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GLOBAL TOPOLOGICAL PROPERTIES OF HOMOGENEOUS VECTOR FIELDS IN R^3***

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Abstract

In this paper, the authors prove that the flows of homogeneous vector field $Q(x)$ at infinity are topologically equivalent to the flows of the tangent vector field $Q_T(u)$ ($u \in S^2$) on the sphere S^2, and show the theorems for the global topological classification of $Q(x)$. They derive the necessary and sufficient conditions for the global asymptotic stability and the boundedness of vector field $Q(x)$, and obtain the criterion for the global topological equivalence of two homogeneous vector fields.

Keywords Tangent vector field, Invariant cone, Global topological equivalence
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§0. Introduction

Let $\mathbf{X}$ be the set of homogeneous polynomial vector fields of degree $m(m > 1)$ in R^3, for each $Q(x) \in \mathbf{X}$, we have

$$Q(x) = Q_1(x)\frac{\partial}{\partial x_1} + Q_2(x)\frac{\partial}{\partial x_2} + Q_3(x)\frac{\partial}{\partial x_3}, \tag{0.1}$$

where $x = (x_1, x_2, x_3) \in R^3$, each $Q_i(x)$ being a homogeneous polynomial of degree m.

In this paper, we always assume that the origin $O(0, 0, 0)$ is the only isolated singularity of $Q(x)$.

[1] studied the geometric properties of trajectories of $Q(x)$ in the neighbourhood of the origin O. [2] discussed the topological classification of trajectories of $Q(x)$. Since [2] did not analyze the geometric properties of the flows of $Q(x)$ at infinity, therefore, the topological classification is incomplete. For example, in [2], there are seven kinds of different invariant cones of $Q(x)$, which are parabolic cone, elliptic cone, hyperbolic cone, three kinds of cones of type P, and center-type cone. Nevertheless, by the global analysis of $Q(x)$, we have discovered that there are at least sixteen kinds of different invariant cones about $Q(x)$ (in §2).

If $Q(x) \in \mathbf{X}$, [3] studied the stability of $Q(x)$ and gave the necessary conditions for the stability of $Q(x)$. We shall ask the following problem:

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Problem 1. If $Q(x) \in \mathbf{X}$, how about the necessary and sufficiant conditions for the global stability of $Q(x)$?

If $Q(x)$, $\bar{Q}(x) \in \mathbf{X}$, let $Q_T(u)$, $\bar{Q}_T(u)$ be Morse-Smale, [4] researched the condition for the topologocal equivalence of $Q(x)$ and $\bar{Q}(x)$. The conditon that $Q_T(u)$ and $\bar{Q}_T(u)$ are Morse-Smale is very strong. If we eliminate the condition, we have the following problem.

Problem 2. For arbitrary $Q(x)$, $\bar{Q}(x)$, how about the condition for the global equivalence of $Q(x)$, and $\bar{Q}(x)$?

In order to answer the problems above, it is necessary for us to study the global topological classifiction of the flows of $Q(x)$ again. In §1, we have proved that the flows of $Q(x)$ at infinity are topologically equivalent to those of the tangent vector field $Q_T(u)$ on the sphere $S^2 = \{u = (u_1, u_2, u_3) : \|u\| = 1\}$. In §2, we study the theorems for the global topological classification of $Q(x)$, derive the necessary and sufficient conditions for the global asymptotic stability and the bounded vector field of $Q(x)$, obtain the criterion for the global topological equivalence of two homogenous vector fields.

§1. Analysis of $Q(x)$ at Infinity

$Q(rx_0) = r^m Q(x_0)$ as $x_0 \in R^3 - \{0\}$ and $r > 0$, thus the direction of vector $Q(x_0)$ and $Q(rx_0)$ is the same. For arbitrary $x \in R^3 - \{0\}$, let

$$r = \|x\|, \quad x = ru, \tag{1.1}$$

where $\|x\| = \sqrt{\langle x, x\rangle}$, $\langle \cdot, \cdot \rangle$ being scalar product, $u \in S^2$, then (0.1) can be turned into

$$\left.\begin{array}{ll} \text{(a)} & du/dt = r^{m-1}(Q(u) - u\langle u, Q(u)\rangle), \\ \text{(b)} & dr/dt = r^m \langle u, Q(u)\rangle. \end{array}\right\} \tag{1.2}$$

