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HITTING TIME AND PLACE TO A SPHERE OR SPHERICAL SHELL FOR BROWNIAN MOTION***

YIN Chuancun* WU Rong**

Abstract

In this paper, the authors compute the explicit formulas for the joint distributions of the hitting time and place for a sphere or concentric spherical shell by Brownian motion, when the process starts either outside the sphere or the region bounded by concentric spheres.

Keywords Brownian motion, Hitting time, Hitting place, Spherical shell, Sphere
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§1. Introduction

Let $X = \{X_t(\omega), t \geq 0\}$ be a standard d-dimensional Brownian motion in $R^d (d \geq 2)$. The first hitting time of X for a Borel set B in R^d is defined to be

$$T_B = \begin{cases} \inf\{t > 0, X_t \in B\}, & \text{if } \{t > 0, X_t \in B\} \neq \emptyset, \\ \infty, & \text{if } \{t > 0, X_t \in B\} = \emptyset. \end{cases}$$

The first hitting place is $X(T_B)$.

In this paper, we mainly consider the sphere $\sum^{d-1}(0,r) = \{x : x \in R^d, |x| = r\}$ and the spherical shell $\sum^{d-1}(0,a) \cup \sum^{d-1}(0,b) = \{x : x \in R^d, |x| = a \text{ or } |x| = b\}$, where $r > 0, a > 0, b > 0$ and $a < b$. For simplicity, we shall write T_r for the hitting time of $\sum^{d-1}(0,r)$ and T_{ab} for the hitting time of $\sum^{d-1}(0,a) \cup \sum^{d-1}(0,b)$.

A lot of work on the distributions for T_r or/and $X(T_r)$ has been done since 1962. Ciesielski and Taylor[1] computed the P_0 distribution function of T_r, and shown that the P_0 distribution of the total time spent by a $(d+2)$-dimensional Brownian motion in the ball $\{x : |x| < r, x \in R^{d+2}\}$ is the same as the P_0 distribution of the hitting time of the sphere $\sum^{d-1}(0,r)$ by a d-dimensional Brownian motion. For more on this phenomenon (see [2,3]). Recently we obtained the P_x distribution functions of T_r for any x such that $|x| < r$ or $|x| > r$ (see [4]). The distribution of the hitting place of a sphere, when Brownian motion starts at any point in space can be found in [5]. The Laplace-Gegenbauer transform for the joint distribution for T_r and $X(T_r)$ was obtained by Wendel[6] (see also [7,8]). It seems impossible to obtain the joint density function or joint distribution function by inverting the Laplace-Gegenbauer

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*Department of Mathematics, Qufu Normal University, Qufu 273165, China.
**Department of Mathematics, Nankai University, Tianjin 300071, China.

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transform. The joint distribution of hitting time and place of a sphere by Brownian motion which starts at any point inside the sphere, was obtained by Hsu Pei[9]. Previously, Wang Zikun[10] obtained the joint distributions of the hitting time, hitting place, last exit time and place of a sphere, when Brownian motion starts inside the sphere, but when Brownian motion starts outside the sphere the corresponding result was not obtained. Generally speaking, it is much harder when the process starts outside than inside. One of the aims of this paper is to obtain the joint distribution of hitting time and place of a sphere by Brownian motion which starts at any point outside the sphere (§ 3).

When $D = (a, b)$ is a finite interval in R^1, the joint distribution function for the first exit time and place can be found in [5]. But the analogous result for $d \geq 2$ has not been obtained. The other aim of this paper is that to compute the joint distribution for T_{ab} and $X(T_{ab})$ explicitly (§4).

In what follows, let J_ν and N_ν denote the first and second Bessel function of order ν, respectively. Let K_ν denote the Bessel function "of purely imaginary argument". Let C_m^ν be the Gegenbauer polynomial of degree m and order ν. It is customary to take $C_0^0 = 1$, $C_0^\nu = 1$, $C_m^0 = \lim\limits_{\nu\to 0} \nu^{-1} C_m^\nu = 2T_m/m$, here T_m is the mth Tchebycheff polynomial $T_m(\cos\theta) = \cos m\theta$. Set $h = \frac{d-2}{2}$.

