regions separated by them, we call this pair of contacting trajectories backward.*

If two inner contacting trajectories of both type I, *or both type* III, *or one type* I *and another type* III, *after passing through inner contacting points, and still staying in G, lie in the same region, these two inner contacting trajectories are called to be face to face.*

Remark 2. From the Definition 2, trajectories passing through A and B in Fig.4 are both backward, and those passing through C and D are face to face.

(1)

(2) (3) (4)

Fig.4

(3) $\nu_1 \geq 2$, $\sigma_1 = \nu_1 + 2$. There are exactly two inner contacting trajectories l_1 and l_2, they are backward and of type I in G, and no other trajectory of type I (by Remark 1, there are at least two inner contacting trajectories of type I. Because G is a triply connected region, if there exists a third trajectory of type I in G, it must locate between L_2 and L_3. Thus, G must contain at least one critical point. This is a contradiction). Furthermore, if there exist trajectories of type II, they can only lie between l_1 and l_2. The trajectories of type IV can only lie in the region constructed by L_1 and the trajectories of type II or III, containing no inner boundaries of G. Otherwise, $\sigma_1 < \nu_1 + 2$, this contradicts the assumption (a). Therefore, we can get two Bendixson annulus each with L_2 or L_3 as its inner boundary, the simple closed curve constructed by l_1, l_2, L_1 and the trajectories of type III are their outer boundaries. So, in this case, the conjecture is also correct by Bendixson annulus theorem.

(B) Assume $n = 4$, $\sigma_1 = \nu_1 + 4$. The domain G has three inner boundaries L_2, L_3 and L_4.

(1) $\nu_1 = 0$, $\sigma_1 = 4$. Since there is no outer contacting point on L_1, from the assumption (c) G contains four trajectories of type I, and no other trajectories. We know from the above proof that there are exactly two pairs of inner boundaries of G which are separated by exactly two backward trajectories of type I, respectively, their phase portraits with different topological structures have only two cases (see Fig.5). Thus, we still can obtain three Bendixson's annuluses with L_2, L_3 and L_4 as their inner boundaries, respectively. So, G must contain three stable limit cycles surrounding L_2, L_3 and L_4, respectively.

(1) (2)

Fig.5

(2) $\nu_1 \geq 1$, $\sigma_1 = \nu_1 + 4$. We know from Definition 1 and Remark 1 that G contains at least four trajectories of type I. In a way similar to the previous analysis, there are exactly two pairs of inner boundaries which are separated by exactly two backward trajectories of type I, respectively, between the third pair of boundaries there are exactly four trajectories mentioned above. So, there are only four trajectories of type I in G. These four trajectories of type I separate the inner boundaries of G into three regions, and the trajectories of type II can only locate between two backward trajectories of type I, the trajectories of type IV can only lie in the region constructed by L_1, and the trajectories of type II or III, which contains no inner boundaries of G. Thus, all the trajectories of type I and III and the parts of boundary L_1 construct outer boundaries of three Bendixson's annulus, their inner boundaries are those of G. Hence, there exist stable limit cycles surrounding L_2, L_3 and L_4, respectively.

(C) Assume that the conjecture is correct for the $n-1$ multiply-connected region, we will prove that it is also correct for the n multiply-connected region.

Since $\sigma_1 = \nu_1 + 2(n-2)$, and every inner contact trajectories of type II, or III or IV must correspond to at least one outer contact point, so there are at least $2(n-2)$ trajectories of type I in G. Because an n multiply-connected region has $n-1$ inner boundaries, hence exactly $n-2$ trajectories of type I can divide each of the inner boundaries of G from others. Now, since different trajectories do not intersect and contact, and G does not contain critical point, so there exist other $n-2$ trajectories of type I, which and the above ones construct $n-2$ pairs of trajectories of type I with backward inner contact points. Furthermore, there is no other trajectory of type I which can locate between any pairs of trajectories. So, there are exactly $2(n-2)$ inner contact trajectories of type I in G, they divide G into $n-1$ different regions, each of them contains an inner boundary of G; and the sum of the number of the trajectories of type II, III and IV equals ν_1.

Let l_1 and l_2 be two backward trajectories of type I which divide some inner boundary of G (denoted by L_2) from others, and l_2 is neighbouring L_2. Since in the region G_1 formed by l_2 and the parts of the boundary L_1, which contains L_2, there is no trajectory of type I, so there is also no trajectories of type II, the trajectories of type IV can only locate in the region constructed by L_1 and the trajectories of type III. Thus, l_2, the parts of boundary L_1 and the trajectories of type III (maybe, do not exist) form the outer boundary L_1^* of a Bendixson's annulus (the number of inner contacting points of the trajectories of type III and IV with L_1^* equals that of outer contacting points), its inner boundary is L_2.

Let $G_2 = G - G_1$, $\overline{L}_1$ be the outer boundary of G_2 formed by l_1 and the other part of the boundary L_1, $\overline{\sigma}_1$ and $\overline{\nu}_1$ be the numbers of inner and outer contacting points on $\overline{L}_1$, respectively, then G_2 is an $n-1$ multiply-connected region. Moreover, $\overline{\sigma}_1 = \overline{\nu}_1 + 2(n-3)$. We know from the inductive assumption that there exist $n-2$ stable limit cycles surrounding each of the $n-2$ inner boundaries, respectively. This proves that there are $n-1$ stable limit cycles which distribute around each of the $n-1$ inner boundaries.

The induction axioms guarantee the correction of the conjecture.

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