Introducing a new time t_1 by means of relation $dt_1 = r^{m-1}dt$ (the time variable is still t), we can obtain

$$\left.\begin{array}{ll} \text{(a)} & du/dt = Q_T(u), \\ \text{(b)} & dr/dt = rR(u), \end{array}\right\} \tag{1.3}$$

where $Q_T(u) \equiv Q(u) - u\langle u, Q(u)\rangle$, $R(u) \equiv \langle u, Q(u)\rangle$. It can be easily proved that $Q_T(u)$ is a tangent vector field on the sphere S^2. We also call $Q_T(u)$ an induced tangent vector field of $Q(x)$ (see [5]). To analyze the geometric properties of $Q_T(u)$ on two-dimensional manifold S^2, we choose an atlas (V_i, ϕ_i), (V_i', ϕ_i') $(i = 1, 2, 3)$, where

$$\begin{aligned} V_i &\equiv \left\{u \in S^2 : u_i > 0\right\}, \quad V_i' \equiv \left\{u \in S^2 : u_i < 0\right\}, \\ \phi_i(u) &= u/u_i : V_i \to \Pi_i \equiv \left\{y = (y_1, y_2, y_3) \in R^3 : y_i = 1\right\}, \\ \phi_i'(u) &= -u/u_i : V_i' \to \Pi_i' = \{y' : y_i' = -1\}, \end{aligned}$$

i.e.,

$$\left.\begin{array}{ll} \text{(a)} & y = \phi_i(u), \\ \text{(b)} & y_i' = \phi_i'(u). \end{array}\right\} \tag{1.4}$$

We can turn (1.3a) into

$$\left.\begin{array}{ll}\text{(a)} & dy/dt = Q(y) - yQ_i(y), \\ \text{(b)} & dy'/dt = Q(y') + y'Q_i(y'),\end{array}\right\} \quad (i = 1, 2, 3), \tag{1.5}$$

where $y \in \Pi_i$, $y' \in \Pi'_i$.

Proposition 1.1. *The flows of the tangent vector field $Q_T(u)$ in region V_i (or V'_i) are topologically equivalent to the flows of vector field $Q(y) - yQ_i(y)$ (or $Q(y') + y'Q_i(y')$) in plane Π_i (or Π'_i).*

If we know the topological classification of vector field in plane Π_i and Π'_i $(i = 1, 2, 3)$, then we know the topological classifiction of $Q_T(u)$ on S^2 by Proposition 1.1. If a planar vector field has only isolated singularities and the ω (or α) limit set of each trajectory is bounded, then the ω (or α) limit set of each trajectory is a singularity, a closed orbit or a graph which consists of singularities and trajectories[6,p.49 or 9]. Therefore, we have Proposition 1.2.

Proposition 1.2. *If the tangent vector field $Q_T(u)$ has only isolated singularities on S^2, then the ω (or α) limit set of a trajectory is a singularity, a closed orbit or a graph.*

Proposition 1.2 is the Theorem 7 of [7] which gave a detailed proof.

To analyze the geometric properties of $Q(x)$ at infinity, we first consider R^3 embeded onto the hyperplane $\Pi_4 = \{z = (z_1, z_2, z_3, z_4) \in R^4 : z_4 = 1\}$ in such a way that $\tilde{Q}(z) \equiv (Q_1(z), Q_2(z), Q_3(z), 0)$ in Π_4 and $Q(x)$ in R^3 are identical, and map

$$f(z) = z/\|z\| : \Pi_4 \to S^3 \equiv \{\tilde{u} = (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3, \tilde{u}_4) : \|\tilde{u}\| = 1\},$$

i.e.

$$\tilde{u} = f(z), \tag{1.6}$$

where $Q_i(z) = Q_i(z_1, z_2, z_3)$ $(i = 1, 2, 3)$. Set $V_4 \equiv \{u \in S^3 : \bar{u}_4 > 0\}$, then f is a diffeomorphism from Π_4 onto V_4. By map (1.6), in Π_4 the system corresponding to vector field $Q(z)$ can be commuted into

$$d\bar{u}/dt = \bar{Q}_T(\bar{u}), \tag{1.7}$$

where $\bar{Q}_T(\bar{u}) \equiv \bar{Q}_(u) - \bar{u}\langle\bar{u}, \bar{Q}(\bar{u})\rangle$. Let $u = (\bar{u}_1, \bar{u}_2, \bar{u}_3)$. We can easily prove that

$$\langle\bar{u}, \bar{Q}(\bar{u})\rangle = R(u), \quad \bar{Q}_T(\bar{u}) = (Q_T(u), \bar{u}_4 R(u)),$$

and $\bar{Q}_T(\bar{u})$ is a tangent vector field on S^3. Moreover, the flows of $\bar{Q}_T(\bar{u})$ on S^3 can be regarded as the extension of the flows of $\bar{Q}(z)$ in Π_4. If we define that the set of the points in Π_4 at infinity is the equator $S^2 = \{(u, \bar{u}_4) \in S^3 : \bar{u}_4 = 0\}$ on S^3, then the equator S^2 (see [8]) (i.e. $\bar{u}_4 = 0$) is an invariant set of $\bar{Q}_T(u)$, and $Q_T(u)$ is also a tangent vector field on the equator S^2.