§2. Lemmas

Let D be a domain of C^3 boundary, $P_D(t, x, y)$ be the transition density function of the Brownian motion killed at time T_{D^c}. It is well known that (see e.g. [11]) $P_D(t, x, y)$ is the unique solution of the following problems:

$$\begin{cases} \frac{\partial}{\partial t} P_D(t,x,y) = \frac{1}{2}\Delta_y P_D(t,x,y), & t>0, x\in D, y\in D, \\ P_D(t,x,y) = 0, & t>0, x\in D, y\in\partial D, \\ \lim\limits_{t\to 0} P_D(t,x,y) = \delta_x(y), & x\in D,\ y\in D. \end{cases} \tag{2.1}$$

Lemma 2.1. *Let D be a bounded domain of C^3 boundary, and set $D_e = R^d \setminus \bar{D}$. Then*

(1) for $x \in D$, $P_x(T_{\partial D} \in dt, X(T_{\partial D}) \in dy) = \frac{1}{2}\frac{\partial P_D(t,x,y)}{\partial n_y} dt\sigma(dy)$;

(2) for $x \in D_e$, $P_x(T_{\partial D} \in dt, X(T_{\partial D}) \in dy, T_{\partial D} < \infty) = -\frac{1}{2}\frac{\partial P_{D_e}(t,x,y)}{\partial n_y} dt\sigma(dy)$,

where n_y is the inward normal direction at $y \in \partial D$ and σ is the $d-1$ dimensional volume measure on ∂D.

Proof. (1) This is a well-known result[9]. Next, we prove (2). For $x \in D_e$ and Borel set $A \subset \partial D$, by the Markov property,

$$\begin{aligned} &P_x(T_{\partial D} \in \mathrm{d}t, X(T_{\partial D}) \in A, T_{\partial D} < \infty) \\ &= -\Big(\int_{D_e} P_y(X(T_{\partial D}) \in A, T_{\partial D} < \infty) \frac{\partial}{\partial t} P_{D_e}(t,x,y)dy\Big)dt \\ &= -\frac{1}{2}\Big(\int_{D_e} P_y(X(T_{\partial D}) \in A, T_{\partial D} < \infty)\Delta_y P_{D_e}(t,x,y)dy\Big)dt. \end{aligned} \tag{2.2}$$

Taking a sphere $\sum^{d-1}(0, R)$ of large radius R such that $R > |x|$. Denote by $\Omega(R)$ the region bounded by ∂D and $\sum^{d-1}(0, R)$. Set $u(y) = P_y(X(T_{\partial D}) \in A, T_{\partial D} < \infty)$ and $v(y) = P_{D_e}(t, x, y)$. After applying Green's second theorem and note that $u(y)$ $(y \notin \partial D)$ is harmonic (see [5]) and $v(y)=0, y\in\partial D$, we have

$$\int_{\Omega(R)} u(y)\Delta v(y)dy=\int_{\partial D} u(y)\frac{\partial v}{\partial n}\sigma(dy)-\int_{\sum^{d-1}(0,R)}\Big(u(y)\frac{\partial v}{\partial n}-v(y)\frac{\partial u}{\partial n}\Big)\sigma(dy). \tag{2.3}$$

It is well-known that $P_{D_e}(t,x,y)=P_{D_e}(t,y,x), P_{D_e}(t,x,y)\le(2\pi t)^{-\frac{d}{2}}\exp(-\frac{|x-y|^2}{2t})$. By the property of harmonic function[12] we know that $|\mathrm{grad}\, u(y)|\le M/|y|$ for large $|y|$, where $M>0$ is a constant and grad stands for gradient. From [13, Chapter 1, (6.13)], there exists a constant λ_0 such that

$$\Big|\frac{\partial}{\partial x_i}P_{D_e}(t,x,y)\Big|\le \text{const.}t^{-\frac{d+1}{2}}\exp\Big(-\lambda_0\frac{|x-y|^2}{4t}\Big).$$

It follows from the above facts that

$$\lim_{R\to\infty}\int_{\sum^{d-1}(0,R)}\Big(u(y)\frac{\partial v}{\partial n}-v(y)\frac{\partial u}{\partial n}\Big)\sigma(dy)=0.$$

Hence by (2.2) and (2.3) we obtain

$$\begin{aligned}P_x(T_{\partial D}\in dt, X(T_{\partial D})\in A, T_{\partial D}<\infty)&=-\frac{1}{2}\int_{\partial D}1_A(y)\frac{\partial}{\partial n_y}P_{D_e}(t,x,y)\sigma(dy)dt\\&=-\frac{1}{2}\int_A\frac{\partial}{\partial n_y}P_{D_e}(t,x,y)\sigma(dy)dt.\end{aligned}$$

So that for $x\in D_e$,

$$P_x(T_{\partial D}\in dt, X(T_{\partial D})\in dy, T_{\partial D}<\infty)=-\frac{1}{2}\frac{\partial}{\partial n_y}P_{D_e}(t,x,y)\sigma(dy)dt.$$

This completes the proof of Lemma 2.1.

For the Brownian motion X_t we let $\theta_t=\angle x0X_t$ if $x\ne 0$, $\theta_t=\angle u0X_t$ for an arbitrary but fixed nonzero vector u, in case $x=0$.