Theorem 1.1. *The flows of $Q(x)$ at infinity are topologically equivalent to the flows of $Q_T(u)$ on S^2.*

Remark. Theorem 1.1 also holds for arbitrary dimension.

§2. Geometric Properties of $Q(x)$

For convenience of the following discussion, we first introduce several useful notations and definitions.

We shall write g, θ, Γ for a singularity, a closed orbit, a trajectory of $Q_T(u)$ on the sphere S^2 respectively; $\Omega_\Gamma(A_\Gamma)$ for the $\omega(\alpha)$ limit set of the trajectory Γ; $G = g_1 \cup g_2 \cup \cdots \cup g_k \cup \Gamma_1 \cup \cdots \cup \Gamma_l (l \geq k)$ for a graph of $Q_T(u)$ on S^2, where $A_{\Gamma_i}, \Omega_{\Gamma_i} \in \{g_1, \cdots, g_k\}$ $(i = 1, \cdots, l)$; $C(\Gamma) \equiv \{r \cdot u : r \in R^+, u \in \Gamma\}$ (R^+ is a set of non-negative numbers) for a conical surface, which is an invariant conical surface of vector field $Q(x)$ by map (1.1) and system (1.3). Similarly, we shall write w for a trajectory of $Q(x)$ on conical surface $C(\Gamma)$; $\Omega_w(A_w)$ for the $\omega(\alpha)$ limit set of the trajectory w; $\theta^\star$ for a closed orbit of $Q(x)$ on the closed conical surface $C(\theta)$. On the basis of Theorem 1.1, we define $(g, +\infty)$, $(\theta, +\infty)$, $(\Gamma, +\infty)$, $(G, +\infty)$ for a singularity, a closed orbit, a trajetory, a graph of $Q(x)$ at infinity respectively, where $+\infty$ stands for $r \to +\infty$.

Definition 2.1. *$C(\Gamma)$ is a parabolic cone of the 1st kind if each $w \in C(\Gamma)$ such that $\Omega_w = O$, $A_w = (g, +\infty)$, or $\Omega_w = (g, +\infty)$, $A_w = O$; $C(\Gamma)$ is a parabolic cone of the 2nd kind if each $w \in C(\Gamma)$ such that $\Omega_w = O$, $A_w = (\theta, +\infty)$, or $\Omega_w = (\theta, +\infty)$, $A_w = O$; $C(\Gamma)$ is a parabolic cone of the 3rd kind if each $w \in C(\Gamma)$ such that $\Omega_w = O$, $A_w = (G, +\infty)$, or $\Omega_w = (G, +\infty)$, $A_w = O$.*

Definition 2.2. *$C(\Gamma)$ is a hyperbolic cone of the 1st kind if each $w \in C(\Gamma)$ such that $\Omega_w = (g_1, +\infty)$, $A_w = (g_2, +\infty)$;*

$C(\Gamma)$ is a hyperbolic cone of the 2nd kind if each $w \in C(\Gamma)$ such that $\Omega_w = (g, +\infty)$, $A_w = (\theta, +\infty)$, or $\Omega_w = (\theta, +\infty)$, $A_w = (g, +\infty)$;

$C(\Gamma)$ is a hyperbolic cone of the 3rd kind if each $w \in C(\Gamma)$ such that $\Omega_w = (\theta_1, +\infty)$, $A_w = (\theta_2, +\infty)$;

$C(\Gamma)$ is a hyperbolic cone of the 4th kind if each $w \in C(\Gamma)$ such that $\Omega_w = (g, +\infty)$, $A_w = (G, +\infty)$, or $\Omega_w = (G, +\infty)$, $A_w = (g, +\infty)$;

$C(\Gamma)$ is a hyperbolic cone of the 5th kind if each $w \in C(\Gamma)$ such that $\Omega_w = (\theta, +\infty)$, $A_w = (G, +\infty)$, or $\Omega_w = (G, +\infty)$, $A_w = (\theta, +\infty)$;

$C(\Gamma)$ is a hyperbolic cone of the 6th kind if each $w \in C(\Gamma)$ such that $\Omega_w = (G_1, +\infty)$, $A_w = (G_2, +\infty)$.

Definition 2.3. Let $C(\theta)$ be a center-type cone.

$C(\Gamma)$ is a cone of type P of the 4th kind if each $w \in C(\Gamma)$, there exists an $\theta^\star \in C(\theta)$ such that $\Omega_w = \theta^\star$, $A_w = (\theta_1, +\infty)$, or $\Omega_w = (\theta_1, +\infty)$, $A_w = \theta^\star$;

$C(\Gamma)$ is a cone of type P of the 5th kind if each $w \in C(\Gamma)$, there exists an $\theta^\star$ such that $\Omega_w = \theta^\star$, $A_w = (G, +\infty)$; or $\Omega_w = (G, +\infty)$, $A_w = \theta^\star$.

We shall discuss the global topological classification of $Q(x)$ as $Q_T(u)$ has only isolated singularities. If $Q_T(u)$ has non-isolated singularities, then

$$Q_T(u) = h(u) Q'_T(u),$$

where $h(u)$ is a polynomial, $Q'_T(u)$ has only isolated singularities.

Lemma 2.1. *Provided that g is a singularity of $Q_T(u)$, then the two rays which start at the origin O through the point g and $-g$ are invariant about $Q(x)$. If m is odd (or even), then the stability of the two rays is the same (or opposite).*

Proof. $Q_T(-g) = (-1)^m Q_T(g) = 0$, $R(-g) = (-1)^{m+1} R(g)$. We recognize that $R(g) \neq$

0, or else $Q(g)=0$, and this is a contradiction. By system (1.3) and map (1.1),

$$r(t,g)=r(0)e^{R(g)t}, \quad r(t,-g)=r(0)e^{R(-g)t},$$
$$x(t,g)=gr(0)e^{R(g)t}, \quad x(t,-g)=-gr(0)e^{R(-g)t},$$

which are two invariant rays of $Q(x)$. If m is odd and $R(g)<0$ (>0), then the rays of $x(t,g)$ and $x(t,-g)$ are stable (unstable). If m is even, then the stability of $x(t,g)$ and $x(t,-g)$ is opposite.

Theorem 2.1. *Let* $\Omega_\Gamma = g_1$, $A_\Gamma = g_2$, $\Gamma=\{u(t): t\in R\}$. *Then*

(1) $C(\Gamma)$ *is a parabolic cone of the* 1*st kind if* $R(g_1)\cdot R(g_2)>0$ (Fig.1);

(2) $C(\Gamma)$ *is a hyperbolic cone of the* 1*st kind if* $R(g_1)>0$, $R(g_2)<0$ (Fig.2);

(3) $C(\Gamma)$ *is an elliptic cone if* $R(g_1)<0$, $R(g_2)>0$ (Fig.3).

Proof. We prove only (1). The proof of (2) and (3) is similar. Without loss of generality, we assume that $R(g_1)>0$, $R(g_2)>0$. Since $R(u)$ is a continuous function about variable u, by setting $\varepsilon_i = R(g_i)/2>0$ (i=1,2), there exists a neighbourhood $N_i(g_i,\varepsilon_i)$ of the point g_i on S^2 such that each $u\in N_i(g_i,\varepsilon_i)$ satisfying $\mid R(u)-R(g_i)\mid < \varepsilon_i$, i.e.,

$$R(g_i)/2 < R(u) < 3R(g_i)/2.$$

Condition $\Omega_\Gamma = g_1$ indicates that there exists $T_1>0$ such that $u(t)\in N_1(g_1,\varepsilon_1)$ if $t>T_1$. The solutions of system (1.3) on conical surface $C(\Gamma)$ can be expressed as

$$u=u(t), \quad r(t)=r(0)e^{\int_0^t R(u(s))ds}.$$

When $t>T_1$, then

$$\begin{aligned}\int_0^t R(u(s))ds &= \int_0^{T_1} R(u(s))ds + \int_{T_1}^t R(u(s))ds \\ &> \int_0^{T_1} R(u(s))ds + R(g_1)(t-T_1)/2.\end{aligned}$$

Thus, $\lim\limits_{t\to+\infty}\int_0^t R(u(s))ds=+\infty$, i.e.

$$\lim_{t\to+\infty} r(t)=+\infty, \quad \lim_{t\to+\infty} x(t)=(g_1,+\infty).$$

Similarly, we can prove that $\lim\limits_{t\to-\infty} R(u(s))ds=-\infty$, i.e.

$$\lim_{t\to-\infty} r(t)=0, \quad \lim_{t\to-\infty} x(t)=0.$$

Corollary 2.1. *If* $\Omega_\Gamma = A_\Gamma = g$, *then* $C(\Gamma)$ *is a parabolic cone of the* 1*st kind.*

Theorem 2.2. *Provided that θ is a T-periodic orbit of $Q_T(u)$, and $I(\theta) \equiv \int_0^T R(\theta(s))ds$, then*

(1) $C(\theta)$ is a center-type cone if $I(\theta) = 0$;

(2) $C(\theta)$ is a parabolic cone of the 2nd kind if $I(\theta) \neq 0$.