Lemma 2.2. *Let $\sigma(dy)$ be the area measure on $\sum^{d-1}(0,r)$, then*

$$\int_{\sum^{d-1}(0,r)}C_m^h(\cos\theta)\sigma(dy)=\begin{cases}\frac{2\pi^{\frac{d}{2}}r^{d-1}}{\Gamma(\frac{d}{2})}, & m=0,\\ 0, & m\ge 1,\end{cases}$$

where $\theta=\angle x0y, x\in R^d$.

Proof. It is an immediate consequence of the following identity due to Wendel[6]

$$E_x(C_m^h(\cos\theta_{T_r}))=\Big(\frac{|x|}{r}\Big)^m C_m^h(1),\quad |x|<r.$$

Lemma 2.3.[14] *Let $\sigma(dy)$ be the area measure on $\sum^{d-1}(0,r)$, then*

$$\int_{\sum^{d-1}(0,r)}C_m^h(\cos\theta)C_k^h(\cos\theta)\sigma(dy)=\begin{cases}\frac{2\pi^{\frac{d}{2}}r^{d-1}h}{(m+h)\Gamma(\frac{d}{2})}C_m^h(1), & m=k, d\ge 3,\\ \frac{2\pi r}{m}C_m^0(1), & m=k\ne 0, d=2,\\ 2\pi r, & m=k=0, d=2,\\ 0, & m\ne k, d\ge 2,\end{cases}$$

where $\theta=\angle x0y, x\in R^d$.

Lemma 2.4. *For $|x|>r, \alpha>0$ and $d\ge 2$, then*

$$\int_0^\infty\frac{\lambda(J_{m+h}(\lambda|x|)N_{m+h}(\lambda r)-J_{m+h}(\lambda r)N_{m+h}(\lambda|x|))}{(\lambda^2+2\alpha)(J_{m+h}^2(\lambda r)+N_{m+h}^2(\lambda r))}d\lambda=-\frac{\pi}{2}\frac{K_{m+h}(\sqrt{2\alpha}|x|)}{K_{m+h}(\sqrt{2\alpha}\, r)},$$

where $m\ge 0$ is an integer.

Proof. By using recurrence formulas[15]

$$\frac{d}{dx}(x^\nu K_\nu) = -x^\nu K_{\nu-1}, \quad \frac{d}{dx}(x^{-\nu} K_\nu) = -x^{-\nu} K_{\nu+1},$$
$$\frac{d}{dx}(x^\nu Z_\nu) = x^\nu Z_{\nu-1}, \quad \frac{d}{dx}(x^{-\nu} Z_\nu) = -x^{-\nu} Z_{\nu-1},$$

where $Z_\nu = J_\nu$ or N_ν, one obtains

$$\int_0^\infty -\frac{\pi}{2}\frac{K_{m+h}(\sqrt{2\alpha}R)}{K_{m+h}(\sqrt{2\alpha}\ r)}(J_{m+h}(\lambda R)N_{m+h}(\lambda r) - J_{m+h}(\lambda r)N_{m+h}(\lambda R))RdR$$
$$= \frac{1}{2\alpha+\lambda^2}.$$

Lemma 2.4 now follows immediately from the Weber's inversion transform[16].

§3. Hitting Spheres from the Exterior

In this section we will give the joint density of the hitting time and place when the starting point of the Brownian motion is outside the sphere. The main technique involves computing the transition density function of Brownian motion in the exterior of a ball by solving boundary value problems.

Theorem 3.1. *Let $B_r^e = \{x : x \in R^d, |x| > r\}$ and $P_{B_R^e}(t,x,y)$ be the transition density function for the killed Brownian motion in B_r^e, then*

(1) *for $d \geq 3$,*

$$P_{B_r^e}(t,x,y) = \frac{\Gamma(\frac{d}{2})}{2\pi^{d/2}h(|x||y|)^h}\sum_{m=0}^\infty (m+h)C_m^h(\cos\theta)$$
$$\cdot \int_0^\infty \frac{\lambda G_{m+h}(\lambda,|x|,r)G_{m+h}(\lambda,|y|,r)}{J_{m+h}^2(\lambda r)+N_{m+h}^2(\lambda r)}\exp\Big(-\frac{1}{2}\lambda^2 t\Big)d\lambda;$$

(2) *for $d = 2$,*

$$P_{B_r^e}(t,x,y) = \sum_{m=0}^\infty |x|D(m,|x|)C_m^0(\cos\theta)$$
$$\cdot \int_0^\infty \frac{\lambda G_m(\lambda,|x|,r)G_m(\lambda,|y|,r)}{J_m^2(\lambda r)+N_m^2(\lambda r)}\exp\Big(-\frac{1}{2}\lambda^2 t\Big)d\lambda,$$

where

$$\theta = \angle x0y, \quad G_{m+h}(\lambda,a,r) = J_{m+h}(\lambda a)N_{m+h}(\lambda r) - J_{m+h}(\lambda r)N_{m+h}(\lambda a),$$
$$D(m,|x|) = \frac{m}{2\pi|x|}, \quad \text{if } m \neq 0; \quad D(m,|x|) = \frac{1}{2\pi|x|}, \quad \text{if } m = 0.$$

Proof. For fixed $x \in B_r^e$, we choose a spherical coordinate system $y = (r, \theta_1, \cdots, \theta_{d-1})$:

$$\begin{cases} y_1 = R\sin\theta_1\sin\theta_2\cdots\sin\theta_{d-1}, \\ y_2 = R\sin\theta_1\sin\theta_2\cdots\cos\theta_{d-1}, \\ \qquad \cdots\cdots, \\ y_d = R\cos\theta_1 \end{cases}$$

so that $x = (|x|, 0, \cdots, 0)$. By symmetry, $P_{B_r^e}(t,x,y)$ is a function of $(t, R, \theta) = (t, |y|, \angle x0y)$, where $\theta = \theta_1$. So that there exists a function Q such that $P_{B_r^e}(t,x,y) = Q(t,R,\theta)$. It follows from (2.1) that $Q(t,R,\theta)$ is the solution of the following initial-boundary value problems:

$$\frac{\partial Q}{\partial t}=\frac{1}{2}\frac{\partial^2 Q}{\partial R^2}+\frac{d-1}{2R}\frac{\partial Q}{\partial t}+\frac{1}{2R^2\sin^{d-2}\theta}\frac{\partial}{\partial\theta}\Big(\sin^{d-2}\theta\frac{\partial Q}{\partial\theta}\Big),\quad t>0, R>r,\quad 0<\theta<\pi, \tag{3.1}$$

$$Q(t,r,\theta)=0,\quad t>0,\quad 0<\theta<\pi, \tag{3.2}$$

$$\lim_{t\to 0}Q(t,R,\theta)=\delta_{(|x|,0)}(R,\theta),\quad R>r,\quad 0<\theta<\pi. \tag{3.3}$$

Setting $Q(t,R,\theta)=S(t,R)\Theta(\theta)$, we are led to two equations

$$\frac{d}{d\theta}\Big(\frac{d\Theta}{d\theta}\sin^{d-2}\theta\Big)+\mu\Theta\sin^{d-2}\theta=0,\quad 0<\theta<\pi, \tag{3.4}$$

$$\frac{\partial S}{\partial t}=\frac{1}{2}\Big(\frac{\partial^2 S}{\partial R^2}+\frac{d-1}{R}\frac{\partial S}{\partial R}-\frac{1}{R^2}\mu S\Big),\quad t>0,\quad R>r, \tag{3.5}$$

where μ is a separation constant. From [15] we know that equation (3.4) has a nonzero solution if and only if $\mu=m(m+d-2)$ and the solution is $C_m^h(\cos\theta)$, where $m\ge 0$ is an integer. By (3.2) one sees $S(t,r)=0, t>0$. By using

$$\delta_{(|x|,0)}(R,\theta)=\frac{\Gamma(\frac{d-1}{2})}{2\pi^{\frac{d-1}{2}}R^{d-1}\sin^{d-2}\theta}\delta_{|x|}(R)\delta_0(\theta), \tag{3.6}$$

and

$$\int_0^\pi \sin^{2h}\theta C_k^h(\cos\theta)C_n^h(\cos\theta)d\theta=\begin{cases}\frac{\pi\Gamma(2h+n)}{2^{2h-1}n!(n+h)\Gamma^2(h)}, & n=k,\\ 0, & n\ne k\end{cases} \tag{3.7}$$

and $Q_m(t,R,\theta)=S_m(t,R)C_m^h(\cos\theta)$ one obtains

$$\lim_{t\to 0}S_m(t,R)=M(m,R)\delta_{|x|}(R),$$

where

$$M(m,R)=\frac{2^{d-4}(m+h)\Gamma^2(h)\Gamma(\frac{d-1}{2})}{\pi^{\frac{d+1}{2}}R^{d-1}\Gamma(2h)}. \tag{3.8}$$

Hence, $S_m(t,R)$ is the solution of

$$\begin{cases}\dfrac{\partial S_m}{\partial t}=\dfrac{1}{2}\Big(\dfrac{\partial^2 S_m}{\partial R^2}+\dfrac{d-1}{R}\dfrac{\partial S_m}{\partial R}-\dfrac{1}{R^2}m(m+2h)S_m\Big),\quad R>r,\quad t>0,\\ S_m(t,r)=0,\quad t>0,\\ \lim\limits_{t\to 0}S_m(t,R)=M(m,R)\delta_{|x|}(R),\quad R>r.\end{cases} \tag{3.9}$$