We omit the proof which is similar to that of Theorem 2.1.

Theorem 2.3. *Let $A_\Gamma = g$, $\Omega_\Gamma = \theta$, $\Gamma = \{u(t) : t \in R\}$, then*

(1) $C(\Gamma)$ is a parabolic cone of the 1st kind if $I(\theta) < 0$, $R(g) < 0$; or $I(\theta) = 0$, $\lim\limits_{t\to+\infty} \int_0^t R(u(s))ds = -\infty$, $R(g) < 0$;

(2) $C(\Gamma)$ is a parabolic cone of the 2nd kind if $I(\theta) > 0$, $R(g) > 0$; or $I(\theta) = 0$, $\lim\limits_{t\to+\infty} \int_0^t R(u(s))ds = +\infty$, $R(g) > 0$;

(3) $C(\Gamma)$ is a hyperbolic cone of the 2nd kind if $I(\theta) > 0$, $R(g) < 0$; or $I(\theta) = 0$, $\lim\limits_{t\to+\infty} \int_0^t R(u(s))ds = +\infty$, $R(g) < 0$;

(4) $C(\Gamma)$ is an elliptic cone if $I(\theta) < 0$, $R(g) > 0$; or $I(\theta) = 0$, $\lim\limits_{t\to+\infty} \int_0^t R(u(s))ds = -\infty$, $R(g) > 0$;

(5) $C(\Gamma)$ is a cone of type P of the 1st kind if $I(\theta) = 0$, $R(g) > 0$, $\lim\limits_{t\to+\infty} \int_0^t R(u(s))ds \neq \pm\infty$; $C(\Gamma)$ is a cone of type P of the 2nd kind if $I(\theta) = 0$, $R(g) < 0$, $\lim\limits_{t\to+\infty} \int_0^t R(u(s))ds \neq \pm\infty$.

Proof. Conclusions (1)–(4) are obvious. Thus, we prove only (5).

If $I(\theta) = 0$, $\lim\limits_{t\to+\infty} \int_0^t R(u(s))ds \neq \pm\infty$, then $C(\theta)$ is a center-type cone, $\lim\limits_{t\to+\infty} r(t) \neq +\infty$ and $\lim\limits_{t\to+\infty} r(t) \neq 0$. Thus, Ω_w is bounded. By [9] we know that the Ω_w is a connected, and closed invariant set. Hence, the Ω_w consists of only a closed orbit $\theta^\star$ of $Q(x)$ on $C(\Gamma)$. On the basis of the property of homogeneous vector fields, all $w \in C(\Gamma)$ cut each ray that is not an invariant in the same direction and are inclined to it at the same nonzero angle, then two different trajectories w_1, w_2 on $C(\Gamma)$ correspond to two different limit sets Ω_{w_1}, Ω_{w_2}. Hence, each closed orbit $\theta^\star$ of $Q(x)$ on $C(\theta)$ is an ω-limit set of a trajectory on $C(\Gamma)$, i.e. $\Omega_w = \theta^\star$. If $R(g) > 0$ (or $R(g) < 0$), by the process of the proof on Theorem 2.1, we can recognize that

$$\lim_{t\to-\infty} \int_0^t R(u(s))ds = -\infty \ \Big(\text{or } \lim_{t\to-\infty} \int_0^t R(u(s))ds = +\infty\Big),$$

i.e.

$$\lim_{t\to-\infty} r(t) = 0 \ (\text{or } \lim_{t\to-\infty} r(t) = +\infty).$$

Thus, $A_w = O\ ((g, +\infty))$. By the Difinition 2.3, we complete the proof.

Example. The following homogeneous system of degree two in R^3 is

$$dx_1/dt = -x_2x_3, \quad dx_2/dt = x_1x_3, \quad dx_3/dt = x_1^2 + x_2^2 - x_3^2, \tag{2.1}$$

and the corresponding

$$R(u) = u_3(u_1^2 + u_2^2 - u_3^2),$$
$$Q_T(u) = (-u_2u_3 - u_1R(u), u_1u_3 - u_2R(u), (1 - u_3^2)(u_1^2 + u_2^2 - u_3^2)),$$

then (2.1) is topologically equivalent to

$$du/dt = Q_T(u), \quad dr/dt = rR(u). \tag{2.2}$$

The induced system in Π_3 is

$$\begin{cases} dy_1/d\tau = -y_2 + y_1(1 - y_1^2 - y_2^2), \\ dy_2/d\tau = y_1 + y_2(1 - y_1^2 - y_2^2), \end{cases} \tag{2.3}$$