Setting $S_m=R^{-h}u_m$, then equations above can be written as in the forms

$$\frac{\partial u_m}{\partial t}=\frac{1}{2}\Big(\frac{\partial^2 u_m}{\partial R^2}+\frac{1}{R}\frac{\partial u_m}{\partial R}-\frac{1}{R^2}(m+h)^2u_m\Big),\quad t>0,\quad R>r, \tag{3.10}$$

$$u_m(t,r)=0,\quad t>0, \tag{3.11}$$

$$\lim_{t\to 0}u_m(t,R)=R^hM(m,R)\delta_{|x|}(R), R>r. \tag{3.12}$$

The solution to (3.10)–(3.12) is

$$u_m(t,R)=|x|^{h+1}M(m,|x|)\cdot\int_0^\infty\frac{\lambda G_{m+h}(\lambda,|x|,r)G_{m+h}(\lambda,|y|,r)}{J_{m+h}^2(\lambda r)+N_{m+h}^2(\lambda r)}\exp\Big(-\frac{1}{2}\lambda^2 t\Big)d\lambda,$$

where

$$G_{m+h}(\lambda,a,r)=J_{m+h}(\lambda a)N_{m+h}(\lambda r)-J_{m+h}(\lambda r)N_{m+h}(\lambda a).$$

Hence

$$\begin{aligned}P_{B_r^e}(t,x,y)&=\sum_{m=0}^{\infty}S_m(t,R)C_m^h(\cos\theta)\\&=\frac{\Gamma(\frac{d}{2})}{2\pi^{\frac{d}{2}}h(|x||y|)^h}\sum_{m=0}^{\infty}(m+h)C_m^h(\cos\theta)\\&\quad\cdot\int_0^\infty\frac{\lambda G_{m+h}(\lambda,|x|,r)G_{m+h}(\lambda,|y|,r)}{J_{m+h}^2(\lambda r)+N_{m+h}^2(\lambda r)}\exp\Big(-\frac{1}{2}\lambda^2t\Big)d\lambda.\end{aligned}$$

For $d=2$, we only note that

$$\delta_{(|x|,0)}(R,\theta)=\frac{1}{R}\delta_{|x|}(R)\delta_0(\theta)$$

and

$$\int_0^\pi C_k^0(\cos\theta)C_n^0(\cos\theta)d\theta=\begin{cases}0, & n\neq k,\\ \pi, & k=n=0,\\ \frac{2\pi}{n^2}, & k=n\neq 0.\end{cases}$$

From these facts one obtains

$$\lim_{t\to0}S_m(t,R)=D(m,R)\delta_{|x|}(R),$$

where

$$D(m,R)=\begin{cases}\dfrac{m}{2\pi R}, & m\neq 0,\\ \dfrac{1}{2\pi R}, & m=0.\end{cases}$$

The rest proof can be proved along the same lines as the case of $d\geq 3$ and will be omitted.

Combining Theorem 3.1 and Lemma 2.1 we have

Theorem 3.2. *For $|x|>r,t>0$, and $|y|=r$, then*

(1) *for $d\geq 3$,*

$$\begin{aligned}&P_x(T_r\in dt,X(T_r)\in dy,T_r<\infty)/dt\sigma(dy)\\&=-\frac{\Gamma(\frac{d}{2})}{2\pi^{d/2+1}rh(r|x|)^h}\sum_{m=0}^{\infty}(m+h)C_m^h(\cos\theta)\int_0^\infty\frac{\lambda G_{m+h}(\lambda,|x|,r)}{J_{m+h}^2(\lambda r)+N_{m+h}^2(\lambda r)}\exp\Big(-\frac{1}{2}\lambda^2t\Big)d\lambda;\end{aligned}$$

(2) *for $d=2$,*

$$\begin{aligned}&P_x(T_r\in dt,X(T_r)\in dy)/dt\sigma(dy)\\&=-\sum_{m=0}^{\infty}\frac{|x|D(m,|x|)}{\pi r}C_m^0(\cos\theta)\int_0^\infty\frac{\lambda G_m(\lambda,|x|,r)}{J_m^2(\lambda r)+N_m^2(\lambda r)}\exp\Big(-\frac{1}{2}\lambda^2t\Big)d\lambda,\end{aligned}$$

where σ is the area measure on $\sum^{d-1}(0,r)$, θ and $G_{m+h}(\lambda,|x|,r)$ and $D(m,|x|)$ are defined as that in Theorem 3.1.

The following Corollary follows immediately from Theorem 3.2 and Lemma 2.2.

Corollary 3.1. *For $|x|>r,d\geq 2$, then*

$$P_x(T_r\in dt,T_r<\infty)/dt=-\frac{1}{\pi}\Big(\frac{r}{|x|}\Big)^h\int_0^\infty\frac{\lambda G_h(\lambda,|x|,r)}{J_h^2(\lambda r)+N_h^2(\lambda r)}\exp\Big(-\frac{1}{2}\lambda^2t\Big)d\lambda.$$

Remark 3.1. This agrees with the corresponding result in [4].

The following theorem is due to Wendel[6], for a different proof see [7,8].

Theorem 3.3. *For $x : |x| > r, d \geq 2, \alpha > 0$, then*

$$E_x[e^{-\alpha T_r} C_m^h(\cos\theta_{T_r})] = C_m^h(1)\Big(\frac{r}{|x|}\Big)^h \frac{K_{m+h}(\sqrt{2\alpha}|x|)}{K_{m+h}(\sqrt{2\alpha}\ r)}.$$

Proof. By using Theorem 3.2, Lemma 2.3 and Lemma 2.4 one gets

$$\begin{aligned} E_x[e^{-\alpha T_r} C_m^h(\cos\theta_{T_r})] &= \int_0^\infty \int_{\sum^{d-1}(0,r)} e^{-\alpha t} C_m^h(\cos\theta) P_x(T_r \in dt, X(T_r) \in dy, T_r < \infty) \\ &= -\frac{1}{\pi} C_m^h(1)\Big(\frac{r}{|x|}\Big)^h \int_0^\infty \frac{2\lambda G_{m+h}(\lambda, |x|, r)}{(\lambda^2 + 2\alpha)(J_{m+h}^2(\lambda r) + N_{m+h}^2(\lambda r))} d\lambda \\ &= C_m^h(1)\Big(\frac{r}{|x|}\Big)^h \frac{K_{m+h}(\sqrt{2\alpha}|x|)}{K_{m+h}(\sqrt{2\alpha}\ r)}. \end{aligned}$$

§4. The Joint Distribution of the Hitting Time and Place for a Shell

In this section,we state and prove the explicit formulas for the joint densities of the first hitting time and place for a concentric spherical shell.

Theorem 4.1. *Let $D = \{x : x \in R^d, a < |x| < b\}, a < b$ and $P_D(t, x, y)$ be the transition density function for the killed Brownian motion in D, then*

(1) *for $d \geq 3$,*

$$\begin{aligned} P_D(t, x, y) = \sum_{m=0}^\infty \sum_{n=1}^\infty & \frac{(m+h)\Gamma(h)\lambda_{m,n,h}^2 J_{m+h}^2(\lambda_{m,n,h} b)}{4\pi^{\frac{d-4}{2}}(J_{m+h}^2(\lambda_{m,n,h} a) - J_{m+h}^2(\lambda_{m,n,h} b))} C_m^h(\cos\theta) \\ & \cdot R(m, n, h, |x|) R(m, n, h, |y|) \exp\Big(-\frac{1}{2}\lambda_{m,n,h}^2 t\Big); \end{aligned}$$

(2) *for $d = 2$,*

$$\begin{aligned} P_D(t, x, y) = \sum_{m=0}^\infty \sum_{n=1}^\infty & |x| D(m, |x|) \frac{\pi^2 \lambda_{m,n,0}^2 J_m^2(\lambda_{m,n,0} b)}{2(J_m^2(\lambda_{m,n,0} a) - J_m^2(\lambda_{m,n,0} b))} C_m^0(\cos\theta) \\ & \cdot R(m, n, 0, |x|) R(m, n, 0, |y|) \exp\Big(-\frac{1}{2}\lambda_{m,n,0}^2 t\Big), \end{aligned}$$

where $\theta = \angle x0y$,

$$R(m, n, h, r) = r^{-h}(J_{m+h}(\lambda_{m,n,h} r) N_{m+h}(\lambda_{m,n,h} a) - J_{m+h}(\lambda_{m,n,h} a) N_{m+h}(\lambda_{m,n,h} r)),$$

$$D(m, |x|) = \frac{m}{2\pi|x|}, \quad \textit{if } m \neq 0;$$

$$D(m, |x|) = \frac{1}{2\pi|x|}, \quad \textit{if } m = 0;$$

$\lambda_{m,1,h}, \lambda_{m,2,h}, \cdots, \lambda_{m,n,h}, \cdots$ *are the positive roots of equation*

$$\frac{J_{m+h}(\lambda a)}{J_{m+h}(\lambda b)} = \frac{N_{m+h}(\lambda a)}{N_{m+h}(\lambda b)}.$$