where $d\tau = u_3 dt$. The phase portrait in chart (V_3, ϕ_3) is Fig.4

where $\theta = \left\{u_0(t) = ((\cos t/\sqrt{2})/\sqrt{2}, (\sin t/\sqrt{2})/\sqrt{2}, 1/\sqrt{2}) : 0 \le t < 2\sqrt{2}\pi\right\}$ is a closed orbit (or a stable limit cycle) of $Q_T(u)$,

$$\Gamma = \Big\{u(t) = (\cos\tau, \sin\tau, \sqrt{(1+re^{-2\tau})/(2+re^{-2\tau})} :$$
$$t = \int_0^\tau \sqrt{(2+re^{-2s})/(1+re^{-2s})}ds, t \in R\Big\}$$

stands for trajectories of $Q_T(u)$ (Fig.4). Since $r > 0$, it follows that $\Omega_\Gamma = \theta$, $A_\Gamma = g(0,0,1)$, and $I(\theta) = 0$,

$$\lim_{t\to+\infty} \int_0^t R(u(s))ds = (\ln 2 - \ln(r+2))/2.$$

Thus, $C(\theta)$ is a center-type cone, and $C(\Gamma)$ is a cone of type P of the 2nd kind.

Theorem 2.4. *Let* $\Omega_\Gamma = \theta_1$, $A_\Gamma = \theta_2$, $\Gamma = \{u(t) : t \in R\}$. *Then*

(1) $C(\Gamma)$ *is a parabolic cone of the* 2*nd kind if one of the following conditions holds:*

(a) $I(\theta_1) \cdot I(\theta_2) > 0$;

(b) $I(\theta_1) = 0$, $\lim\limits_{t\to+\infty} \int_0^t R(u(s))ds = +\infty$, $I(\theta_2) > 0$;

(c) $I(\theta_1) = 0$, $\lim\limits_{t\to+\infty} \int_0^t R(u(s))ds = -\infty$, $I(\theta_2) < 0$;

(d) $I(\theta_1) > 0$, $I(\theta_2) = 0$, $\lim\limits_{t\to-\infty} \int_0^t R(u(s))ds = -\infty$;

(e) $I(\theta_1) < 0$, $I(\theta_2) = 0$, $\lim\limits_{t\to-\infty} \int_0^t R(u(s))ds = +\infty$;

(f) $I(\theta_1) = I(\theta_2) = 0$, $\lim\limits_{t\to+\infty} \int_0^t R(u(s))ds = +\infty$, $\lim\limits_{t\to-\infty} \int_0^t R(u(s))ds = -\infty$;

(g) $I(\theta_1) = I(\theta_2) = 0$, $\lim\limits_{t\to+\infty} \int_0^t R(u(s))ds = -\infty$, $\lim\limits_{t\to-\infty} \int_0^t R(u(s))ds = +\infty$;

(2) $C(\Gamma)$ *is a hyperbolic cone of the* 3*rd kind if one of the following conditions holds:*

(a) $I(\theta_1) > 0$, $I(\theta_2) < 0$;

(b) $I(\theta_1) = 0$, $\lim\limits_{t\to+\infty} \int_0^t R(u(s))ds = +\infty$, $I(\theta_2) < 0$;

(c) $I(\theta_1) > 0$, $I(\theta_2) = 0$, $\lim\limits_{t\to-\infty} \int_0^t R(u(s))ds = +\infty$;

(d) $I(\theta_1) = I(\theta_2) = 0$, $\lim\limits_{t\to+\infty}\int_0^t R(u(s))ds = +\infty$, $\lim\limits_{t\to-\infty}\int_0^t R(u(s))ds = +\infty$;

(3) $C(\Gamma)$ *is an elliptic cone if one of the following conditions holds:*

(a) $I(\theta_1) < 0$, $I(\theta_2) > 0$;

(b) $I(\theta_1) = 0$, $\lim\limits_{t\to+\infty}\int_0^t R(u(s))ds = -\infty$, $I(\theta_2) > 0$;

(c) $I(\theta_1) < 0$, $I(\theta_2) = 0$, $\lim\limits_{t\to-\infty}\int_0^t R(u(s))ds = -\infty$;

(d) $I(\theta_1) = I(\theta_2) = 0$, $\lim\limits_{t\to\pm\infty}\int_0^t R(u(s))ds = -\infty$;

(4) $C(\Gamma)$ *is a cone of type* P *of the* 1*st kind if one of the following conditions holds:*

(a) $I(\theta_1) = 0$, $\lim\limits_{t\to+\infty}\int_0^t R(u(s))ds \neq \pm\infty$, $I(\theta_2) > 0$;

(b) $I(\theta_1) < 0$, $I(\theta_2) = 0$, $lim_{t\to-\infty}\int_0^t R(u(s))ds \neq \pm\infty$;