Proof. (1) Since Theorem 4.1 can be proved along the same lines as the proof of Theorem 3.1, we only indicate the main steps. By using the separation variable technique, it follows from the proof of Theorem 3.1 that

$$P_d(t,x,y)=\sum_{m=0}^{\infty}R^{-h}C_m^h(\cos\theta)u_m(t,R),$$

where $u_m(t,R)$ is the solution of

$$\begin{cases}\frac{\partial u_m}{\partial t}=\frac{1}{2}\Big(\frac{\partial^2 u_m}{\partial R^2}+\frac{1}{R}\frac{\partial u_m}{\partial R}-\frac{(m+h)^2}{R^2}u_m\Big), & a<R<b,\quad t>0,\\ u_m(t,a)=u_m(t,b)=0, & t>0,\\ \lim\limits_{t\to 0}u_m(t,R)=R^hM(m,R)\delta_{|x|}(R), & a<R<b,\end{cases}\tag{4.1}$$

where $R=|y|$, $M(m,R)$ as defined in (3.8). Setting $u_m(t,R)=T_m(t)V_m(R)$, we have

$$T_m'(t)+\frac{1}{2}\beta T_m(t)=0,\quad t>0,\tag{4.2}$$

$$\frac{d^2V_m}{dR^2}+\frac{1}{R}\frac{dV_m}{dR}+\Big(\beta-\frac{(m+h)^2}{R^2}\Big)V_m(R)=0,\quad a<R<b,\tag{4.3}$$

where β is a separation constant. Then one gets the following eigenvalues problems

$$\begin{cases}\frac{d^2V_m}{dR^2}+\frac{1}{R}\frac{dV_m}{dR}+\Big(\beta-\frac{(m+h)^2}{R^2}\Big)V_m(R)=0,\quad a<R<b,\\ V_m(a)=0,\\ V_m(b)=0.\end{cases}\tag{4.4}$$

If $\beta\le 0$ one can easily check that (4.4) has only zero solution. In the sequel, we suppose $\beta>0$, and set $\sqrt{\beta}=\lambda$. The solution of equation in (4.4) which satisfies $V_m(a)=0$ is

$$V_m(R)=J_{m+h}(\lambda R)N_{m+h}(\lambda a)-N_{m+h}(\lambda R)J_{m+h}(\lambda a),$$

where λ is determined by $V_m(b)=0$:

$$\frac{J_{m+h}(\lambda a)}{J_{m+h}(\lambda b)}=\frac{N_{m+h}(\lambda a)}{N_{m+h}(\lambda b)}.\tag{4.5}$$

Denote by $\lambda_{m,1,h},\lambda_{m,2,h},\cdots,\lambda_{m,n,h},\cdots$ the positive roots of equation (4.5), and inserting $\lambda_{m,n,h}$ into (4.2) one gets

$$T_{m,n}(t)=C_{m,n}\exp\Big(-\frac{1}{2}\lambda_{m,n,h}^2t\Big).$$

Thus the solution of (4.1) is

$$u_m(t,R)=\sum_{n=1}^{\infty}C_{m,n}V_m(n,h,R)\exp\Big(-\frac{1}{2}\lambda_{m,n,h}^2t\Big),$$

where

$$V_m(n,h,R)=J_{m+h}(\lambda_{m,n,h}R)N_{m+h}(\lambda_{m,n,h}a)-N_{m+h}(\lambda_{m,n,h}R)J_{m+h}(\lambda_{m,n,h}a).$$

From the general theory of Sturm-Liouville problems, it follows that

$$\int_a^b RV_m(n,h,R)V_m(k,h,R)dR=0,\quad\text{for } n\neq k.$$

For $n=k$, by using the initial conditon in (4.1) one obtains

$$C_{m,n}=\frac{|x|^{h+1}M(m,|x|)V_m(n,h,|x|)}{\int_a^b RV_m^2(n,h,R)dR}.\tag{4.6}$$

By using formula (see [17])

$$\int_a^b RZ_\nu^2(\lambda R)dR=\frac{(\lambda R)^2[Z_\nu'(\lambda R)]^2+[(\lambda R)^2-\nu^2](Z_\nu(\lambda R))^2}{2\lambda^2}\Big|_a^b,$$

where Z_ν is any cylinder function with order ν, and also Wronskian relation

$$J_\nu(z)N'_\nu(z) - J'_\nu(z)N_\nu(z) = \frac{2}{\pi z},$$

one obtains after a straightforward but tedious calculation

$$\int_a^b R(V_m(n,h,R))^2 dR = \frac{2}{\pi^2\lambda^2_{m,n,h}}\Big(\frac{J^2_{m+h}(\lambda_{m,n,h}a)}{J^2_{m+h}(\lambda_{m,n,h}b)} - 1\Big).$$

Substituting above into (4.6) we obtain

$$C_{m,n} = |x|^{h+1}M(m,|x|)V_m(n,h,|x|)\frac{\pi^2\lambda^2_{m,n,h}J^2_{m+h}(\lambda_{m,n,h}b)}{2(J^2_{m+h}(\lambda_{m,n,h}a) - J^2_{m+h}(\lambda_{m,n,h}b))}.$$