(c) $I(\theta_1) = I(\theta_2) = 0$, $\lim\limits_{t\to+\infty}\int_0^t R(u(s))ds \neq \pm\infty$, $\lim\limits_{t\to-\infty}\int_0^t R(u(s))ds = -\infty$;

(d) $I(\theta_1) = I(\theta_2) = 0$, $\lim\limits_{t\to+\infty}\int_0^t R(u(s))ds = -\infty$, $\lim\limits_{t\to-\infty}\int_0^t R(u(s))ds \neq \pm\infty$;

(5) $C(\Gamma)$ *is a cone of type* P *of the* 3*rd kind if* $I(\theta_1) = I(\theta_2) = 0$, $\lim\limits_{t\to\pm\infty}\int_0^t R(u(s))ds \neq \pm\infty$;

(6) $C(\Gamma)$ *is a cone of type* P *of the* 4*th kind if one of the following conditions holds:*

(a) $I(\theta_1) = 0$, $\lim\limits_{t\to+\infty}\int_0^t R(u(s))ds \neq \pm\infty$, $I(\theta_2) < 0$;

(b) $I(\theta_1) > 0$, $I(\theta_2) = 0$, $\lim\limits_{t\to-\infty}\int_0^t R(u(s))ds \neq \pm\infty$;

(c) $I(\theta_1) = I(\theta_2) = 0$, $\lim\limits_{t\to+\infty}\int_0^t R(u(s))ds \neq \pm\infty$, $\lim\limits_{t\to-\infty}\int_0^t R(u(s))ds = +\infty$;

(d) $I(\theta_1) = I(\theta_2) = 0$, $\lim\limits_{t\to+\infty}\int_0^t R(u(s))ds = +\infty$, $\lim\limits_{t\to-\infty}\int_0^t R(u(s))ds \neq \pm\infty$.

Theorem 2.5. *Let* $\Omega_\Gamma = G$, $A_\Gamma = g_0$, $\Gamma = \{u(t) : t \in R\}$. *Then*

(1) $C(\Gamma)$ *is a parabolic cone of the* 1*st kind if all* $g_i \in G$ *such that* $R(g_i) < 0$, $R(g_0) < 0$; *or*

$$\lim_{t\to+\infty}\int_0^t R(u(s))ds = -\infty, \quad R(g_0) < 0;$$

(2) $C(\Gamma)$ *is a parabolic cone of the* 3*rd kind if all* $g_i \in G$ *such that* $R(g_i) > 0$, $R(g_0) > 0$; *or*

$$R(g_0) > 0, \quad \lim_{t\to+\infty}\int_0^t R(u(s))ds = +\infty;$$

(3) $C(\Gamma)$ *is a hyperbolic cone of the* 4*th kind if all* $g_i \in G$ *such that* $R(g_i) > 0$, $R(g_0) < 0$; *or*

$$R(g_0) < 0, \quad \lim_{t\to+\infty}\int_0^t R(u(s))ds = +\infty;$$

(4) $C(\Gamma)$ *is an elliptic cone if all* $g_i \in G$ *such that* $R(g_i) < 0$, $R(g_0) > 0$; *or*

$$R(g_0) > 0, \quad \lim_{t\to+\infty}\int_0^t R(u(s))ds = -\infty.$$

Theorem 2.6. *Let* $\Omega_\Gamma = G$, $A_\Gamma = \theta$, $\Gamma = \{u(t) : t \in R\}$. *Then*

(1) $C(\Gamma)$ *is a parabolic cone of the* 2*nd kind if one of the following conditions holds:*

(a) *all* $g_i \in G$ *such that* $R(g_i) < 0$, $I(\theta) < 0$;

(b) $\lim\limits_{t\to+\infty}\int_0^t R(u(s))ds = -\infty$, $I(\theta) < 0$;

(c) $\lim\limits_{t\to+\infty}\int_0^t R(u(s))ds = -\infty$, $I(\theta) = 0$, $\lim\limits_{t\to-\infty}\int_0^t R(u(s))ds = +\infty$;

(2) $C(\Gamma)$ *is a parabolic cone of the* 3rd *kind if one of the following conditions holds:*

(a) *all* $g_i \in G$ *such that* $R(g_i) > 0$, $I(\theta) > 0$;

(b) $\lim_{t\to+\infty} \int_0^t R(u(s))ds = +\infty$, $I(\theta) > 0$;

(c) $I(\theta) = 0$, $\lim_{t\to+\infty} \int_0^t R(u(s))ds = +\infty$; $\lim_{t\to-\infty} \int_0^t R(u(s))ds = -\infty$;