Finally, we have

$$\begin{aligned} P_D(t,x,y) &= \sum_{m=0}^\infty R^{-h}C^h_m(\cos\theta)u_m(t,R) \\ &= \sum_{m=0}^\infty\sum_{n=1}^\infty \frac{(m+h)\Gamma(h)\lambda^2_{m,n,h}J^2_{m+h}(\lambda_{m,n,h}b)}{4\pi^{\frac{d-4}{2}}(J^2_{m+h}(\lambda_{m,n,h}a) - J^2_{m+h}(\lambda_{m,n,h}b))}C^h_m(\cos\theta) \\ &\quad \cdot R(m,n,h,|x|)R(m,n,h,|y|)\exp\Big(-\frac{1}{2}\lambda^2_{m,n,h}t\Big), \end{aligned}$$

where

$$R(m,n,h,r) = r^{-h}(J_{m+h}(\lambda_{m,n,h}r)N_{m+h}(\lambda_{m,n,h}a) - J_{m+h}(\lambda_{m,n,h}a)N_{m+h}(\lambda_{m,n,h}r)).$$

For the case $d = 2$, the proof is almost identical with that of the case $d \geq 3$ and will be omitted.

The following two theorems follow immediately from Theorem 4.1 and Lemma 2.1(1).

Theorem 4.2. *For $x \in D = \{x : x \in R^d, a < |x| < b\}, a < b, y \in \partial D, d \geq 3$, then*

(1) *for $|y| = b$,*

$$\begin{aligned} &P_x(T_{ab} \in dt, X(T_{ab}) \in dy)/dt\sigma(dy) \\ &= \sum_{m=0}^\infty\sum_{n=1}^\infty \frac{(m+h)\Gamma(h)\lambda^2_{m,n,h}J_{m+h}(\lambda_{m,n,h}a)J_{m+h}(\lambda_{m,n,h}b)}{4\pi^h b^{(h+1)}(J^2_{m+h}(\lambda_{m,n,h}a) - J^2_{m+h}(\lambda_{m,n,h}b))} \\ &\quad \cdot C^h_m(\cos\theta)R(m,n,h,|x|)\exp\Big(-\frac{1}{2}\lambda^2_{m,n,h}t\Big); \end{aligned}$$

(2) *for $|y| = a$,*

$$\begin{aligned} &P_x(T_{ab} \in dt, X(T_{ab}) \in dy)/dt\sigma(dy) \\ &= -\sum_{m=0}^\infty\sum_{n=1}^\infty \frac{(m+h)\Gamma(h)\lambda^2_{m,n,h}J^2_{m+h}(\lambda_{m,n,h}b)}{4\pi^h a^{(h+1)}(J^2_{m+h}(\lambda_{m,n,h}a) - J^2_{m+h}(\lambda_{m,n,h}b))} \\ &\quad \cdot C^h_m(\cos\theta)R(m,n,h,|x|)\exp\Big(-\frac{1}{2}\lambda^2_{m,n,h}t\Big), \end{aligned}$$

where $\theta = \angle x0y, \sigma(dy)$ is the area measure on $\sum^{d-1}(0,a) \cup \sum^{d-1}(0,b), R(m,n,h,r)$ and $\lambda_{m,n,h}$ as defined in Theorem 4.1.

Theorem 4.2′. *For $x \in D = \{x : x \in R^2, a < |x| < b\}, a < b, y \in \partial D$, then*

(1) *for* $|y| = b$,

$$\begin{aligned} &P_x(T_{ab} \in dt, X(T_{ab}) \in dy)/dt\sigma(dy) \\ &= \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{|x| D(m,|x|)\pi\lambda_{m,n,0}^2 J_m(\lambda_{m,n,0}a) J_m(\lambda_{m,n,0}b)}{2b(J_m^2(\lambda_{m,n,0}a) - J_m^2(\lambda_{m,n,0}b))} \\ &\quad \cdot C_m^0(\cos\theta) R(m,n,0,|x|) \exp\Big(-\frac{1}{2}\lambda_{m,n,0}^2 t\Big); \end{aligned}$$

(2) *for* $|y| = a$,

$$\begin{aligned} &P_x(T_{ab} \in dt, X(T_{ab}) \in dy)/dt\sigma(dy) \\ &= -\sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{|x| D(m,|x|)\pi\lambda_{m,n,0}^2 J_m^2(\lambda_{m,n,0}b)}{2a(J_m^2(\lambda_{m,n,0}a) - J_m^2(\lambda_{m,n,0}b))} \\ &\quad \cdot C_m^0(\cos\theta) R(m,n,0,|x|) \exp\Big(-\frac{1}{2}\lambda_{m,n,0}^2 t\Big), \end{aligned}$$

where $\sigma(dy)$ *is the area measure on*

$$\sum\nolimits^{d-1}(0,a) \bigcup \sum\nolimits^{d-1}(0,b),$$

$\theta, \lambda_{m,n,0}, D(m,|x|)$ *and* $R(m,n,0,r)$ *as defined in Theorem* 4.1.

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