(3) $C(\Gamma)$ *is an elliptic cone if one of the following conditions holds:*

(a) *all* $g_i \in G$ *such that* $R(g_i) < 0$, $I(\theta) > 0$;

(b) $\lim_{t\to+\infty} \int_0^t R(u(s))ds = -\infty$, $I(\theta) > 0$;

(c) $\lim_{t\to\pm\infty} \int_0^t R(u(s))ds = -\infty$, $I(\theta) = 0$;

(4) $C(\Gamma)$ *is a hyperbolic cone of the* 5th *kind if one of the following conditions holds:*

(a) *all* $g_i \in G$ *such that* $R(g_i) > 0$, $I(\theta) < 0$;

(b) $\lim_{t\to+\infty} \int_0^t R(u(s))ds = +\infty$, $I(\theta) < 0$;

(c) $\lim_{t\to\pm\infty} \int_0^t R(u(s))ds = +\infty$, $I(\theta) = 0$;

(5) $C(\theta)$ *is a cone of type* P *of the* 1st *kind if all* $g_i \in G$ *such that* $R(g_i) < 0$, $I(\theta) = 0$, $\lim_{t\to-\infty} \int_0^t R(u(s))ds \neq \pm\infty$; *or*

$$\lim_{r\to+\infty} \int_0^t R(u(s))ds = -\infty, \quad I(\theta) = 0, \quad \lim_{t\to-\infty} \int_0^t R(u(s))ds \neq \pm\infty.$$

(6) $C(\theta)$ *is a cone of type* P *of the* 5th *kind if all* $g_i \in G$ *such that* $R(g_i) > 0$, $I(\theta) = 0$, $\lim_{t\to-\infty} \int_0^t R(u(s))ds \neq \pm\infty$; *or*

$$\lim_{t\to+\infty} \int_0^t R(u(s))ds = +\infty, \quad I(\theta) = 0, \quad \lim_{t\to-\infty} \int_0^t R(u(s))ds \neq \pm\infty.$$

Theorem 2.7. *Let* $\Omega_\Gamma = G_1$, $A_\Gamma = G_2$. *Then*

(1) $C(\Gamma)$ *is a parabolic cone of the* 3rd *kind if all* $g_i \in G_1$, $g_j \in G_2$ *such that* $R(g_i)\cdot R(g_j) > 0$; *or*

$$\lim_{t\to+\infty} \int_0^t R(u(s))ds = +\infty(-\infty), \quad \lim_{t\to-\infty} \int_0^t R(u(s))ds = -\infty(+\infty);$$

(2) $C(\Gamma)$ *is a hyperbolic cone of the* 6th *kind if all* $g_i \in G_1$ *such that* $R(g_i) > 0$, *all* $g_j \in G_2$ *such that* $R(g_j) < 0$; *or*

$$\lim_{t\to\pm\infty} \int_0^t R(u(s))ds = +\infty;$$

(3) $C(\Gamma)$ *is an elliptic cone if all* $g_i \in G_1$ *such that* $R(g_i) < 0$, *and all* $g_j \in G_2$ *such that* $R(g_j) > 0$; *or*

$$\lim_{t\to\pm\infty} \int_0^t R(u(s))ds = -\infty.$$

The proofs of Theorems 2.4–2.7 are similar to that of Theorems 2.1 and 2.3, so we omit them. If we summarize up Theorems 2.1–2.7, we can derive the following properties of homogeneous vector field $Q(x)$ of degree m in R^3.

Corollary 2.2. *The necessary and sufficient conditions for the global asymptotic stability of* $Q(x)$ *are:*

(a) m *is odd;*

(b) *all singularities* g_i *of* $Q_T(u)$ *satisfy* $R(g_i) < 0$;

(c) *all closed orbits θ of $Q_T(u)$ saitsfy $I(\theta_i) < 0$.*

Corollary 2.3. *The necessary and sufficient conditions for the boundedness of $Q(x)$ are*

(a) *m is odd;*

(b) *all singularities g_i of $Q_T(u)$ satisfy $R(g_i) < 0$;*

(c) *all colsed orbits θ_i satisfy $I(\theta_i) < 0$; or $I(\theta) = 0$, $\lim\limits_{t\to+\infty} \int_0^t R(\Gamma(s))ds \neq +\infty$ if $\Omega_\Gamma = \theta$.*

Corollary 2.4. *If two homogeneous vector fields $Q(x)$, $\bar{Q}(x)$ have topological equivalent tangent vector fields $Q_T(u)$, $\bar{Q}_T(u)$ on S^2 and the kinds of all the correspondent invariant cones are the same, then the two homogeneous vector fields $Q(x)$, $\bar{Q}(x)$ are global topopogical equivalence.*

Thus, we have also solved the Problem 1 and the Problem 2.

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